Essays on Games on Networks

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The Faculty of Business, Economics and Informatics of the University of Zurich hereby authorizes the printing of this dissertation, without indicating an opinion of the views expressed in the work.

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The Chairman of the Doctoral Board: Prof. Dr. Steven Ongena

To my parents

Contents

Preface

This thesis consists of three essays on games on networks, each forming a chapter in this work. The first essay is entitled *Conformist and Anti-conformist Behavior in Social Networks* (Chapter 1), the second essay *Conformism under Incomplete Information* (Chapter 2), and the third essay *Local Key Player Analysis* (Chapter 3). The second essay is co-authored with Theodoros Rapanos (Stockholm University) and Yves Zenou (Stockholm University and Monash University). An earlier version of the third essay circulated under the title *Centrality with Vertex Idiosyncracy: Comparative Statics and Vertex-weighted Key-player Analysis*.

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Abbreviations

BNE Bayesian Nash equilibrium

LD local differencing

NAHLA nonaffine heterogeneous local average

NALA nonaffine local average

NE Nash equilibrium

NRD normally regular digraph

Symbols

Below is a list of frequently used symbols and their definitions.

Number systems

$\mathbb{C}$	set (field) of complex numbers
$\mathbb{R}$	set (field) of real numbers
$\mathbb{R}_+$	set of nonnegative real numbers
$\mathbb{R}_{++}$	set of positive real numbers
$\mathbb{Z}_+$	set of nonnegative integers
$\mathbb{Z}_{++}$	set of positive integers
$\mathcal{I}(n)$	set of integers $\{1,\ldots,n\}$

Set theory

$\partial\,\mathcal{S}$	boundary of set $\mathcal{S}$ (with respect to Euclidean topology)
$\lvert\mathcal{S}\rvert$	cardinality of set $\mathcal{S}$
$\complement_{\mathcal{U}}\mathcal{S}$	(absolute) complement of set $\mathcal{S}$ in universe $\mathcal{U}$
$\mathrm{int}(\mathcal{S})$	interior of set $\mathcal{S}$ (with respect to Euclidean topology)
$\subset$	weak set inclusion

Mappings

$\partial^k f$	derivative of function f of order k
$\mathrm{id}_{\mathcal{S}}$	identity mapping with domain and codomain $\mathcal{S}$
$\mathrm{im}(f)$	image of mapping f
$\mathbb{1}_{\mathcal{S}}$	indicator function of set $\mathcal{S}$
$\delta_{i,j}$	Kronecker's delta of i and j
$f{\restriction}_{\mathcal{S}}$	restriction of mapping f to subset $\mathcal{S}$ of its domain
sgn	sign function

Matrices

$\mathcal{M}(m,n,\mathbb{F})$	set of all $m \times n$ matrices over field $\mathbb{F}$
$\mathcal{M}(n,\mathbb{F})$	set of all square matrices of order n over field $\mathbb{F}$
$\geq_c$	componentwise defined weak partial order on $\mathcal{M}(n,\mathbb{R})$
$>_c$	componentwise defined strong partial order on $\mathcal{M}(n,\mathbb{R})$
$\lVert\cdot\rVert$	a sub-multiplicative matrix norm
$\lVert\cdot\rVert_1$	maximum absolute column sum norm
$\lVert\cdot\rVert_\infty$	maximum absolute row sum norm

$\boldsymbol{O}_n$	zero matrix of order n
$\boldsymbol{I}_n$	identity matrix of order n
$\lvert\boldsymbol{A}\rvert$	matrix the components of which are the absolute values of the components of $\boldsymbol{A}$
$[\boldsymbol{A}]_{i,j}$	component in row i and column j of matrix $\boldsymbol{A}$
$[\boldsymbol{A}]_{i,\bullet}$	row i of matrix $\boldsymbol{A}$
$[\boldsymbol{A}]_{\bullet,j}$	column j of matrix $\boldsymbol{A}$
$[\boldsymbol{A}]_{\mathcal{R},\mathcal{C}}$	submatrix of square matrix $\boldsymbol{A}$ of order n formed by rows of $\boldsymbol{A}$ with indices in $\mathcal{R}$ and columns of $\boldsymbol{A}$ with indices in $\mathcal{C}$, where $\mathcal{R}$ and $\mathcal{C}$ are nonempty, proper subsets of $\{1,\ldots,n\}$
$[\boldsymbol{A}]_{-\mathcal{R},-\mathcal{C}}$	submatrix of square matrix $\boldsymbol{A}$ of order n obtained by deleting rows with indices in $\mathcal{R}$ and columns with indices in $\mathcal{C}$, where $\mathcal{R}$ and $\mathcal{C}$ are nonempty, proper subsets of $\{1,\ldots,n\}$
$\operatorname{adj}(\boldsymbol{A})$	adjugate of square matrix $\boldsymbol{A}$
$\text{c-sp}(\boldsymbol{A})$	column space of matrix $\boldsymbol{A}$
$\det(\boldsymbol{A})$	determinant of square matrix $\boldsymbol{A}$
$\boldsymbol{A}^{-1}$	inverse of nonsingular square matrix $\boldsymbol{A}$
$\ker(\boldsymbol{A})$	kernel of matrix $\boldsymbol{A}$
$\rho(\boldsymbol{A})$	spectral radius of square matrix $\boldsymbol{A}$
$\sigma(\boldsymbol{A})$	spectrum of square matrix $\boldsymbol{A}$
$\boldsymbol{A}^{\mathsf{T}}$	transpose of matrix $\boldsymbol{A}$
$\boldsymbol{A}\circ\boldsymbol{B}$	Hadamard product of conformable matrices $\boldsymbol{A}$ and $\boldsymbol{B}$
$\boldsymbol{A}\otimes\boldsymbol{B}$	Kronecker product of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$

Vectors

$\mathbb{R}^n$	n-dimensional Euclidean space
$\geq_c$	componentwise defined weak partial order on $\mathbb{R}^n$
$>_c$	componentwise defined strong partial order on $\mathbb{R}^n$
$\langle\cdot,\cdot\rangle$	standard inner product
$\lVert\cdot\rVert_2$	Euclidean vector norm
$\mathbf{0}_n$	zero (column) vector in $\mathbb{R}^n$
$\mathbf{1}_n$	(column) vector of ones in $\mathbb{R}^n$
$(a_1,\ldots,a_n)$	(column) vector in $\mathbb{R}^n$
$\lvert\boldsymbol{a}\rvert$	(column) vector the components of which are the absolute values of the components of $\boldsymbol{a}$
$[\boldsymbol{a}]_i$	component i of (column) vector $\boldsymbol{a}$
$[\boldsymbol{a}]_{-i}$	(column) vector $\boldsymbol{a}$ without ith component
$[\boldsymbol{a}]_{\mathcal{R}}$	subvector of (column) vector $\boldsymbol{a}$ in $\mathbb{R}^n$ formed by rows of $\boldsymbol{a}$ with indices in $\mathcal{R}$, where $\mathcal{R}$ is a nonempty, proper subset of $\{1,\ldots,n\}$
$\{\boldsymbol{e}_i\}_{i=1}^n$	canonical basis of $\mathbb{R}^n$ with $\boldsymbol{e}_i := (\delta_{i,1},\ldots,\delta_{i,n})$
$\operatorname{span}\mathcal{S}$	linear subspace of $\mathbb{R}^n$ spanned by set of vectors $\mathcal{S}$ in $\mathbb{R}^n$

Graph theory

$\dot{\boldsymbol{A}}(D)$	adjacency matrix of digraph D

$\mathcal{A}(D)$	arc set of digraph D
$\operatorname{diam}(D)$	diameter of digraph D
$\operatorname{sl}(D)$	directed pseudograph derived from digraph D by adding self-loops to all vertices with zero out-degree
$\operatorname{dist}_D(u,v)$	distance from vertex u to vertex v in digraph D
$\deg_D^-(v)$	in-degree of vertex v in digraph D
$\mathcal{N}_D^-(v)$	in-neighborhood of vertex v in digraph D
$\deg_D^+(v)$	out-degree of vertex v in digraph D
$\mathcal{N}_D^+(v)$	out-neighborhood of vertex v in digraph D
$D^\top$	transpose of digraph D
$\mathcal{V}(D)$	vertex set of digraph D

Miscellanea

$\mathcal{O}$	Landau's big-O symbol
o	Landau's little-o symbol
0^0	undefined expression

Introduction

Many aspects of our lives are governed by social networks. For example, the decision to buy a new product, to attend a party, or to commit a crime is often influenced by acquaintances and friends. The influence can take many forms, for example, the choices or information sharing of others: A teenager might attend a party only if her best friend is attending. The decision to upgrade to a new software version might depend on the expertise of informed acquaintances.

The analysis of social networks is a growing field within economics. Economics provides an analytical framework to study the behavior of individuals embedded in a social network. One strand of the literature focuses on how behavior is shaped by the individuals' preferences and the structure of their network. The influence of preferences and network on behavior is often modeled by what is called *games on networks* or *network games*. A network game is a strategic game where finitely many players (for example, individuals, enterprises, or countries) are connected by a network. Each player chooses an action (for example, to buy a product, to choose a level of education, to decide on the time to share with friends or to spend on sports activities, to engage in criminal activities) in order to maximize his or her utility or payoff, given the behavior of other players. Players take thereby explicitly or implicitly the interdependencies induced by the structure of the network into account.

The present thesis consists of three essays on games on networks, each forming a chapter in this work. A brief summary of each chapter is given below.

Chapter 1: Conformist and Anti-conformist Behavior in Social Networks

This paper proposes a network game of complete information where the players have a preference for either conformist or anti-conformist behavior. The network game admits of nonaffine best reply functions and players without out-neighbors. Admitting of players without out-neighbors demonstrates that an endogenous effects matrix is always row-normalized, even in the presence of isolated players. The paper provides new results on the identification of endogenous and exogenous effects in a statistical model that is derived from the system of best reply functions at the Nash equilibrium of the network game. A novel nonidentifying condition in the form of a kernel condition has its origin in the notion of weakly ex ante homogeneous players. It is shown that the typical linear independence condition involving products or powers of the endogenous and exogenous effects matrices is in general only necessary but not sufficient for identification through the conditional mean of endogenous and exogenous effects. Linear independence conditions are, however, sufficient for identification through the conditional variance and entail restrictions on the topologies of the digraphs from which the endogenous and exogenous effects matrices are derived. While such restrictions are in general difficult to characterize, a complete characterization is possible and related to the notion of a normally regular digraph in two cases where the conditions involve three matrices. The nonaffine nature of the network game is instrumental in aligning the support of the statistical model with the action space of the underlying network game. An important class of existing linear social interactions models can be reconciled with the network game by imposing a parameter restriction that is not without loss of generality.

Chapter 2: Conformism under Incomplete Information

Although conformism has been studied in a network setting before, this is one of the first papers to examine conformism under incomplete information, and it is the first to provide and discuss a comprehensive theoretical framework. Social interaction is modelled as a Bayesian network game, which is the natural setting for analyzing decisions whose potential returns or costs are ex ante uncertain (like, for example, in education and crime). The paper establishes existence and uniqueness of the equilibrium, characterizes the optimal decisions, and examines conditions under which policy interventions can be welfare-improving.

Chapter 3: Local Key Player Analysis

Identifying key players is an important aspect in network analysis. Key players are vertices in a network who are considered important in a certain sense. A precise definition depends on the context and the purpose to which their identification is put. In the context of crime, for example, the key player is defined as the criminal who, once removed from the criminal network, reduces criminal activity most. This paper introduces formally the concept of a local key player in the context of a network game where the social planner's objective is to reduce aggregate activity of only those players who reside in a certain local area or part of the network. In the context of crime, where networks of criminals spread across different police areas, local key player analysis provides a means to identify and neutralize the key criminal in each police area in order to reduce criminal activity locally (that is, in each police area) most, thereby taking the criminals' cross-area connections into account.

Chapter 1

Conformist and Anti-conformist Behavior in Social Networks

Abstract

This paper proposes a static, noncooperative network game of complete information where the players have a preference for either conformist or anti-conformist behavior. The network game admits of nonaffine best reply functions and players without out-neighbors. A special case of the game entails an asymmetric loss function and endogenous social norms that are asymptotically geometric means. Admitting of players without out-neighbors demonstrates that an endogenous effects matrix is always row-normalized, even in the presence of isolated players. The paper provides new results on the identification of endogenous and exogenous effects in a statistical model that is derived from the system of best reply functions at the Nash equilibrium of the network game. A nonidentifying condition in the form of a kernel condition has its origin in the notion of weakly ex ante homogeneous players. It is shown that the typical linear independence condition involving products or powers of the endogenous and exogenous effects matrices (see, for example, Bramoullé, Djebbari, and Fortin 2009, Propositions 1, 4, and 5; Blume et al. 2015, Theorem 3) is in general only necessary but not sufficient for identification through the conditional mean of endogenous and exogenous effects. Linear independence conditions are, however, sufficient for identification through the conditional variance and entail restrictions on the topologies of the digraphs from which the endogenous and exogenous effects matrices are derived. While such restrictions are in general difficult to characterize, a complete characterization is possible and related to the notion of a normally regular digraph (Jørgensen 2015) in two cases where the conditions involve three matrices. The nonaffine nature of the network game is instrumental in aligning the support of the statistical model with the action space of the underlying network game. An important class of existing linear social interactions models can be reconciled with the network game by imposing a parameter restriction that is not without loss of generality.

1.1 Introduction

Social interactions are ubiquitous and an important feature of modern life.[1] Examples include cooperative learning in the classroom, the exchange of information, and the use of social media by groups of individuals with a common interest in order to find and meet other like-minded people. Social interactions often manifest in interdependent or even correlated behavior of socially adjacent individuals. One important example is conformist behavior where an individual's well-being or utility is decreasing in the distance between his own behavior and the behavior of a group of socially adjacent individuals, frequently referred to as peers. To give an instance of conformist behavior, excessive alcohol consumption among Canadian and U.S. American college students is strongly influenced by peers (Borsari and Carey 2001).

A rich body of literature has emerged in the last two decades on the economics of social interactions (for surveys, see, for example, Manski 2000; Blume and Durlauf 2001; Glaeser and Scheinkman 2001; Moffitt 2001; Durlauf 2004; Durlauf and Ioannides 2010; Ioannides 2012). One main strand of the literature is concerned with identifying and measuring the origins and natures of correlated behavior of individuals connected in social space (see Blume et al. 2011 for a comprehensive survey) and has experienced a strong growth ever since Manski (1993) advanced a taxonomy of hypotheses, referred to as effects, and discussed a related identification problem. Manski's (1993) taxonomy distinguishes between three different hypotheses for correlated behavior in a group of socially connected individuals: *endogenous effects*, *exogenous effects*, also called *contextual effects*, and *correlated effects*. Endogenous effects arise if an individual's behavior tends to vary with a statistic of the behavior of the group; exogenous effects arise if an individual's behavior tends to vary with a statistic of the exogenous characteristics of the group; and correlated effects arise if individuals of the same group tend to behave similarly because they have similar exogenous characteristics or face a common environment. As pointed out by Manski (1993) and others, the distinction between endogenous, exogenous, and correlated effects is important because they have different implications for policy interventions, in particular, only endogenous effects can generate a social multiplier.[2] Manski's (1993) taxonomy of hypotheses is reflected in the structural form of the statistical model, dubbed linear-in-means model, within which he discusses identification of the three effects, in particular, the non-identification of endogenous and exogenous effects. Within the class of linear statistical models that have an uncountable support, most of the models put forward to identify and measure the origins and natures of correlated behavior of individuals connected in social space are variations of Manski's (1993) linear-in-means model. A typical

1. In economics social interactions refer to direct interactions of individuals connected in social space who exhibit interdependencies in their preferences or beliefs or in the constraints they face. They are therefore different and distinct from interactions mediated by a market and as such are sometimes considered externalities (see, for example, Ioannides 2012, chapters 3, 4, and 5).

2. For a discussion of the social multiplier see, for example, Glaeser, Scheinkman, and Sacerdote (2003).

linear model relates an individual's behavioral outcome to his characteristics, to arithmetic means of the characteristics of socially adjacent individuals, and to a statistic of their outcomes. The models differ by the statistic used to summarize the behavioral outcomes of socially adjacent individuals, the constraints imposed on the topology of the social network by which the individuals are connected, and the dependence structure and distribution of the error terms. In Manski's (1993) model, the statistic is the arithmetic mean, which explains the term linear-in-means model. The statistic is also the arithmetic mean in numerous extensions and generalizations of the linear-in-means model (see, for example, Bramoullé, Djebbari, and Fortin 2009; Patacchini and Zenou 2012; Boucher et al. 2014; Fortin and Yazbeck 2015; Lin 2015; Tatsi 2015). Other statistics than the arithmetic mean include the sum (see, for example, Calvó-Armengol, Patacchini, and Zenou 2009; Tatsi 2015) and the minimum or maximum (Tao and Lee 2014; Tatsi 2015). The mode and the median are also possible statistics but have not been discussed in the literature yet. So-called hybrid models have also been considered where the behavior of socially adjacent individuals enters through at least two statistics, for example, two different sums in case adjacent individuals are partitioned into two sets (Patacchini, Rainone, and Zenou 2015), the arithmetic mean and the sum (Liu, Patacchini, and Zenou 2014; Liu et al. 2015; Lindquist, Sauermann, and Zenou 2016), or the mean and the maximum (Tao and Lee 2014; Tatsi 2015).

As noted by Blume et al. (2015, pp. 445–46), the aforementioned strand of the literature is not well integrated with respect to economic and econometric theory. This applies in particular to linear models of social interactions (which precludes discrete choice models) where the individuals have a preference or taste for conformist behavior. As regards this class of models, the discrepancy between economic models and econometric or statistical models takes various forms.[3] First and foremost, the predictions of economic models, for example, the system of best reply functions at the Nash equilibrium of a network game, are distinct from the statistical models used to study parameter identification and estimation and taken to data in applied work with regard to structural characteristics. This concerns, for example, the way in which statistical models admit of individuals who are not affected by other individuals' actions but whose actions may affect other individuals, which hereinafter are referred to as players without out-neighbors. Statistical models used to study parameter identification typically admit of players without out-neighbors (Bramoullé, Djebbari, and Fortin 2009; Blume et al. 2015), whereas economic models either do not admit of this type of individuals (Patacchini and Zenou 2012) or they do so in an inconsistent way (Blume et al. 2015).[4] Second, the codomains of behavioral variables in economic models are often not in alignment with the supports of (the response variable(s) in) corresponding statistical models.[5] The

3. The terms *econometric model* and *statistical model* are used synonymously.

4. At the Bayesian Nash equilibrium of the network game of Blume et al. (2015), the action of a player without out-neighbors is decreasing in the endogenous social interactions parameter if his private marginal benefit is positive.

5. A behavioral variable is a mapping that associates with each individual a value in some set, the codomain of the mapping.

supports of linear statistical models have a nonempty intersection with the negative real line, whereas in economic models, behavioral outcomes are predominantly and inherently nonnegative, for example, educational achievement, consumption of substances like alcohol and cigarettes, time spent on recreational activities like doing sports or playing video games, criminal activity, and labor market outcomes. Third, statistical models often admit of negative values of the endogenous social interactions parameter, its domain is often restricted to the open ball (in the real line) with center zero and radius one.[6] It is as yet unclear whether a negative and statistically significant estimate of the endogenous social interactions parameter (see, for example, Tao and Lee 2014, Table 7) is reconcilable with an economic model, specifically, with the existence of a Nash equilibrium of a network game, where the codomain of the behavioral variable is a subset of the nonnegative real line. On top of these discrepancies, little is known about the properties of an economic model that underlies a typical statistical model used in applied work. This concerns in particular results for policy recommendations.

The objective of this paper is to advance the integration of economic and econometric theory for models of social interactions where the players have a preference or taste for either conformist or anti-conformist behavior, where anti-conformist behavior is to be understood as the antonym of conformist behavior, that is, an individual has a preference or taste for anti-conformist behavior if his well-being or utility is increasing in the distance between own behavior and the behavior of a group of socially adjacent individuals.

In order to achieve its objective, this paper proposes a static, noncooperative network game of complete information that admits of nonaffine best reply functions. The players of the game are characterized by a common action space, which is the real line, the nonnegative real line, or a compact interval with zero as its lower bound. Given the majority of behavioral outcomes in applied work is nonnegative, the real line is the least relevant action space, but it is the most convenient and tractable one from a mathematical and statistical point of view. The compact interval is particulary suitable for outcomes that involve a natural upper bound like, for example, hours spent per day on some activity. The nonaffine nature of the game is interesting for two reasons. First, it admits of an asymmetric loss function that maps a player's action and the actions of socially adjacent players to a social cost (in case of conformist behavior) or benefit (in case of anti-conformist behavior) and of endogenous social norms that are asymptotically geometric means (as opposed to arithmetic means in the affine case). Second, it is instrumental in aligning the action space of the network game with the support of the corresponding statistical model in case the action space is not the real line.

This paper contributes to the literature on the economics of social interactions in many respects. First, it demonstrates that a negative endogenous social interactions parameter is reconcilable with equilibrium behavior in a network game where the players have a preference for anti-conformist behavior and a common action space

6. In this paper, the endogenous social interactions parameter is called *social cost parameter* if it is positive and *social benefit parameter* if it is negative.

that is a subset of the nonnegative real line. Second, the network game admits of players without out-neighbors, for example, players that have no role models but may serve as role models to other players. Players without out-neighbors yield new insights on the structural characteristics of a typical linear social interactions model and give rise to new results on parameter identification. As regards structural characteristics, players without out-neighbors entail an endogenous effects matrix (that is, a matrix whose components determine endogenous effects) that is structurally different from an exogenous effects matrix (that is, a matrix whose components determine exogenous effects). Specifically, if player i has no out-neighbors, then the ith row of the endogenous effects matrix is the ith unit vector (with a one in the ith component and zeros elsewhere) and the ith row of the exogenous effects matrix is the zero vector. The endogenous effects matrix is therefore always row-normalized, even in the presence of isolated players. This insight is unprecedented in the literature. Third, the paper introduces the notion of weakly ex ante homogeneous players and recognizes its role for parameter identification in the form of a kernel condition that has not been discussed in the literature yet. Fourth, it is shown that the typical linear independence condition found in the literature involving products or powers of the endogenous and exogenous effects matrices (see, for example, Bramoullé, Djebbari, and Fortin 2009, Propositions 1, 4, and 5; Blume et al. 2015, Theorem 3) is in general only necessary but not sufficient for the identification of endogenous and exogenous effects through the mean.[7] Linear independence conditions are, however, sufficient for identification through the variance and entail restrictions on the topologies of the digraphs from which the endogenous and exogenous effects matrices are derived. While such restrictions are in general difficult to characterize, a complete characterization is possible and related to the notion of a normally regular digraph (Jørgensen 2015) in two cases where the conditions involve three matrices.

The rest of the paper is structured as follows. Section 1.2 discusses related literature. Section 1.3 is concerned with the baseline economic model of social interactions. It introduces the aforementioned network game (Section 1.3.1) and a typology of player heterogeneity (Section 1.3.2). Sufficient conditions for the existence of a unique and interior Nash equilibrium of the network game are given in Section 1.3.3. This is followed by a discussion of the properties of Nash equilibrium actions (Section 1.3.4), a welfare analysis (Section 1.3.5), and a policy analysis (Section 1.3.6). Three extensions of the baseline model are discussed in Section 1.3.7; the most important of all concerns the players' idiosyncrasies (Section 1.3.7.1) and admits of idiosyncrasies with local externalities, which give rise to exogenous effects. Section 1.4 translates the economic model to a statistical model. The assumptions invoked in the translation (Sections 1.4.1 and 1.4.2) are guided by the desire to get a close resemblance of the resulting statistical model with existing models in the social interactions literature. The identification problem is discussed in Section 1.4.3. Section 1.4.4 examines the existence of a statistical model and a nonaffine

7. This result is of course due to differing definitions of the notion of an identified parameter. This paper's discussion of the identification problem is based on the statistical notion of identification where a parameter is called identified if the mapping that associates to any parameter point a probability distribution of the response variable is injective (see, for example, Koopmans and Reiersol 1950).

network game such that the supports of the former are in alignment with the action space of the latter. Section 1.4.5 shows that an important class of existing linear social interactions models can be reconciled with the network game by imposing a parameter restriction that is not without loss of generality. Finally, Section 1.5 concludes. A brief review of basic concepts in graph theory is given in Appendix A. Some basic results in matrix analysis are collected in Appendix B. Tables can be found in Appendix C and the proofs of all main results in Appendix D.

1.2 Related literature

This paper lies at the intersection of two strands of the literatures on game theory and the economics of social interactions: games on networks and the identification of social interactions. Both strands are too vast to survey here.[8] Instead, in order to place this paper into an appropriate context, only selected work is briefly discussed.

As regards games on networks, this paper is closely related to previous work by Patacchini and Zenou (2012), Sommer and Sulger (2012), and Blume et al. (2015). The network game of this paper is an extension of Sommer and Sulger's (2012) *generalized local-average conformity game*, which in turn is a generalization of Patacchini and Zenou's (2012) *local average game*. The latter network game is a static, noncooperative game of complete information where the players have a nonnegative action space, are connected by an undirected network, and have multi-affine best reply functions. It draws on ideas and concepts from the economics literature on conformism (see, among others, Jones 1984; Kandel and Lazear 1992; Bernheim 1994; Akerlof 1997; Fershtman and Weiss 1998). Sommer and Sulger (2012) generalize the local average game to the case of best reply functions that are not multi-affine. The network game of this paper is an extension of Sommer and Sulger's (2012) network game to the case of players without out-neighbors. The network game of Blume et al. (2015) is similar in structure to the local average game, but there are important differences. First, it is a game of incomplete information about the players' private benefits. Second, it involves weaker assumptions about the players' preferences. Third, it is somewhat less appealing because of an action space that is equal to the real line.

As regards identification of social interactions, the problem of identifying endogenous and exogenous effects was first formally studied by Manski (1993) in the context of what is commonly known as the linear-in-means model. Subsequent research has addressed the identification problem in numerous variations and extensions of the linear-in-means model (see, for example, Graham and Hahn 2005; Lee 2007; Graham 2008; Bramoullé, Djebbari, and Fortin 2009; Davezies, D'Haultfoeuille, and Fougère 2009; De Giorgi, Pellizzari, and Redaelli 2010; Blume et al. 2015). Of the more recent research, Bramoullé, Djebbari, and Fortin (2009) and Blume et al. (2015) stand out because their models impose weak assumptions on the individuals' connections. They share some similarities with the models of this paper, but there are also notable differences.

8. See, for example, Jackson and Zenou (2015) for a recent survey on games on networks and Blume et al. (2011) for a survey on the identification of social interactions.

Bramoullé, Djebbari, and Fortin (2009) discuss identification of endogenous and exogenous effects by two statistical models that may be considered variations of Manski's (1993) linear-in-means model and Moffitt's (2001) model, in particular, a model without fixed effects and a model with fixed effects. Both models feature a single so-called interaction matrix whose components determine both endogenous and exogenous effects, that is, the endogenous effects matrix and the exogenous effects matrix are the same.[9] Bramoullé, Djebbari, and Fortin (2009) discuss identification of social effects, which subsumes both endogenous and exogenous effects, by the two models. They show that social effects are identified if a parameter restriction and a linear independence condition involving powers of the interaction matrix are satisfied (see Propositions 1, 4, and 5).

Blume et al. (2015) discuss identification of endogenous and exogenous effects by a statistical model without fixed effects that is derived from a network game. Their model admits of distinct endogenous and exogenous effects matrices.[10] Both matrices have the same structural characteristics, in particular, they are nonnegative with row sums that are equal to zero or one. They study parameter identification for the cases of observable individual data and observable aggregate data under different informational assumptions regarding the two matrices. For example, for the case of observable individual data and known matrices, they show that endogenous and exogenous effects are identified if a parameter restriction and a linear independence condition involving the endogenous and exogenous effects matrices are satisfied (see Theorem 3).

Although they differ in a variety of respects, Bramoullé, Djebbari, and Fortin (2009) and Blume et al. (2015) have two features in common. First, their statistical models admit of individuals without out-neighbors; moreover, their models agree in the manner they admit of this type of individuals. Specifically, an individual without out-neighbors corresponds to a row of zeros in both the endogenous and exogenous effects matrices. Second, their notion of an identified parameter (for the case of observable individual data) is weaker than—and notably not equivalent to—the statistical notation of an identified parameter.[11] As regards the foregoing two features, this paper is different. Similar to Blume et al. (2015), the statistical models admit of distinct endogenous and exogenous effects matrices, but they are *structurally different* in the presence of players without out-neighbors. Specifically, an individual without out-neighbors corresponds to a zero row vector in the exogenous effects matrix and a unit row vector in the endogenous effects matrix whose components are all zero except for the component lying on the matrix's main diagonal, which is equal to one. This characteristic of the endogenous effects matrix may seem counter-intuitive, but it is perfectly consistent with economic

9. Bramoullé, Djebbari, and Fortin's (2009) interaction matrix is structurally identical to what is called an exogenous effects matrix in this paper (see Definition B).

10. In Blume et al. (2015) the endogenous effects matrix is referred to as the peer-effects sociomatrix and the exogenous effects matrix as the contextual-effects sociomatrix.

11. Bramoullé, Djebbari, and Fortin (2009) call "social effects ... identified if and only if the ... structural parameters can be uniquely recovered from the unrestricted reduced-form parameters" (p. 44). For the case of observable individual data, Blume et al.'s (2015) notion of an identified parameter is defined via the injectivity of a certain mapping (see equation (2) and Definition 2).

theory (see Section 1.3.1). The structural difference between the two effects matrices carries profound implications for the identification of endogenous and exogenous effects, the discussion of which is based on the statistical notion of an identified parameter—thereby pointing to the difference in the second feature.

1.3 The economic model

1.3.1 The nonaffine local average game

The network game to be described hereinafter is a static, noncooperative game of complete information, where the players take their decisions simultaneously and independently of one another. The players are assumed to be rational in the sense that they seek to maximize their well-being.

There are $n > 1$ players. In what follows, each mathematical object associated with a particular player will be indexed by an element of the set $\mathcal{I} := \{1, \ldots, n\}$. Even a player will be abstractly represented by an element of $\mathcal{I}$, so that $\mathcal{I}$ corresponds to the set of players of the game.

The players share a common action space, which is denoted by $\mathcal{Y}$. An action of player $i \in \mathcal{I}$ is denoted by y_i. The set of all possible action profiles $\boldsymbol{y} := (y_1, \ldots, y_n)$ is equal to $\mathcal{Y}^n$. With a slight abuse of terminology, the set $\mathcal{Y}^n$ is referred to as the game's action space.

I consider three different types of action spaces and, hence, three different types of games: $\mathbb{R}$, $\mathbb{R}_+$, and $[0, \bar{v}]$, where $\bar{v} > 0$. Although $[0, \bar{v}] \subset \mathbb{R}_+$ and $\mathbb{R}_+ \subset \mathbb{R}$, a game with $\mathcal{Y} = [0, \bar{v}]$ is not a special case of a game with $\mathcal{Y} = \mathbb{R}_+$, and a game with $\mathcal{Y} = \mathbb{R}_+$ is not a special case of a game with $\mathcal{Y} = \mathbb{R}$. This is mainly due to the fact that the three sets $[0, \bar{v}]$, $\mathbb{R}_+$, and $\mathbb{R}$ are different from a topological point of view, specifically, $\partial[0, \bar{v}] = \{0, \bar{v}\}$, $\partial\, \mathbb{R}_+ = \{0\}$, and $\partial\, \mathbb{R} = \emptyset$.

As is characteristic for a game, a player's well-being depends not only on his action but may also depend on the actions of other players. This dependence is made explicit by means of a network by which the players are connected. I assume that this network is fixed and common knowledge and that it can be represented by a digraph G on $\mathcal{I}$. The digraph G encodes information about the identities of the players who directly affect a player's well-being through their actions. For a particular player, the set of players who directly affect his well-being is given by his *out-neighborhood*. For all $i \in \mathcal{I}$, player i's out-neighborhood in G is denoted by $\mathcal{N}_G^+(i)$ and its cardinality, the *out-degree* of i in G, by $\deg_G^+(i)$. Note that, for all $i \in \mathcal{I}$, $i \notin \mathcal{N}_G^+(i)$, which is a consequence of the definition of a digraph. The assumption of a digraph implies that a player is not necessarily an out-neighbor of his out-neighbors, that is, for all $i \in \mathcal{I}$, $j \in \mathcal{N}_G^+(i)$ does not necessarily imply that $i \in \mathcal{N}_G^+(j)$. In short, the dependence of a player's well-being on the actions of his out-neighbors is potentially unidirectional.

Occasionally, I assume that the digraph G is not empty, as stated by the following condition.

Condition G There exists a player $i \in \mathcal{I}$ with $\mathcal{N}_G^+(i) \neq \emptyset$.

Some subsets of $\mathcal{I}$ deserve a special notation: Let $\mathcal{I}_0(G)$ denote the set of all players that are isolated in G, that is, $\mathcal{I}_0(G) := \{i \in \mathcal{I} \mid \mathcal{N}_G^-(i) = \mathcal{N}_G^+(i) = \varnothing\}$, where $\mathcal{N}_G^-(i)$ is player i's in-neighborhood in G, and let $\mathcal{I}_0^+(G)$ denote the set of all players whose out-neighborhood in G is empty, that is, $\mathcal{I}_0^+(G) := \{i \in \mathcal{I} \mid \mathcal{N}_G^+(i) = \varnothing\} = \{i \in \mathcal{I} \mid \deg_G^+(i) = 0\}$. Note that $|\mathcal{I}_0(G)| < n$ and $|\mathcal{I}_0^+(G)| < n$ if Condition G is satisfied.

In order to state the assumption about the players' preferences over $\mathcal{Y}^n$, I introduce a function of quite general but otherwise unspecified form. Let f be a real-valued function with domain $\mathcal{Y}$. I assume that f satisfies the following assumption.

Assumption F The function f has the following four properties: (i) f is surjective; (ii) f is strictly increasing; (iii) f is differentiable on its domain with $\partial f := f' > 0$; and (iv) f is twice differentiable on the interior of its domain.

The set of all real-valued functions with domain $\mathcal{Y}$ that satisfy Assumption F is denoted by $\mathcal{F}(\mathcal{Y})$. Some basic properties of functions in $\mathcal{F}(\mathcal{Y})$ are given in the following result.

Lemma 1.1 *If $g \in \mathcal{F}(\mathcal{Y})$, then (i) $g\colon \mathcal{Y} \to g(\mathcal{Y})$ is continuous; (ii) $g(\mathcal{Y}) \subset \mathbb{R}$ is an interval; (iii) $g\colon \mathcal{Y} \to g(\mathcal{Y})$ is bijective; and (iv) $g^{-1}\colon g(\mathcal{Y}) \to \mathcal{Y}$ is continuous (which implies that $g\colon \mathcal{Y} \to g(\mathcal{Y})$ is open), strictly increasing, and continuously differentiable with $\partial g^{-1} = 1/(\partial g \circ g^{-1}) > 0$.*

Examples of functions in $\mathcal{F}(\mathcal{Y})$ include $\mathrm{id}_{\mathcal{Y}}$, the identity function on $\mathcal{Y}$, and $y \mapsto 1 - \exp(-\lambda y)$, where $\lambda > 0$. For $\mathcal{Y}$ equal to $\mathbb{R}_+$ or $[0, \bar{v}]$, two functions in $\mathcal{F}(\mathcal{Y})$ other than the identity are the two-parameter Box-Cox transformation (see Box and Cox 1964, p. 214),

$$y \mapsto \begin{cases} \dfrac{(y+\epsilon)^\lambda - 1}{\lambda} & \text{if } \lambda \neq 0, \\ \log(y+\epsilon) & \text{if } \lambda = 0, \end{cases}$$

and $y \mapsto \theta y + (1-\theta)\log(\epsilon + y)$, where $\lambda \in \mathbb{R}$, $\epsilon > 0$ is arbitrary small, and $\theta \in (0,1)$. The principal square root and its restriction to $[0, \bar{v}]$ are examples of functions that do not not lie in $\mathcal{F}(\mathbb{R}_+)$ and $\mathcal{F}([0, \bar{v}])$, respectively.[12]

I assume that the players' preferences over $\mathcal{Y}^n$ can be represented by a family of utility functions $\{u_i\colon \mathcal{Y}^n \to \mathbb{R}\}_{i \in \mathcal{I}}$ that satisfies the following assumption.

Assumption U For all $i \in \mathcal{I}$, player i's utility function $u_i\colon \mathcal{Y}^n \to \mathbb{R}$ is given by

$$u_i(y_1, \ldots, y_n) := p\big(f(y_i) \mid \alpha_i, \beta\big) + s_i\big(f(y_1), \ldots, f(y_n) \mid \gamma, G\big),$$

12. Let $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ be the principal square root function, that is, for all $y \in \mathbb{R}_+$, $f(y) = \sqrt{y}$. The function f is surjective, strictly increasing, and twice differentiable on $\mathrm{int}(\mathbb{R}_+) = \mathbb{R}_{++}$. The first derivative, ∂f, is strictly positive on $\mathbb{R}_{++}$. The right first derivative at 0, however, does not exist. Indeed, for all $h > 0$, $(f(0+h) - f(0))/h = 1/\sqrt{h}$, and $\lim_{h \downarrow 0} 1/\sqrt{h}$ does not exist.

where $(\alpha_i, \beta, \gamma) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$ is a triple of parameters, the so-called *private component function* $p(\cdot \mid \alpha_i, \beta)\colon \mathcal{Y} \to \mathbb{R}$ is given by

$$p(y_i \mid \alpha_i, \beta) := \alpha_i y_i - \frac{\beta}{2} y_i^2,$$

and the so-called *social component function* $s_i(\cdot \mid \gamma, G)\colon \mathcal{Y}^n \to \mathbb{R}$ is given by

$$s_i(y_1, \dots, y_n \mid \gamma, G) := \begin{cases} 0, & \text{if } \deg_G^+(i) = 0, \\ -\dfrac{\gamma}{2}\left(y_i - \dfrac{\sum_{j \in \mathcal{N}_G^+(i)} y_j}{\deg_G^+(i)}\right)^2 & \text{if } \deg_G^+(i) > 0. \end{cases}$$

Some comments on Assumption U are in order. To this end, suppose Condition G is satisfied. Let $i \in \mathcal{I} \setminus \mathcal{I}_0^+(G)$, and let $(y_1, \dots, y_n) \in \mathcal{Y}^n$ be an action profile.

Player i's utility function is symmetric in his out-neighbors' actions.[13] It exhibits local strategic complements if $\gamma > 0$ and local strategic substitutes if $\gamma < 0$ because

$$\forall j \in \mathcal{I} \quad \frac{\partial^2 u_i(y_1, \dots, y_n)}{\partial y_i \partial y_j} = \begin{cases} \dfrac{\gamma \partial f(y_i) \partial f(y_j)}{\deg_G^+(i)} & \text{if } j \in \mathcal{N}_G^+(i), \\ 0 & \text{if } j \notin \mathcal{N}_G^+(i), \end{cases}$$

where $\partial f(y_i) \partial f(y_j) > 0$ (Assumption F). It does, however, not exhibit positive or negative local externalities.[14]

The utility that player i ascribes to the action profile $(y_1, \dots, y_n)$ consists of two components: the *private component* and the *social component*.

The private component is given by $\alpha_i f(y_i) - (\beta/2) f(y_i)^2$. Under certain conditions, including $\beta > 0$, the point $f^{-1}(\alpha_i/\beta)$ is defined and lies in the interior of $\mathcal{Y}$, from which it follows that the function $y_i \mapsto \alpha_i f(y_i) - (\beta/2) f(y_i)^2$ is strictly increasing on $(-\infty, f^{-1}(\alpha_i/\beta)) \cap \operatorname{int}(\mathcal{Y})$ and strictly decreasing on $(f^{-1}(\alpha_i/\beta), +\infty) \cap \operatorname{int}(\mathcal{Y})$, with a global maximum point at $f^{-1}(\alpha_i/\beta)$.[15] The private component can in turn be decomposed into two parts: the *private benefit* and the *private cost*. The private benefit is defined as the sum of the positive parts of $\alpha_i f(y_i)$ and $-(\beta/2) f(y_i)^2$ and the private cost is defined as the sum of the negative parts of $\alpha_i f(y_i)$ and $-(\beta/2) f(y_i)^2$. The positive part and the negative part of $\alpha_i f(y_i)$ are denoted by $(\alpha_i f(y_i))^+ := \max\{0, \alpha_i f(y_i)\}$ and $(\alpha_i f(y_i))^- := -\min\{0, \alpha_i f(y_i)\}$, respectively. A similar notation applies to the positive part and the negative part of $-(\beta/2) f(y_i)^2$.

13. That is, for all permutations π of $\mathcal{I}$ with fixed points $\mathcal{I} \setminus \mathcal{N}_G^+(i)$, $u_i(y_{\pi(1)}, \dots, y_{\pi(n)}) = u_i(y_1, \dots, y_n)$.

14. In accordance with the terminology introduced by Galeotti et al. (2010, pp. 226–27), player i's utility function is said to exhibit positive (respectively, negative) local externalities if for all $(y_1, \dots, y_n) \in \mathcal{Y}^n$ and for all $(\tilde{y}_1, \dots, \tilde{y}_n) \in \mathcal{Y}^n$ with $\tilde{y}_i = y_i$ and $\{j \in \mathcal{N}_G^+(i) \mid \tilde{y}_j \geq y_j\} = \mathcal{N}_G^+(i)$, $u_i(\tilde{y}_1, \dots, \tilde{y}_n) \geq u_i(y_1, \dots, y_n)$ (respectively, $u_i(\tilde{y}_1, \dots, \tilde{y}_n) \leq u_i(y_1, \dots, y_n)$).

15. If $\beta > 0$ and $\alpha_i/\beta \in f(\mathcal{Y})$, then $f^{-1}(\alpha_i/\beta)$ is defined. If $f^{-1}(\alpha_i/\beta)$ is defined, then

$$\partial\big(y \mapsto \alpha_i f(y) - (\beta/2) f(y)^2\big)(y_i) \begin{cases} < 0 & \text{if } y_i > f^{-1}(\alpha_i/\beta), \\ = 0 & \text{if } y_i = f^{-1}(\alpha_i/\beta), \\ > 0 & \text{if } y_i < f^{-1}(\alpha_i/\beta). \end{cases}$$

It follows from the foregoing that player i's private component can be written as follows:

$$\alpha_i f(y_i) - \frac{\beta}{2} f(y_i)^2 = \big(\alpha_i f(y_i)\big)^+ - \left(\big(\alpha_i f(y_i)\big)^- + \frac{\beta}{2} f(y_i)^2\right). \tag{1.1}$$

The social component is given by

$$-\frac{\gamma}{2}\left(f(y_i) - \frac{\sum_{j \in \mathcal{N}_G^+(i)} f(y_j)}{\deg_G^+(i)}\right)^2.$$

The social component represents player i's social cost (if $\gamma > 0$) or social benefit (if $\gamma < 0$) from deviating from a *social norm* that is defined via his out-neighbors' actions. If f is the identity function on $\mathcal{Y}$, then player i's social norm is given by the arithmetic mean of his out-neighbors' actions (see Example 1.2 for a discussion). If f is the composition of an infinitesimal positive translation with the natural logarithm, which, by abuse of notation, is denoted by $f \approx \log$, then player i's social norm is essentially given by the geometric mean of his out-neighbors' actions (see Example 1.3 for a discussion). The distance between player i's action and his social norm is referred to as the *social distance* between player i and his out-neighbors. It is important to note that player i's social norm is endogenous because it is defined via his out-neighbors' actions. The players' social norms are potentially heterogeneous (in equilibrium) because the players may vary in their out-neighborhoods and not all players may play the same action (in equilibrium).

I discuss the above concepts in more detail for the case $f = \mathrm{id}_{\mathcal{Y}}$ (Example 1.2) and the case $f \approx \log$ (Example 1.3).

Example 1.2 Suppose $\mathcal{Y} = \mathbb{R}_+$, Condition G is satisfied, $\beta > 0$, $\gamma \neq 0$, and $f = \mathrm{id}_{\mathbb{R}_+}$. Let $i \in \mathcal{I} \setminus \mathcal{I}_0^+(G)$, and suppose $\alpha_i > 0$. The private component of player i's utility is given by $\alpha_i y_i - (\beta/2) y_i^2$. The function $y_i \mapsto \alpha_i y_i - (\beta/2) y_i^2$ is strictly increasing on $[0, \alpha_i/\beta)$ and strictly decreasing on $(\alpha_i/\beta, +\infty)$, with a global maximum point at $\alpha_i/\beta > 0$. See Figure 1.1 for an illustration. The sum of the positive parts and the sum of the negative parts of $\alpha_i y_i$ and $-(\beta/2) y_i^2$ are given by $\alpha_i y_i$ and $(\beta/2) y_i^2$, respectively. The social component of player i's utility is given by

$$-\frac{\gamma}{2}\left(y_i - \frac{\sum_{j \in \mathcal{N}_G^+(i)} y_j}{\deg_G^+(i)}\right)^2.$$

Player i's social norm is given by the arithmetic mean of his out-neighbors' actions, and player i's social cost or benefit is zero if and only if his action is equal to this mean. The social cost or benefit is strictly increasing in the distance between player i's action and his social norm. Two actions that are at the same distance from the social norm, one below and one above, cause the same social cost or benefit. In summary, player i's action and social norm are mapped to a social cost or benefit by means of a quadratic and therefore symmetric loss function. ◇

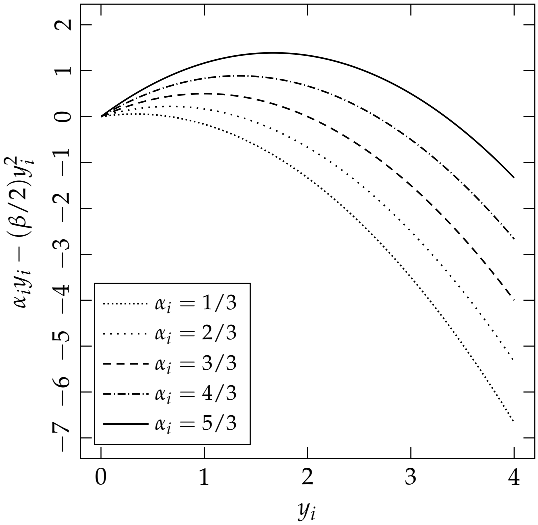


Figure 1.1. Private component function $y_i \mapsto \alpha_i y_i - (\beta/2)y_i^2$ for different positive α_i's and $\beta = 1$ (Example 1.2)

Example 1.3 Suppose $\mathcal{Y} = \mathbb{R}_+$, Condition G is satisfied, $\beta > 0$, $\gamma \neq 0$, and f satisfies $f(y) = \log(\epsilon + y)$, where $\epsilon \in (0,1)$ is arbitrary small. Let $i \in \mathcal{I} \setminus \mathcal{I}_0^+(G)$, and suppose $\alpha_i > \beta \log(\epsilon)$. The private component of player i's utility is given by $\alpha_i \log(\epsilon + y_i) - (\beta/2)(\log(\epsilon + y_i))^2$. The function $y_i \mapsto \alpha_i \log(\epsilon + y_i) - (\beta/2)(\log(\epsilon + y_i))^2$ is strictly increasing on $[0, \exp(\alpha_i/\beta) - \epsilon)$ and strictly decreasing on $(\exp(\alpha_i/\beta) - \epsilon, +\infty)$, with a global maximum point at $\exp(\alpha_i/\beta) - \epsilon > 0$. See Figure 1.2 for an illustration of the case $\alpha_i > 0$ and Figure 1.3 for an illustration of the case $\alpha_i < 0$. The sum of the positive parts and the sum of the negative parts of $\alpha_i \log(\epsilon + y_i)$ and $-(\beta/2)(\log(\epsilon + y_i))^2$ are given by

$$\big(\alpha_i \log(\epsilon + y_i)\big)^+ = \alpha_i \log(\epsilon + y_i)\, \mathbb{1}_{P_\epsilon}\big((\alpha_i, y_i)\big)$$

and

$$\begin{aligned}\big(\alpha_i \log(\epsilon + y_i)\big)^- + \frac{\beta}{2}\big(\log(\epsilon + y_i)\big)^2 = \frac{\beta}{2}&\big(\log(\epsilon + y_i)\big)^2 \\ &- \alpha_i \log(\epsilon + y_i)\big(1 - \mathbb{1}_{P_\epsilon}\big((\alpha_i, y_i)\big)\big),\end{aligned}$$

respectively, where $P_\epsilon := (-\infty, 0) \times (0, 1-\epsilon) \cup [0, +\infty) \times [1-\epsilon, +\infty) \subset \mathbb{R} \times \mathbb{R}_{++}$. The sum of the negative parts depends thus on α_i and β. The social component of player i's utility is given by

$$-\frac{\gamma}{2}\left(\log\left(\frac{\epsilon + y_i}{\left(\prod_{j \in \mathcal{N}_G^+(i)} \epsilon + y_j\right)^{1/\deg_G^+(i)}}\right)\right)^2. \tag{1.2}$$

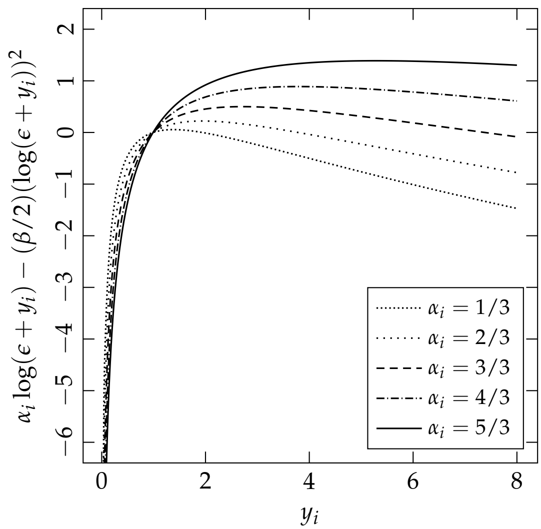


Figure 1.2. Private component function $y_i \mapsto \alpha_i \log(\epsilon + y_i) - (\beta/2)(\log(\epsilon + y_i))^2$ for different positive α_i's, $\beta = 1$, and $\epsilon = 1/100$ (Example 1.3)

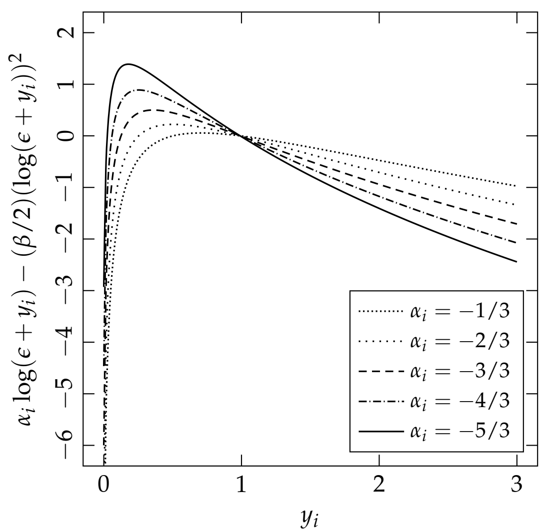


Figure 1.3. Private component function $y_i \mapsto \alpha_i \log(\epsilon + y_i) - (\beta/2)(\log(\epsilon + y_i))^2$ for different negative α_i's, $\beta = 1$, and $\epsilon = 1/100$ (Example 1.3)

For the purpose of discussing the term (1.2), let the function $d\colon \mathbb{R}^2_{++} \to \mathbb{R}$ be defined by $d(x,y) := (\log(x/y))^2$. The function d is a semimetric on $\mathbb{R}_{++}$.[16] In addition, it has the following two properties: (i) for all $y \in \mathbb{R}_{++}$, the function $x \mapsto d(x,y)$ is strictly decreasing on $(0,y)$ and strictly increasing on $(y,+\infty)$; and (ii) for all $(x,\Delta x) \in \mathbb{R}^2_{++}$ with $\Delta x < x$, the function $y \mapsto d(x-\Delta x,y) - d(x+\Delta x,y)$ is negative on $(0, \sqrt{(x-\Delta x)(x+\Delta x)})$ and positive on $(\sqrt{(x-\Delta x)(x+\Delta x)}, +\infty)$.[17]

In the limit $\epsilon \downarrow 0$, provided that $\{y_j \mid j \in \mathcal{N}^+_G(i)\} \subset \mathbb{R}_{++}$, player i's social norm is given by the geometric mean of his out-neighbors' actions, and player i's social cost or benefit is zero if and only if his action is equal to this mean. The social cost or benefit is strictly increasing in the distance between player i's action and his social norm. Two actions that are at the same distance from the social norm, one below and one above, cause in general not the same social cost or benefit. If the social norm exceeds (respectively, falls short of) a certain threshold, an action below the social norm causes a higher (respectively, lower) social cost or benefit than an action above the social norm that is at the same distance from the norm as the action below the norm. In summary, player i's action and social norm are mapped to a social cost or benefit by means of an asymmetric loss function. $\diamond$

I conclude the discussion of Assumption U by extending the existing body of terms surrounding utility functions.

The parameters $\{\alpha_i \mid i \in \mathcal{I}\} \cup \{\beta, \gamma\}$ are referred to as the players' *preference parameters*. For all $i \in \mathcal{I}$, α_i is referred to as player i's *idiosyncrasy*. Note that an idiosyncrasy is the only preference parameter that may vary from one player to another. Note also that a player's idiosyncrasy is equal to the marginal private benefit of his own action if $f = \mathrm{id}_{\mathcal{Y}}$.[18] The parameter β is referred to as the players' common *private cost parameter*. Note that a player's private cost of own action may also depend on his idiosyncrasy.[19] The parameter γ is referred to as the players' common *social cost parameter* (if $\gamma \geq 0$) or common *social benefit parameter* (if $\gamma < 0$). If $\gamma > 0$, it is also referred to as the players' common *preference for conformist behavior*, and if $\gamma < 0$, it is also referred to as the players' common *preference for anti-conformist* behavior. If $\gamma = 0$, the players are said to have a common preference for *nonconformist* behavior.

Based on the preceding considerations, I introduce the notion of a (generic) nonaffine local average game.

Definition G A *nonaffine local average game*, or *NALA game* for short, is a static, noncooperative game of complete information. The set of players, $\mathcal{I}$, is finite with $|\mathcal{I}| = n > 1$. The players are connected to each other by a fixed social network that is represented by a digraph G of order n. The players have a common action space $\mathcal{Y}$ that is equal to $\mathbb{R}$, $\mathbb{R}_+$, or $[0, \bar{v}]$. The players' preferences over $\mathcal{Y}^n$ are represented by a family of utility functions $\{u_i\colon \mathcal{Y}^n \to \mathbb{R}\}_{i \in \mathcal{I}}$ that satisfies Assumption U,

16. The function d is a semimetric on $\mathbb{R}_{++}$ because it satisfies the following three axioms: (A.1) for all $(x,y) \in \mathbb{R}^2_{++}$, $d(x,y) \geq 0$ (nonnegativity); (A.2) for all $(x,y) \in \mathbb{R}^2_{++}$, $d(x,y) = 0$ if and only if $x = y$ (identity of indiscernibles); and (A.3) for all $(x,y) \in \mathbb{R}^2_{++}$, $d(x,y) = d(y,x)$ (symmetry).

17. See Appendix D for a proof of the properties of d.

18. See Example 1.2.

19. See the decomposition of the private component (1.1) and in particular Example 1.3.

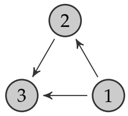


Figure 1.4. A digraph of order 3 (Example 1.4)

which encompasses Assumption F. A NALA game is denoted by the quintuple $(\mathcal{I}, G, \mathcal{Y}, \{(\alpha_i, \beta, \gamma)\}_{i\in\mathcal{I}}, f)$. A NALA game for which $\mathcal{Y}$ is left unspecified is referred to as a *generic NALA game*.[20]

In the remainder of this section, I give a compact representation of the players' utility functions that dispenses with the distinction between empty and nonempty out-neighborhoods. To this end, I state the following definition.

Definition A Let $\mathrm{sl}(G)$ denote the directed pseudograph that is derived from G by adding self-loops to all vertices with zero out-degree.[21] The *endogenous effects matrix* of G is the row-normalized adjacency matrix of $\mathrm{sl}(G)$ with respect to the canonical enumeration of $\mathcal{V}(G)$, $\mathrm{id}_{\mathcal{V}(G)}$, and is denoted by $\bar{\boldsymbol{A}}(G)$. The component in row i and column j of $\bar{\boldsymbol{A}}(G)$ is denoted by $\bar{a}_{i,j}(G)$.

The endogenous effects matrix of G is a square matrix of order n that satisfies

$$\forall(i,j)\in\mathcal{I}^2 \quad \bar{a}_{i,j}(G) = \begin{cases} \delta_{i,j} & \text{if } \deg^+_G(i) = 0, \\ \dfrac{\mathbb{1}_{\mathcal{N}^+_G(i)}(j)}{\deg^+_G(i)} & \text{if } \deg^+_G(i) > 0. \end{cases} \tag{1.3}$$

Some properties of $\bar{\boldsymbol{A}}(G)$ follow directly from its definition: (i) $\bar{\boldsymbol{A}}(G)$ is nonnegative; (ii) $\bar{\boldsymbol{A}}(G)$ is different from $\boldsymbol{O}_n$; (iii) for all $i \in \mathcal{I}$, if $\deg^+_G(i) = 0$, then $\bar{a}_{i,i}(G) = 1$, and if $\deg^+_G(i) > 0$, then $\bar{a}_{i,i}(G) = 0$; (iv) $\bar{\boldsymbol{A}}(G)\mathbf{1}_n = \mathbf{1}_n$; (v) if G is empty, that is, $\mathcal{A}(G) = \emptyset$, then $\bar{\boldsymbol{A}}(G) = \boldsymbol{I}_n$; (vi) $\rho(\bar{\boldsymbol{A}}(G)) = 1$, that is, the spectral radius of $\bar{\boldsymbol{A}}(G)$ is one, because $\bar{\boldsymbol{A}}(G)$ is nonnegative and row-normalized (Lemma B.8).

I illustrate Definition A with three examples.

Example 1.4 Suppose $\mathcal{I} = \{1,2,3\}$ and $\mathcal{A}(G) = \{(1,2),(1,3),(2,3)\}$. See Figure 1.4 for an illustration of G. Evidently, $\mathrm{sl}(G) = (\mathcal{I}, \mathcal{A}(G) \cup \{(3,3)\})$ and

$$\bar{\boldsymbol{A}}(G) = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix}. \qquad \diamond$$

Example 1.5 If $\mathcal{A}(G) = \bigcup_{i\in\mathcal{I}\setminus\{1\}}\{(i,1)\}$, that is, G is star-shaped with arcs from every peripheral player $i \in \mathcal{I}\setminus\{1\}$ to the central player 1, then $\bar{\boldsymbol{A}}(G) = \mathbf{1}_n \boldsymbol{e}_1^\intercal$. $\diamond$

20. The expression "$\mathcal{Y}$ is left unspecified" does not implicate that $\mathcal{Y}$ can be any subset of $\mathbb{R}$; it is to be understood as all that is known about $\mathcal{Y}$ is that it is equal to $\mathbb{R}$, $\mathbb{R}_+$, or $[0, \bar{v}]$.

21. A directed pseudograph is a graph for which multiple arcs or self-loops are admissible (see, for example, Bang-Jensen and Gutin 2009, Section 1.2).

Example 1.6 If G is complete, then $\bar{\boldsymbol{A}}(G) = (1/(n-1))(\mathbf{1}_n\mathbf{1}_n^\top - \boldsymbol{I}_n)$. ◇

Using Definition A, for all $i \in \mathcal{I}$, player i's utility can be written as

$$u_i(y_1, \dots, y_n) = \alpha_i f(y_i) - \frac{\beta}{2} f(y_i)^2 - \frac{\gamma}{2}\left(f(y_i) - \sum_{j \in \mathcal{I}} \bar{a}_{i,j}(G) f(y_j)\right)^2. \tag{1.4}$$

1.3.2 Typology of player heterogeneity

In this section, I discuss different types or, more precisely, degrees of player heterogeneity. To this end, let $\Gamma := (\mathcal{I}, G, \mathcal{Y}, \{(\alpha_i, \beta, \gamma)\}_{i \in \mathcal{I}}, f)$ be a generic NALA game. Let $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_n)$ denote the profile, that is, the (column) vector, of the players' idiosyncrasies.

The players of Γ can be ex ante heterogeneous in two respects: first, with respect to their preferences, that is, their idiosyncracies, and second, with respect to their out-neighborhoods, which define their positions in G, for example, in terms of in-degrees and out-degrees. The following terminology, however, is predicated on the players' idiosyncracies only.

Definition H The players of Γ are called *weakly ex ante homogeneous* if $\boldsymbol{\alpha} = \bar{\boldsymbol{A}}(G)\boldsymbol{\alpha}$, that is, for all $i \in \mathcal{I} \setminus \mathcal{I}_0^+(G)$, $\alpha_i = (1/\deg_G^+(i)) \sum_{j \in \mathcal{N}_G^+(i)} \alpha_j$.[22] The players of Γ are called *strongly ex ante homogeneous* if there exists an $\bar{\alpha} \in \mathbb{R}$ such that $\boldsymbol{\alpha} = \bar{\alpha}\mathbf{1}_n$, that is, for all $i \in \mathcal{I}$, $\alpha_i = \bar{\alpha}$. The players of Γ are called *ex ante heterogeneous* if they are not strongly ex ante homogeneous.

The notion of strongly ex ante homogeneous players is stronger than its weaker form, that is, if the players of Γ are strongly ex ante homogeneous, then they are weakly ex ante homogeneous. The converse is in general not true. An exception is the following result.[23]

Corollary 1.7 *Suppose G is complete. The players of Γ are strongly ex ante homogeneous if and only if they are weakly ex ante homogeneous.*

Corollary 1.7 suggests that the connectedness of G or, more generally speaking, the structure of G affects the extent to which ex ante heterogeneous players can be weakly ex ante homogeneous. Indeed, the players of Γ cannot be weakly but not strongly ex ante homogeneous if G is strongly connected (see Corollary 1.9), which is, for example, the case when G is complete. The following discussion relates the structure of G to the players' (potential) homogeneity, thereby giving substance to the notion of a digraph's structure.

The players of Γ are weakly ex ante homogeneous if and only if $\boldsymbol{\alpha}$ is a fixed point of the linear mapping with domain and codomain $\mathbb{R}^n$ that is represented by the matrix $\bar{\boldsymbol{A}}(G)$ with respect to the standard basis for $\mathbb{R}^n$. The set of fixed points

22. For all $i \in \mathcal{I}_0^+(G)$, the ith equation of the system $\boldsymbol{\alpha} = \bar{\boldsymbol{A}}(G)\boldsymbol{\alpha}$ is the identity $\alpha_i = \alpha_i$.
23. The result is stated as a corollary because if follows from a more fundamental result, which is stated in Proposition 1.8 as Result 1.8.2.

of $\bar{\boldsymbol{A}}(G)$ in $\mathbb{R}^n$ is equal to $\text{Eig}(1, \bar{\boldsymbol{A}}(G))$, the eigenspace of $\bar{\boldsymbol{A}}(G)$ associated with the eigenvalue 1. The eigenspace $\text{Eig}(1, \bar{\boldsymbol{A}}(G))$ is a linear subspace of $\mathbb{R}^n$; it contains $\mathbf{1}_n$ because $\bar{\boldsymbol{A}}(G)$ is row-normalized, from which it follows that its dimension is at least 1: $\text{g.m.}(1, \bar{\boldsymbol{A}}(G)) := \dim_{\mathbb{R}}(\text{Eig}(1, \bar{\boldsymbol{A}}(G))) \geq 1$.[24] Besides this lower bound, $\text{g.m.}(1, \bar{\boldsymbol{A}}(G))$ depends on the digraph G only through its structure, that is, its isomorphism class: if H is a digraph of order n that is isomorphic to G, then $\text{g.m.}(1, \bar{\boldsymbol{A}}(H)) = \text{g.m.}(1, \bar{\boldsymbol{A}}(G))$. The structure of G imposes restrictions on the (potential) homogeneity of ex ante heterogeneous players; specifically, ex ante heterogeneous players cannot be weakly ex ante homogeneous if $\text{g.m.}(1, \bar{\boldsymbol{A}}(G)) = 1$ or, equivalently, $\text{Eig}(1, \bar{\boldsymbol{A}}(G)) = \text{span}\{\mathbf{1}_n\}$. The contrapositive of this statement and two other properties of $\text{Eig}(1, \bar{\boldsymbol{A}}(G))$ are summarized in the following result.

Proposition 1.8 (*1.8.1*) *If the players of Γ are weakly but not strongly ex ante homogeneous, then* $\text{span}\{\mathbf{1}_n\} \subsetneq \text{span}\{\mathbf{1}_n, \boldsymbol{\alpha}\} \subset \text{Eig}(1, \bar{\boldsymbol{A}}(G))$, *that is,*

$$\text{g.m.}\big(1, \bar{\boldsymbol{A}}(G)\big) > 1. \tag{1.5}$$

(*1.8.2*) *If G is strongly connected, then* $\text{Eig}(1, \bar{\boldsymbol{A}}(G)) = \text{span}\{\mathbf{1}_n\}$ *or, equivalently,* $\text{g.m.}(1, \bar{\boldsymbol{A}}(G)) = 1$.

(*1.8.3*) *If player $x \in \mathcal{I}$ is isolated in G, then* $\text{span}\{\mathbf{1}_n, \boldsymbol{e}_x\} \subset \text{Eig}(1, \bar{\boldsymbol{A}}(G))$, *that is,* $\text{g.m.}(1, \bar{\boldsymbol{A}}(G)) > 1$.

The following result follows directly from Result 1.8.2.

Corollary 1.9 *Suppose G is strongly connected. The players of Γ are strongly ex ante homogeneous if and only if they are weakly ex ante homogeneous.*

In order to discuss the role of isolated players for the existence of weakly but not strongly ex ante homogeneous players, consider the following scenario: There exist a player $x \in \mathcal{I}$ and an $\bar{\alpha} \in \mathbb{R}$ such that $\alpha_x \neq \bar{\alpha}$ and for all $i \in \mathcal{I} \setminus \{x\}$, $\alpha_i = \bar{\alpha}$. Let us consider two cases. First, suppose no player of Γ is isolated in G. It follows that the players of Γ are not weakly ex ante homogeneous. Second, suppose player x is isolated in G. It follows that the players of Γ are weakly but not strongly ex ante homogeneous, and all but one of the players of Γ are strongly ex ante homogeneous. The second case demonstrates that isolated players admit of weakly ex ante homogeneous players that are not strongly ex ante homogeneous. As regards inequality (1.5), it may be desirable to take account of this property of isolated players. Specifically, the inequality

$$\text{g.m.}\big(1, \bar{\boldsymbol{A}}(G)\big) - |\mathcal{I}_0(G)| > 1 \tag{1.6}$$

may convey information about the existence of weakly but not strongly ex ante homogeneous players beyond that contained in (1.5).

24. The dimension of the eigenspace $\text{Eig}(1, \bar{\boldsymbol{A}}(G))$ is referred to as the geometric multiplicity of the eigenvalue 1 of $\bar{\boldsymbol{A}}(G)$ and is denoted by $\text{g.m.}(1, \bar{\boldsymbol{A}}(G))$.

The preceding discussion shows that G or, more generally, any digraph of order n admits of players that are weakly but not strongly ex ante homogeneous if at least one of the players is isolated. It is not clear from the outset if this is also the case in the absence of isolated players. It is therefore of high interest to know if there exist digraphs or, more precisely, isomorphism classes of digraphs that satisfy not only inequality (1.5)—and therefore admit of players that are weakly but not strongly ex ante homogeneous—but also inequality (1.6)—and therefore admit of nonisolated players that are weakly but not strongly ex ante homogeneous. The following overview is confined to digraphs of orders at most 4 for reasons of computational complexity; for example, there are 1,048,576 (respectively, 1,073,741,824) digraphs of order 5 (respectively, 6), which are partitioned into 9,608 (respectively, 1,540,944) isomorphism classes (see, for example, Harary 1969, Table A2).[25] For the sake of completeness, the overview includes digraphs of order 1, which is in conflict with the assumption that $n > 1$, and empty digraphs. There is only one digraph of order 1, namely, the empty digraph $(\{1\}, \emptyset)$, which cannot satisfy (1.5) and (1.6). Among the 4 digraphs of order 2, which are partitioned into 3 isomorphism classes, only the empty digraph satisfies (1.5) and (1.6). There are 64 digraphs of order 3. The digraph isomorphism partitions this set into 16 isomorphism classes. Broadly speaking, there are 16 digraphs of order 3 that are not isomorphic. The representatives of 4 of these 16 classes (13 digraphs altogether, which corresponds to a share of approximately 20 per cent of 64) satisfy (1.5). The representatives of one class (3 digraphs altogether, which corresponds to a share of approximately 5 per cent of 64) satisfy (1.6). Details on a representative for each class are given in Table C.1 in Appendix C. There are 4,096 digraphs of order 4, which are partitioned into 218 isomorphism classes. The representatives of 33 classes (486 digraphs altogether, which corresponds to a share of approximately 12 per cent of 4,096) satisfy (1.5). The representatives of 18 classes (266 digraphs altogether, which corresponds to a share of approximately 6 per cent of 4096) satisfy (1.6). See again Table C.1 for more details.

The present discussion of weakly ex ante homogeneous players is rounded off by an example.

Example 1.10 Suppose $\mathcal{I} = \{1, 2, \ldots, 6\}$ and $\mathcal{A}(G) = \{(1,2), (2,1), (3,1), (3,2), (3,4), (4,3), (4,5), (4,6), (5,6), (6,5)\}$. See Figure 1.5 for an illustration of G. Elementary calculations yield $\mathrm{Eig}(1, \bar{\boldsymbol{A}}(G)) = \mathrm{span}\{4\boldsymbol{e}_1 + 4\boldsymbol{e}_2 + 3\boldsymbol{e}_3 + \boldsymbol{e}_4, \mathbf{1}_6\}$ with, for example,

$$\frac{4}{10}\boldsymbol{e}_1 + \frac{4}{10}\boldsymbol{e}_2 + \frac{3}{10}\boldsymbol{e}_3 + \frac{1}{10}\boldsymbol{e}_4 + \frac{15}{10}\mathbf{1}_6 = \frac{1}{10}\begin{pmatrix} 19 \\ 19 \\ 18 \\ 16 \\ 15 \\ 15 \end{pmatrix} \in \mathrm{Eig}\big(1, \bar{\boldsymbol{A}}(G)\big).$$

25. In general, there are $4^{\binom{n}{2}}$ different digraphs of order $n \in \mathbb{Z}_{++}$.

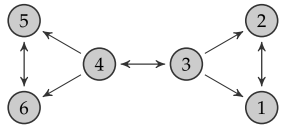


Figure 1.5. A digraph of order 6 (Example 1.10)

Figure 1.6. A linear digraph of order 5 (Example 1.11)

It follows that $\text{g.m.}(1, \bar{A}(G)) = 2$, so that the players of Γ can be weakly but not strongly ex ante homogeneous (Proposition 1.8). ◇

1.3.3 Existence and uniqueness of interior Nash equilibria

This section is concerned with the questions of existence and uniqueness of NEs of NALA games. The focus is on NEs in pure strategies that lie in the interior of the n-ary Cartesian power of the players' common action space.

Let $\Gamma := (\mathcal{I}, G, \mathcal{Y}, \{(\alpha_i, \beta, \gamma)\}_{i \in \mathcal{I}}, f)$ be a generic NALA game, let $\alpha_{\min} := \min\{\alpha_i \mid i \in \mathcal{I}\}$ and $\alpha_{\max} := \max\{\alpha_i \mid i \in \mathcal{I}\}$, and let $\boldsymbol{\alpha}$ be defined as in Section 1.3.2.

For a particular action space $\mathcal{Y}$, the NALA game Γ may have multiple NEs or none at all. This is illustrated in Examples 1.11 and 1.12.

Example 1.11 Suppose $\mathcal{Y} = [0, \bar{v}]$ and $\mathcal{A}(G) = \bigcup_{i=1}^{n-1} \{(i, i+1)\}$, that is, G is a linear digraph. See Figure 1.6 for an illustration of G for the case $n = 5$. In addition, suppose $\alpha_1 = \beta\bar{v}$, $\{\alpha_i \mid i \in \mathcal{I} \setminus \{1\}\} \subset (\beta\bar{v}, +\infty)$, $\beta > 0$, $\gamma = -\beta$, and $f = \mathrm{id}_{[0,\bar{v}]}$. The players' utility functions satisfy

$$\forall i \in \mathcal{I} \quad u_i(y_1, \dots, y_n) = \begin{cases} (\alpha_i + \gamma y_{i+1}) y_i - \dfrac{\gamma}{2} y_{i+1}^2 & \text{if } i < n, \\ \alpha_i y_i + \dfrac{\gamma}{2} y_i^2 & \text{if } i = n. \end{cases}$$

It is straightforward to show that Γ has uncountably many NEs, which are given by $\{(y_1^\star, \bar{v}, \dots, \bar{v}) \mid y_1^\star \in [0, \bar{v}]\}$. ◇

A NE of Γ that involves at least an action at the boundary of the action space, $\partial\mathcal{Y}$, is called a *boundary NE*. For example, $(0, \bar{v}, \dots, \bar{v})$ and $(\bar{v}/2, \bar{v}, \dots, \bar{v})$ are boundary NEs of the NALA game Γ of Example 1.11.

Example 1.12 Suppose $\mathcal{Y} = \mathbb{R}_+$ and $\mathcal{A}(G) = \bigcup_{i=1}^{n-1} \{(i, i+1)\} \cup \{(n, 1)\}$, that is, G is a cycle digraph. See Figure 1.7 for an illustration of G for the case $n = 4$. In addition, suppose $\{\alpha_i \mid i \in \mathcal{I}\} \subset \mathbb{R}_{++}$, $\beta = 0$, $\gamma > 0$, and $f = \mathrm{id}_{\mathbb{R}_+}$. Let π be the permutation of $\mathcal{I}$ defined by

$$\pi(i) := \begin{cases} i + 1 & \text{if } i < n, \\ 1 & \text{if } i = n. \end{cases}$$

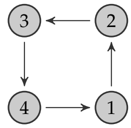


Figure 1.7. A cycle digraph of order 4 (Example 1.12)

The players' utility functions satisfy

$$\forall i \in \mathcal{I} \quad u_i(y_1, \ldots, y_n) = \alpha_i y_i - \frac{\gamma}{2}\left(y_i - y_{\pi(i)}\right)^2.$$

It is straightforward to show that Γ has no NE.[26] ◇

The remainder of this section is concerned with sufficient conditions for the existence of a unique and interior NE of Γ. The conditions depend critically on the topological properties of $\mathcal{Y}$, which is why they are stated separately for each type of action space. Proposition 1.13 covers the case $\mathcal{Y} = \mathbb{R}$ and admits of both a negative and a nonnegative γ. In the interests of clarity and in order to facilitate comparison, the case $\mathcal{Y} = [0, \bar{v}]$ is covered by two propositions: Proposition 1.14 deals with the case of a nonnegative γ and Proposition 1.15 with the case of a negative γ. The same applies to the case $\mathcal{Y} = \mathbb{R}_+$. A discussion of Propositions 1.13, 1.14, 1.15, and 1.17 follows in Sections 1.3.3.1 to 1.3.3.4.

The following notation is used to represent the NE of Γ: To any real-valued function g with domain $\mathcal{D} \subset \mathbb{R}$ and any integer $N > 1$, there is associated a vector field $\boldsymbol{g}_N \colon \mathcal{D}^N \to g(\mathcal{D})^N$ whose kth component is $g \circ \mathrm{proj}_k^N$, where $\mathrm{proj}_k^N \colon \mathcal{D}^N \to \mathcal{D}$ denotes the kth projection function that maps a point in $\mathcal{D}^N$ to its kth component, in particular, for all $\boldsymbol{x} := (x_1, \ldots, x_N) \in \mathcal{D}^N$, $\boldsymbol{g}_N(\boldsymbol{x}) = (g(x_1), \ldots, g(x_N))$. Often N is omitted from the notation $\boldsymbol{g}_N$, and it is simply written as $\boldsymbol{g}$.

Proposition 1.13 *Suppose $\mathcal{Y} = \mathbb{R}$. The NALA game Γ has a unique NE $\boldsymbol{y}^\star \in \mathbb{R}^n$, which is given by*

$$\boldsymbol{f}(\boldsymbol{y}^\star) = \frac{1}{\beta + \gamma}\left(\boldsymbol{I}_n - \frac{\gamma}{\beta + \gamma}\bar{\boldsymbol{A}}(G)\right)^{-1} \boldsymbol{\alpha}, \tag{1.7}$$

if four conditions are satisfied: (1.13.1) $\beta > 0$, (1.13.2) $\beta + \gamma > 0$, (1.13.3) $(\beta + \gamma) \notin \sigma(\gamma \bar{\boldsymbol{A}}(G))$, and (1.13.4) f is not bounded below and above.

Proposition 1.14 *Suppose $\mathcal{Y} = [0, \bar{v}]$. The NALA game Γ has a unique and interior NE $\boldsymbol{y}^\star \in (0, \bar{v})^n$, which is given by (1.7), if four conditions are satisfied: (1.14.1) $\beta > 0$, (1.14.2) $\gamma \geq 0$, (1.14.3) $\beta f(0) < \alpha_{\min}$, and (1.14.4) $\alpha_{\max} < \beta f(\bar{v})$.*

Proposition 1.15 *Suppose $\mathcal{Y} = [0, \bar{v}]$. The NALA game Γ has a unique and interior NE $\boldsymbol{y}^\star \in (0, \bar{v})^n$, which is given by (1.7), if four conditions are satisfied: (1.15.1) $\beta > 0$, (1.15.2) $-\beta/2 < \gamma < 0$, (1.15.3) $\beta f(0) + \gamma(f(0) - f(\bar{v})) < \alpha_{\min}$, and (1.15.4) $\alpha_{\max} < \beta f(\bar{v}) + \gamma(f(\bar{v}) - f(0))$.*

26. See Appendix D for details.

Proposition 1.16 *Suppose $\mathcal{Y} = \mathbb{R}_+$. The NALA game Γ has a unique and interior NE $\boldsymbol{y}^\star \in \mathbb{R}^n_{++}$, which is given by (1.7), if four conditions are satisfied: (1.16.1) $\beta > 0$, (1.16.2) $\gamma \geq 0$, (1.16.3) $\beta f(0) < \alpha_{\min}$, and (1.16.4) f is not bounded above.*[27]

Proposition 1.17 *Suppose $\mathcal{Y} = \mathbb{R}_+$. The NALA game Γ has a unique and interior NE $\boldsymbol{y}^\star \in \mathbb{R}^n_{++}$, which is given by (1.7), if six conditions are satisfied: (1.17.1) $\beta > 0$, (1.17.2) $-\beta/2 < \gamma < 0$, (1.17.3) $\beta f(0) < \alpha_{\min}$, (1.17.4) $\beta(\beta + 2\gamma) f(0) < (\beta+\gamma)\alpha_{\min} + \gamma\alpha_{\max}$, (1.17.5) $|\gamma|(\beta+\gamma)(\alpha_{\max} - \beta f(0)) < \beta(\beta+2\gamma)(\alpha_{\min} - \beta f(0))$, and (1.17.6) f is not bounded above.*[28]

The systems of equations that govern equilibrium actions, that is, the systems of best reply functions, are the same for all three types of action spaces for the focus is on an interior NE. An equivalent but more compact representation of the common system of equations (1.7) is given by

$$\boldsymbol{f}(\boldsymbol{y}^\star) = \Big(\beta \boldsymbol{I}_n - \gamma\big(\bar{\boldsymbol{A}}(G) - \boldsymbol{I}_n\big)\Big)^{-1} \boldsymbol{\alpha}. \tag{1.8}$$

In order to discuss equilibrium actions, suppose $\beta > 0$ and $\gamma > -\beta/2$, so that $\beta + \gamma > 0$ and $|\gamma/(\beta+\gamma)| < 1$. In addition, suppose Γ has a unique and interior NE $\boldsymbol{y}^\star := (y_1^\star, \ldots, y_n^\star)$, which is given by the system of equations (1.7). It follows that

$$\boldsymbol{f}(\boldsymbol{y}^\star) = \frac{1}{\beta+\gamma}\boldsymbol{\alpha} + \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \tag{1.9}$$

that is,

$$\forall i \in \mathcal{I} \quad f(y_i^\star) = \begin{cases} \dfrac{\alpha_i}{\beta} & \text{if } \deg_G^+(i) = 0, \\[2ex] \dfrac{\alpha_i}{\beta+\gamma} + \dfrac{\gamma}{\beta+\gamma} \dfrac{\sum_{j \in \mathcal{N}_G^+(i)} f(y_j^\star)}{\deg_G^+(i)} & \text{if } \deg_G^+(i) > 0. \end{cases} \tag{1.10}$$

Apart from a missing error term, the system of equations (1.9) has the same form as a stationary spatial autoregressive process of order one with a row-normalized spatial weights matrix. An explicit expression for $\boldsymbol{y}^\star$ is given by[29]

$$\forall i \in \mathcal{I} \quad y_i^\star = \begin{cases} f^{-1}\left(\dfrac{\alpha_i}{\beta}\right) & \text{if } \deg_G^+(i) = 0, \\[2ex] f^{-1}\left(\dfrac{\alpha_i}{\beta+\gamma} + \dfrac{\gamma}{\beta+\gamma} \dfrac{\sum_{j \in \mathcal{N}_G^+(i)} f(y_j^\star)}{\deg_G^+(i)}\right) & \text{if } \deg_G^+(i) > 0. \end{cases}$$

27. Sommer and Sulger (2012) prove the same result for the case where all players of Γ have at least one out-neighbor in G (see Proposition 2.1).

28. For the case where all players of Γ have at least one out-neighbor in G, Sommer and Sulger (2012) show that Γ has a *unique interior* NE if Conditions 1.17.1 and 1.17.2 and the inequality

$$\frac{\beta(\beta+2\gamma)}{\beta+\gamma} f(0)\boldsymbol{1}_n <_c \left(\boldsymbol{I}_n - \frac{|\gamma|}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)\boldsymbol{\alpha}$$

are satisfied (see Appendix B.2). Sommer and Sulger (2012) do, however, not provide sufficient conditions for the existence of a *unique and interior* NE of Γ.

29. The function f is bijective (Lemma 1.1).

In equilibrium, a player's action depends on the mean of his out-neighbors' (transformed) actions, provided that the player's out-neighborhood is not empty. In general, this average is a *local* average because it involves only the actions of a player's out-neighbors. It is for this reason that the game is called a generic nonaffine *local average* game. The game is called a generic *nonaffine* local average game because it admits of best reply functions that are not multi-affine, as opposed to the so-called local average game of Patacchini and Zenou (2012), which is characterized by multi-affine best reply functions.[30] Patacchini and Zenou's (2012) local average game is a special case of a NALA game with $\mathcal{Y} = \mathbb{R}_+$, where G is symmetric, $\mathcal{I}_0^+(G) = \varnothing$ (that is, all players have at least one out-neighbor), $\gamma > 0$, and $f = \mathrm{id}_{\mathbb{R}_+}$.[31]

I illustrate the preceding results with an example.

Example 1.18 Suppose $\mathcal{Y} = [0, \bar{v}]$, G is complete, $\beta > 0$, $\{\alpha_i \mid i \in \mathcal{I}\} \subset (0, \beta\bar{v})$, $\gamma > 0$, and $f = \mathrm{id}_{[0,\bar{v}]}$. The NALA game Γ has a unique and interior NE $\boldsymbol{y}^\star := (y_1^\star, \dots, y_n^\star)$ (Proposition 1.14), which is given by[32]

$$\boldsymbol{y}^\star = \frac{n}{(n-1)\beta + n\gamma}\left(\frac{\gamma}{\beta}\frac{\langle \mathbf{1}_n, \boldsymbol{\alpha}\rangle}{n}\mathbf{1}_n + \left(1 - \frac{1}{n}\right)\boldsymbol{\alpha}\right), \tag{1.11}$$

where $(1/n)\langle \mathbf{1}_n, \boldsymbol{\alpha}\rangle$ is the arithmetic mean of the players' idiosyncrasies. It follows that

$$\max\{y_i^\star \mid i \in \mathcal{I}\} - \min\{y_i^\star \mid i \in \mathcal{I}\} = \frac{1}{\beta + \dfrac{n}{n-1}\gamma}(\alpha_{\max} - \alpha_{\min}).$$

Thus, the range of the players' actions decreases with γ, that is, the stronger the players' common preference for conformist behavior or, in other words, the higher the cost to deviate from the social norm, the more the players conform to each other. The players' actions are identical if and only if the players of Γ are strongly ex ante homogeneous or, equivalently (Corollary 1.7), weakly ex ante homogeneous. $\diamond$

1.3.3.1 Discussion of Proposition 1.13

Suppose $\mathcal{Y} = \mathbb{R}$ and Conditions 1.13.1 to 1.13.4 are satisfied. Assumption F and Condition 1.13.4 imply that $f(\mathbb{R}) = \mathbb{R}$. Note that $\partial f > 0$ (Assumption F) and f is bijective with strictly increasing f^{-1} (Lemma 1.1).

In order to discuss Conditions 1.13.1 and 1.13.2, let $k \in \mathcal{I}$. I consider two cases. First, suppose player k has no out-neighbors. It follows that his utility from playing action $y_k \in \mathbb{R}$ equals $\alpha_k f(y_k) - (\beta/2) f(y_k)^2$ and his marginal utility at y_k equals $(\alpha_k - \beta f(y_k))\partial f(y_k)$. In light of the above, Condition 1.13.1 implies that the function $y_k \mapsto \alpha_k f(y_k) - (\beta/2) f(y_k)^2$ is strictly increasing on $(-\infty, f^{-1}(\alpha_k/\beta))$ and strictly decreasing on $(f^{-1}(\alpha_k/\beta), +\infty)$, where $f^{-1}(\alpha_k/\beta)$ is defined because $\beta > 0$ and

30. Let $i \in \mathcal{I}$. Player i's best reply function $b_i \colon \mathcal{Y}^{n-1} \to \mathcal{Y}$ is called *multi-affine* if for all $j \in \mathcal{I} \setminus \{i\}$ and for all $\{\tilde{y}_k \mid k \in \mathcal{I} \setminus \{i, j\}\} \subset \mathcal{Y}$, $y_j \mapsto b_i(\tilde{y}_1, \dots, \tilde{y}_{j-1}, y_j, \tilde{y}_{j+1}, \dots, \tilde{y}_n)$ is an affine function.

31. The players of Patacchini and Zenou's (2012) local average game are connected by an undirected graph, which corresponds to the case of a symmetric digraph.

32. See Appendix D for details.

$f(\mathbb{R}) = \mathbb{R}$. As for the hypothetical case $\beta = 0$, the function $y_k \mapsto \alpha_k f(y_k)$ is strictly decreasing on $\mathbb{R}$ if $\alpha_k < 0$, constant on $\mathbb{R}$ if $\alpha_k = 0$, and strictly increasing on $\mathbb{R}$ if $\alpha_k > 0$. As for the hypothetical case $\beta < 0$, the function $y_k \mapsto \alpha_k f(y_k) - (\beta/2) f(y_k)^2$ is strictly decreasing on $(-\infty, f^{-1}(\alpha_k/\beta))$ and strictly increasing on $(f^{-1}(\alpha_k/\beta), +\infty)$. Hence, Condition 1.13.1 rules out the possibility that player k has a preference for decreasing or increasing his action without limits or that he is indifferent about which action to play. For the special case $f = \mathrm{id}_{\mathbb{R}}$, Condition 1.13.1 implies that the function $y_k \mapsto \alpha_k f(y_k) - (\beta/2) f(y_k)^2 = \alpha_k y_k - (\beta/2) y_k^2$ is strictly concave. Second, suppose player k's out-neighborhood is not empty and the actions played by his out-neighbors satisfy $\{y_j \mid j \in \mathcal{N}_G^+(k)\} \subset \mathbb{R}$.[33] It follows that his utility from playing action $y_k \in \mathbb{R}$ equals

$$\alpha_k f(y_k) - \frac{\beta}{2} f(y_k)^2 - \frac{\gamma}{2}\left(f(y_k) - \frac{\sum_{j \in \mathcal{N}_G^+(k)} f(y_j)}{\deg_G^+(k)}\right)^2 \tag{1.12}$$

and his marginal utility at y_k equals

$$\left(\alpha_k - (\beta + \gamma) f(y_k) + \gamma \frac{\sum_{j \in \mathcal{N}_G^+(k)} f(y_j)}{\deg_G^+(k)}\right) \partial f(y_k).$$

In light of the above, Condition 1.13.2 implies that the function $y_k \mapsto$ (1.12) is strictly increasing on $(-\infty, c_k)$ and strictly decreasing on $(c_k, +\infty)$, where

$$c_k := f^{-1}\left(\frac{\alpha_k}{\beta + \gamma} + \frac{\gamma}{\beta + \gamma} \frac{\sum_{j \in \mathcal{N}_G^+(k)} f(y_j)}{\deg_G^+(k)}\right) \in \mathbb{R}$$

is defined because $\beta + \gamma > 0$ and $f(\mathbb{R}) = \mathbb{R}$. Analogous to the first case, Condition 1.13.2 rules out the possibility that player k has a preference for decreasing or increasing his action without limits or that he is indifferent about which action to play.

Given the preceding considerations, Conditions 1.13.1, 1.13.2, and 1.13.4 imply that, for all $i \in \mathcal{I}$, player i's best reply function $b_i \colon \mathbb{R}^{n-1} \to \mathbb{R}$ is given by

$$\begin{aligned} & b_i(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n) \\ &= \begin{cases} f^{-1}\left(\dfrac{\alpha_i}{\beta}\right) & \text{if } \deg_G^+(i) = 0, \\ f^{-1}\left(\dfrac{\alpha_i}{\beta + \gamma} + \dfrac{\gamma}{\beta + \gamma} \dfrac{\sum_{j \in \mathcal{N}_G^+(i)} f(y_j)}{\deg_G^+(i)}\right) & \text{if } \deg_G^+(i) > 0. \end{cases} \end{aligned}$$

It is important to note that the best reply functions are in general not multi-affine if Condition G is satisfied and $f \neq \mathrm{id}_{\mathbb{R}}$. A NE $\boldsymbol{y}^\star := (y_1^\star, \ldots, y_n^\star) \in \mathbb{R}^n$ of Γ satisfies

$$\forall i \in \mathcal{I} \quad y_i^\star = b_i(y_1^\star, \ldots, y_{i-1}^\star, y_{i+1}^\star, \ldots, y_n^\star). \tag{1.13}$$

33. The condition $\{y_j \mid j \in \mathcal{N}_G^+(k)\} \subset \mathbb{R}$ rules out the possibility that one of player k's out-neighbors has a preference for decreasing of increasing his action without limits.

For all $i \in \mathcal{I}$, let the function $h_i\colon \mathbb{R}^n \to \mathbb{R}$ be defined by

$$h_i(y_1, \ldots, y_n) := f^{-1}\left(\frac{\alpha_i}{\beta+\gamma} + \frac{\gamma}{\beta+\gamma}\sum_{j\in\mathcal{I}} \bar{a}_{i,j}(G) f(y_j)\right)$$
$$= \begin{cases} f^{-1}\left(\frac{\alpha_i}{\beta+\gamma} + \frac{\gamma}{\beta+\gamma} f(y_i)\right) & \text{if } \deg_G^+(i) = 0, \\ f^{-1}\left(\frac{\alpha_i}{\beta+\gamma} + \frac{\gamma}{\beta+\gamma}\frac{\sum_{j\in\mathcal{N}_G^+(i)} f(y_j)}{\deg_G^+(i)}\right) & \text{if } \deg_G^+(i) > 0. \end{cases}$$

Note that (1.13) is equivalent to, for all $i \in \mathcal{I}$, $y_i^\star = h_i(y_1^\star, \ldots, y_n^\star)$, so that $\boldsymbol{y}^\star$ is a fixed point of the mapping $\boldsymbol{h} := (h_1, \ldots, h_n)\colon \mathbb{R}^n \to \mathbb{R}^n$, that is, $\boldsymbol{y}^\star = \boldsymbol{h}(\boldsymbol{y}^\star)$. Also note that $\boldsymbol{y}^\star = \boldsymbol{h}(\boldsymbol{y}^\star)$ is equivalent to

$$\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)\boldsymbol{f}(\boldsymbol{y}^\star) = \frac{1}{\beta+\gamma}\boldsymbol{\alpha}. \tag{1.14}$$

Condition 1.13.3 implies that the system of linear equations

$$\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)\boldsymbol{z}^\star = \frac{1}{\beta+\gamma}\boldsymbol{\alpha} \tag{1.15}$$

has a unique solution $\boldsymbol{z}^\star := (z_1, \ldots, z_n) \in \mathbb{R}^n$ (Lemma B.3). It is clear from a comparison of (1.14) and (1.15) that a NE of Γ is a solution to the system of linear (if f is linear) or nonlinear (if f is nonlinear) equations $\boldsymbol{z}^\star = \boldsymbol{f}(\boldsymbol{y}^\star)$. This system has a unique solution $\boldsymbol{y}^\star \in \mathbb{R}^n$ because f is bijective (Lemma 1.1) and $f(\mathbb{R})^n = \mathbb{R}^n$ (Assumption F and Condition 1.13.4); in particular, $\boldsymbol{y}^\star = \boldsymbol{f}^{-1}(\boldsymbol{z}^\star)$. An explicit expression for the players' equilibrium actions is given by

$$\forall i \in \mathcal{I} \quad y_i^\star = f^{-1}\left(\frac{1}{\beta+\gamma}\sum_{j\in\mathcal{I}}\left\langle \alpha_j \boldsymbol{e}_i, \left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1}\boldsymbol{e}_j\right\rangle\right).$$

If Condition 1.13.4 is not satisfied, which implies that $f \neq \mathrm{id}_\mathbb{R}$ and $f(\mathbb{R})^n$ is a proper subset of $\mathbb{R}^n$, then $\boldsymbol{z}^\star = \boldsymbol{f}(\boldsymbol{y}^\star)$ may have no solution $\boldsymbol{y}^\star$.

1.3.3.2 Discussion of Proposition 1.14

Suppose $\mathcal{Y} = [0, \bar{v}]$ and Conditions 1.14.1 to 1.14.4 are satisfied. Assumption F implies that $f([0,\bar{v}]) = [f(0), f(\bar{v})]$. Note that $\rho(\bar{\boldsymbol{A}}(G)) = 1$ because $\bar{\boldsymbol{A}}(G)$ is non-negative and row-normalized (Lemma B.8).

Conditions 1.14.3 and 1.14.4 impose an upper bound on the range of the players' idiosyncrasies, that is, they entail a restriction as regards the players' ex ante heterogeneity. Indeed, the two conditions are sufficient for $\alpha_{\max} - \alpha_{\min} < \beta(f(\bar{v}) - f(0))$, where $\beta(f(\bar{v}) - f(0)) > 0$ because $\beta > 0$ (Condition 1.14.1) and f is strictly increasing (Assumption F).

An interior NE $\boldsymbol{y}^\star$ of Γ is a solution to the system of equations (1.14). Analogous to the case $\mathcal{Y} = \mathbb{R}$, the existence and uniqueness of $\boldsymbol{y}^\star$ is shown in two steps. First, by showing that the system of linear equations (1.15) has a unique solution $\boldsymbol{z}^\star \in \mathbb{R}^n$. Second, by showing that the system of equations $\boldsymbol{z}^\star = \boldsymbol{f}(\boldsymbol{y}^\star)$ has a unique solution $\boldsymbol{y}^\star$ in the interior of $[0, \bar{v}]^n$. As to the first step, Conditions 1.14.1 and 1.14.2 imply that $0 \leq \gamma/(\beta+\gamma)\rho(\bar{\boldsymbol{A}}(G)) < 1$, which in turn implies that (1.15) has a unique solution $\boldsymbol{z}^\star \in \mathbb{R}^n$ (Lemma B.3), where

$$\boldsymbol{z}^\star = \frac{1}{\beta+\gamma}\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1}\boldsymbol{\alpha}. \tag{1.16}$$

As to the second step, Conditions 1.14.1, 1.14.2, and 1.14.3 imply $\boldsymbol{z}^\star \in (f(0), +\infty)^n$. Similarly, Conditions 1.14.1, 1.14.2, and 1.14.4 imply that $\boldsymbol{z}^\star \in (-\infty, f(\bar{v}))^n$. It follows that $\boldsymbol{z}^\star = \boldsymbol{f}(\boldsymbol{y}^\star)$ has a unique solution $\boldsymbol{y}^\star \in (0, \bar{v})^n$ because f is bijective (Lemma 1.1) and $\boldsymbol{z}^\star \in (f(0), f(\bar{v}))^n = (-\infty, f(\bar{v}))^n \cap (f(0), +\infty)^n$. The proof of $\boldsymbol{z}^\star \in (f(0), +\infty)^n$ is as follows. The two inequalities $0 \leq \gamma/(\beta+\gamma)\rho(\bar{\boldsymbol{A}}(G)) < 1$ do not only imply that $\boldsymbol{I}_n - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G)$ is nonsingular but also that its inverse is nonnegative and bounded below by $\boldsymbol{I}_n$ (Lemma B.6). This result is trivial if $\gamma = 0$. If $\gamma > 0$, the result follows from the fact that $\boldsymbol{I}_n - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G)$ is a positive scalar multiple of $(1+\beta/\gamma)\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)$, which is an M-matrix.[34] Condition 1.16.3 implies that $\beta f(0)\mathbf{1}_n <_c \boldsymbol{\alpha}$. The two inequalities $\boldsymbol{I}_n \leq_c (\boldsymbol{I}_n - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G))^{-1}$ and $\beta f(0)\mathbf{1}_n <_c \boldsymbol{\alpha}$ imply that $\boldsymbol{z}^\star \in (f(0), +\infty)^n$ (Lemma B.1). Indeed,

$$\frac{1}{\beta+\gamma}\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1}\boldsymbol{\alpha} >_c \frac{\beta f(0)}{\beta+\gamma}\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1}\mathbf{1}_n = f(0)\mathbf{1}_n$$

because $(\boldsymbol{I}_n - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G))^{-1}\mathbf{1}_n = ((\beta+\gamma)/\beta)\mathbf{1}_n$. This concludes the proof of $\boldsymbol{z}^\star \in (f(0), +\infty)^n$. The proof of $\boldsymbol{z}^\star \in (-\infty, f(\bar{v}))^n$ is similar.

Finally, note that Γ has no NEs that involve actions at the boundary of the action space, $\partial[0, \bar{v}] = \{0, \bar{v}\}$, in particular, neither $\mathbf{0}_n$ nor $\bar{v}\mathbf{1}_n$ is a boundary NE of Γ.[35]

1.3.3.3 Discussion of Proposition 1.15

Suppose $\mathcal{Y} = [0, \bar{v}]$ and Conditions 1.15.1 to 1.15.4 are satisfied. Note that Conditions 1.15.1 and 1.15.2 imply that $\beta + \gamma > 0$ and $-1 < \gamma/(\beta+\gamma) < 0$, where the latter inequality is sufficient for $|\gamma/(\beta+\gamma)| < 1$. Note also that Condition 1.15.3 is stronger than Condition 1.14.3 and Condition 1.15.4 is stronger than Condition 1.14.4, in particular, $\beta f(0) < \beta f(0) + \gamma(f(0) - f(\bar{v}))$ and $\beta f(\bar{v}) + \gamma(f(\bar{v}) - f(0)) < \beta f(\bar{v})$ because $\gamma < 0$ and f is strictly increasing (Assumption F).

Conditions 1.15.3 and 1.15.4 impose an upper bound on the range of the players' idiosyncrasies that is lower than the bound imposed by Conditions 1.14.3 and 1.14.4, that is, the conditions of Proposition 1.15 are more restrictive than those of Proposition 1.14 as regards the players' ex ante heterogeneity. Indeed, Conditions 1.15.3 and 1.15.4 are sufficient for $\alpha_{\max} - \alpha_{\min} < (\beta + 2\gamma)(f(\bar{v}) - f(0))$, where

34. For the definition of M-matrices and their properties see, for example, Berman and Plemmons (1994, Definition 1.2 on p. 133 and Theorem 2.3 on pp. 134–38).

35. See the proof of Proposition 1.14 for details.

$\beta(f(\bar{v}) - f(0)) > (\beta + 2\gamma)(f(\bar{v}) - f(0)) > 0$ because $\beta > 0$ (Condition 1.15.1), $\gamma > -\beta/2$ (Condition 1.15.2), and f is strictly increasing (Assumption F).

An interior NE $\boldsymbol{y}^\star$ of Γ is a solution to the system of equations (1.14). Analogous to the case $\mathcal{Y} = \mathbb{R}$, the existence and uniqueness of $\boldsymbol{y}^\star$ is shown in two steps. First, by showing that the system of linear equations (1.15) has a unique solution $\boldsymbol{z}^\star \in \mathbb{R}^n$. Second, by showing that the system of equations $\boldsymbol{z}^\star = \boldsymbol{f}(\boldsymbol{y}^\star)$ has a unique solution $\boldsymbol{y}^\star$ in the interior of $[0, \bar{v}]^n$. As to the first step, Conditions 1.15.1 and 1.15.2 imply that $|\gamma/(\beta+\gamma)| < 1$, which in turn implies that $\boldsymbol{I}_n - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G)$ is nonsingular (Lemma B.3). It follows that (1.15) has a unique solution $\boldsymbol{z}^\star \in \mathbb{R}^n$, which is given by (1.16). As to the second step, the inequality $|\gamma/(\beta+\gamma)| < 1$ does not only imply that $\boldsymbol{I}_n - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G)$ is nonsingular but also that $\boldsymbol{I}_n - \gamma^2/(\beta+\gamma)^2\bar{\boldsymbol{A}}(G)^2$ is nonsingular with a nonnegative inverse that is bounded below by $\boldsymbol{I}_n$ because $0 < \gamma^2/(\beta+\gamma)^2\rho(\bar{\boldsymbol{A}}(G)^2) = \gamma^2/(\beta+\gamma)^2 < 1$ (Lemma B.6). The foregoing result implies that $\boldsymbol{z}^\star$ has an alternative representation that is given by

$$\boldsymbol{z}^\star = \frac{1}{\beta+\gamma}\left(\boldsymbol{I}_n - \frac{\gamma^2}{(\beta+\gamma)^2}\bar{\boldsymbol{A}}(G)^2\right)^{-1}\left(\boldsymbol{I}_n - \frac{|\gamma|}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)\boldsymbol{\alpha}.$$

The following result forms the basis for establishing $\boldsymbol{z}^\star \in (f(0), f(\bar{v}))^n$.

Lemma 1.19 *If Conditions 1.15.1 to 1.15.4 are satisfied, then*

$$\frac{\beta(\beta+2\gamma)}{\beta+\gamma}f(0)\mathbf{1}_n <_c \left(\boldsymbol{I}_n - \frac{|\gamma|}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)\boldsymbol{\alpha} <_c \frac{\beta(\beta+2\gamma)}{\beta+\gamma}f(\bar{v})\mathbf{1}_n \tag{1.17}$$

and $\{\bar{\boldsymbol{\alpha}} \in \mathbb{R}^n \mid \bar{\boldsymbol{\alpha}}$ *satisfies* (1.17)$\}$ *is a convex set.*

The inequality $\boldsymbol{I}_n \leq_c (\boldsymbol{I}_n - \gamma^2/(\beta+\gamma)^2\bar{\boldsymbol{A}}(G)^2)^{-1}$ and the left inequality of (1.17) imply that $\boldsymbol{z}^\star \in (f(0), +\infty)^n$ (Lemma B.1). Indeed,

$$\begin{aligned}\frac{1}{\beta+\gamma}&\left(\boldsymbol{I}_n - \frac{\gamma^2}{(\beta+\gamma)^2}\bar{\boldsymbol{A}}(G)^2\right)^{-1}\left(\boldsymbol{I}_n - \frac{|\gamma|}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)\boldsymbol{\alpha}\\ &>_c \frac{\beta(\beta+2\gamma)f(0)}{(\beta+\gamma)^2}\left(\boldsymbol{I}_n - \frac{\gamma^2}{(\beta+\gamma)^2}\bar{\boldsymbol{A}}(G)^2\right)^{-1}\mathbf{1}_n = f(0)\mathbf{1}_n\end{aligned}$$

because $(\boldsymbol{I}_n - \gamma^2/(\beta+\gamma)^2\bar{\boldsymbol{A}}(G)^2)^{-1}\mathbf{1}_n = (\beta+\gamma)^2/(\beta(\beta+2\gamma))\mathbf{1}_n$. Similarly, the inequality $\boldsymbol{I}_n \leq_c (\boldsymbol{I}_n - \gamma^2/(\beta+\gamma)^2\bar{\boldsymbol{A}}(G)^2)^{-1}$ and the right inequality of (1.17) imply that $\boldsymbol{z}^\star \in (-\infty, f(\bar{v}))^n$. It follows that $\boldsymbol{z}^\star = \boldsymbol{f}(\boldsymbol{y}^\star)$ has a unique solution $\boldsymbol{y}^\star \in (0, \bar{v})^n$ because f is bijective (Lemma 1.1) and $\boldsymbol{z}^\star \in (f(0), f(\bar{v}))^n$.

Finally, note that Γ has no NEs that involve actions at the boundary of the action space, $\partial[0, \bar{v}] = \{0, \bar{v}\}$, in particular, neither $\mathbf{0}_n$ nor $\bar{v}\mathbf{1}_n$ is a boundary NE of Γ.[36]

The preceding discussion demonstrates that Γ has a unique interior NE if conditions less restrictive than those of Proposition 1.15 are satisfied.[37] The corresponding

36. See the proof of Proposition 1.15 for details.
37. Note the distinction between a unique and interior NE of Γ and a unique interior NE of Γ.

result is stated as Remark 1.20, which also highlights the importance of Conditions 1.15.3 and 1.15.4 for the absence of NEs that involve actions at the boundary of the action space.

Remark 1.20 Suppose $\mathcal{Y} = [0, \bar{v}]$. The NALA game Γ has a unique interior NE $\boldsymbol{y}^\star \in (0, \bar{v})^n$, which is given by (1.7), if three conditions are satisfied: Conditions 1.15.1 and 1.15.2 and the two inequalities (1.17).

It should be emphasized that Conditions 1.15.1 and 1.15.2 and the two inequalities (1.17) are not too restrictive in the sense that the set of all families $\{\alpha_i\}_{i\in\mathcal{I}}$ that satisfy (1.17) is empty. A simple example for a $\{\alpha_i\}_{i\in\mathcal{I}}$ that satisfies (1.17) is $\{(\beta/2)(f(0) + f(\bar{v}))\}_{i\in\mathcal{I}}$, that is, $\boldsymbol{\alpha} = (\beta/2)(f(0) + f(\bar{v}))\mathbf{1}_n$.[38] Examples with ex ante heterogeneous players can be constructed on the basis of Lemma 1.19.

In the remainder of this section, I discuss the two inequalities (1.17) in the context of Remark 1.20, thereby assuming that $f = \mathrm{id}_{[0,\bar{v}]}$. To this end, I assume that Conditions 1.15.1 and 1.15.2 and the two inequalities (1.17) are satisfied. As stated above, Conditions 1.15.1 and 1.15.2 imply that $\boldsymbol{y}^\star$ can be written as

$$\boldsymbol{y}^\star = \frac{1}{\beta+\gamma}\left(\boldsymbol{I}_n - \frac{\gamma^2}{(\beta+\gamma)^2}\bar{\boldsymbol{A}}(G)^2\right)^{-1}\left(\boldsymbol{I}_n - \frac{|\gamma|}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)\boldsymbol{\alpha},$$

where $\beta + \gamma > 0$, $0 < |\gamma|/(\beta+\gamma) < 1$, and $\boldsymbol{I}_n - \gamma^2/(\beta+\gamma)^2\bar{\boldsymbol{A}}(G)^2$ has an inverse that is nonnegative and bounded below by $\boldsymbol{I}_n$. This shows that $\boldsymbol{y}^\star$ is positive, that is, $\mathbf{0}_n <_c \boldsymbol{y}^\star$, because (1.17) implies that

$$\mathbf{0}_n <_c \left(\boldsymbol{I}_n - \frac{|\gamma|}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)\boldsymbol{\alpha} \tag{1.18}$$

or, equivalently,

$$\forall i \in \mathcal{I} \quad \alpha_i > \begin{cases} \dfrac{|\gamma|}{\beta+\gamma}\alpha_i & \text{if } \deg^+_G(i) = 0, \\ \dfrac{|\gamma|}{\beta+\gamma}\dfrac{\sum_{j\in\mathcal{N}^+_G(i)}\alpha_j}{\deg^+_G(i)} & \text{if } \deg^+_G(i) > 0. \end{cases} \tag{1.19}$$

It is important to note that (1.18) is sufficient but not necessary for $\mathbf{0}_n <_c \boldsymbol{\alpha}$, that is, $\{\alpha_i \mid i \in \mathcal{I}\} \subset \mathbb{R}_{++}$.[39] For the purpose of interpreting (1.19), let $k \in \mathcal{I}$. First, consider the case $\deg^+_G(k) = 0$. The inequality $\alpha_k > |\gamma|/(\beta+\gamma)\alpha_k$ is equivalent to $\alpha_k > 0$, where $\alpha_k > 0$ is both necessary and sufficient for $y^\star_k > 0$ because $y^\star_k = \alpha_k/\beta$. Second, consider the case $\deg^+_G(k) > 0$. In this case, too, (1.19) entails a lower

38. Suppose $\boldsymbol{\alpha} = (\beta/2)(f(0) + f(\bar{v}))\mathbf{1}_n$. It follows that

$$\left(\boldsymbol{I}_n - \frac{|\gamma|}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)\boldsymbol{\alpha} = \left(1 - \frac{|\gamma|}{\beta+\gamma}\right)\beta\frac{f(0)+f(\bar{v})}{2}\mathbf{1}_n = \frac{\beta(\beta+2\gamma)}{\beta+\gamma}\frac{f(0)+f(\bar{v})}{2}\mathbf{1}_n$$

because $\bar{\boldsymbol{A}}(G)\mathbf{1}_n = \mathbf{1}_n$. Consequently, $\boldsymbol{\alpha}$ satisfies the two inequalities (1.17).

39. Indeed, premultiplying both sides of (1.18) by the inverse of $\boldsymbol{I}_n - |\gamma|/(\beta+\gamma)\bar{\boldsymbol{A}}(G)$ yields $\mathbf{0}_n <_c \boldsymbol{\alpha}$ (Lemma B.1).

bound for α_k. This lower bound (i) depends on β, γ, and the average idiosyncrasy of player k's out-neighbors; (ii) is strictly decreasing in β and γ; and (iii) can be well above zero because $\{\alpha_j \mid j \in \mathcal{N}_G^+(k)\} \subset \mathbb{R}_{++}$. Besides imposing a lower bound on α_k, for each maximal subset of weakly connected players of Γ, (1.19) entails a restriction with regard to the range of these players' idiosyncrasies. This is illustrated in Example 1.21.

Example 1.21 Suppose $k \in \mathcal{I}$, $\beta > 0$ (Condition 1.15.1), $-\beta/2 < \gamma < 0$ (Condition 1.15.2), and $d := \deg_G^+(k) > 0$. In addition, suppose $\mathcal{I}_k := \mathcal{N}_G^+(k) \cup \{k\}$ constitutes the vertex set of a complete weakly connected component G_k of G, so that for all $i \in \mathcal{I}_k$, $\mathcal{N}_G^+(i) = \mathcal{I}_k \setminus \{i\}$ and $\deg_G^+(i) = d$. This means that every player in G_k is an out-neighbor of every other player in G_k. Note that G_k is of order $d+1$ and the endogenous effects matrix of G_k (that is, the row-normalized adjacency matrix of $\mathrm{sl}(G_k)$ with respect to the unique order isomorphism $h\colon \mathcal{I}_k \to \{1,\dots,d+1\}$), $\bar{\boldsymbol{A}}(G_k)$, is equal to $(1/d)(\mathbf{1}_{d+1}\mathbf{1}_{d+1}^\top - \boldsymbol{I}_{d+1})$. Let $\alpha_{\min,\mathcal{I}_k} := \min\{\alpha_i \mid i \in \mathcal{I}_k\}$ and $\alpha_{\max,\mathcal{I}_k} := \max\{\alpha_i \mid i \in \mathcal{I}_k\}$. Finally, suppose $\alpha_{\min,\mathcal{I}_k} > 0$ and the α_i's are uniformly distributed between $\alpha_{\min,\mathcal{I}_k}$ and $\alpha_{\max,\mathcal{I}_k}$, that is,

$$\{\alpha_i \mid i \in \mathcal{I}_k\} = \left\{ \alpha_{\min,\mathcal{I}_k} + \frac{\alpha_{\max,\mathcal{I}_k} - \alpha_{\min,\mathcal{I}_k}}{d}(i-1) \;\middle|\; i \in \{1,\dots,d+1\} \right\}.$$

It bears mentioning that this assumption is compatible with the extreme case where $\alpha_{\min,\mathcal{I}_k} = \alpha_{\max,\mathcal{I}_k}$. It is straightforward to show that[40]

$$\begin{aligned} & \forall i \in \mathcal{I}_k \qquad \alpha_i > \frac{|\gamma|}{\beta+\gamma} \frac{\sum_{j\in\mathcal{N}_G^+(i)} \alpha_j}{\deg_G^+(i)} \\ \Leftrightarrow \quad & \alpha_{\max,\mathcal{I}_k} - \alpha_{\min,\mathcal{I}_k} < \frac{\beta+2\gamma}{|\gamma|} \frac{2d}{d+1} \alpha_{\min,\mathcal{I}_k} && (1.20) \\ \Leftrightarrow \quad & \frac{\alpha_{\max,\mathcal{I}_k}}{\alpha_{\min,\mathcal{I}_k}} < 1 + \frac{\beta+2\gamma}{|\gamma|} \frac{2d}{d+1}. && (1.21) \end{aligned}$$

It follows that $(\mathcal{I}_k, G_k, \{(\alpha_i, \beta, \gamma)\}_{i\in\mathcal{I}_k})$ satisfies (1.19) if and only if the range of $\{\alpha_i \mid i \in \mathcal{I}_k\}$ is bounded above by a threshold that is proportional to $\alpha_{\min,\mathcal{I}_k}$, where the factor of proportionality is strictly increasing in β, γ, and d. This restriction can be rephrased in terms of an upper bound for $\alpha_{\max,\mathcal{I}_k}/\alpha_{\min,\mathcal{I}_k}$, as is evident from (1.21). Loosely speaking, $\alpha_{\max,\mathcal{I}_k}$ must not exceed a multiple of $\alpha_{\min,\mathcal{I}_k}$. Figure 1.8 illustrates this upper bound for $\alpha_{\max,\mathcal{I}_k}/\alpha_{\min,\mathcal{I}_k}$ as a function of γ for different values of d, thereby assuming that $\beta = 1$. The graphs for $d \in \{1,2,3,6,9\}$ and the limiting case $d \to \infty$ indicate that the upper bound for $\alpha_{\max,\mathcal{I}_k}/\alpha_{\min,\mathcal{I}_k}$ is close to one for γ in a small right neighborhood of $-1/2$. In order that inequality (1.21) is true, the players must therefore be nearly strongly ex ante homogeneous if γ is close to its lower bound, $-1/2$. ◇

40. See Appendix D for the proofs of (1.20) and (1.21).

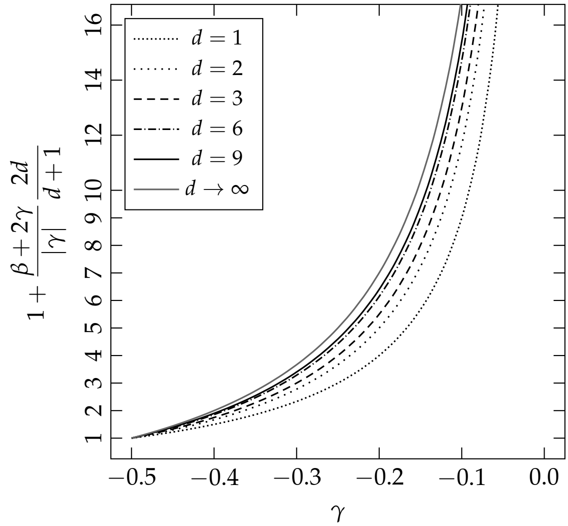


Figure 1.8. Upper bound for $\alpha_{\max}/\alpha_{\min}$ as a function of γ for $\beta = 1$ and $d \in \{1, 2, 3, 6, 9\}$, including the limiting case $d \to \infty$ (Example 1.21)

So far, the discussion has emphasized the role of the left inequality of (1.17) for the positivity of $\boldsymbol{y}^\star$. The right inequality of (1.17),

$$\left(\boldsymbol{I}_n - \frac{|\gamma|}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)\boldsymbol{\alpha} <_c \frac{\beta(\beta+2\gamma)}{\beta+\gamma}\bar{v}\mathbf{1}_n \tag{1.22}$$

or, equivalently,

$$\forall i \in \mathcal{I} \quad \alpha_i < \begin{cases} \dfrac{|\gamma|}{\beta+\gamma}\alpha_i + \dfrac{\beta(\beta+2\gamma)}{\beta+\gamma}\bar{v} & \text{if } \deg_G^+(i) = 0, \\ \dfrac{|\gamma|}{\beta+\gamma}\dfrac{\sum_{j\in\mathcal{N}_G^+(i)}\alpha_j}{\deg_G^+(i)} + \dfrac{\beta(\beta+2\gamma)}{\beta+\gamma}\bar{v} & \text{if } \deg_G^+(i) > 0, \end{cases} \tag{1.23}$$

ensures that $\boldsymbol{y}^\star <_c \bar{v}\mathbf{1}_n$. Similar to (1.18) and (1.19), (1.22) and (1.23) entail a restriction with regard to the players' idiosyncrasies. This is demonstrated in the continuation of Example 1.21.

Example 1.21 (cont'd) It is straightforward to show that[41]

$$\begin{aligned} \forall i \in \mathcal{I}_k \qquad & \alpha_i < \frac{|\gamma|}{\beta+\gamma}\frac{\sum_{j\in\mathcal{N}_G^+(i)}\alpha_j}{\deg_G^+(i)} + \frac{\beta(\beta+2\gamma)}{\beta+\gamma}\bar{v} \\ \Leftrightarrow \quad & \alpha_{\max,\mathcal{I}_k} - \alpha_{\min,\mathcal{I}_k} < \frac{\beta+2\gamma}{|\gamma|}\frac{2d}{d+1}(\beta\bar{v} - \alpha_{\max,\mathcal{I}_k}). \end{aligned} \tag{1.24}$$

41. See Appendix D for a proof of (1.24).

It follows that $(\mathcal{I}_k, G_k, \bar{v}, \{(\alpha_i, \beta, \gamma)\}_{i\in\mathcal{I}_k})$ satisfies (1.23) if and only if the range of $\{\alpha_i \mid i \in \mathcal{I}_k\}$ is bounded above by a threshold. If $(\mathcal{I}_k, G_k, \bar{v}, \{(\alpha_i, \beta, \gamma)\}_{i\in\mathcal{I}_k})$ satisfies (1.24), then $\beta\bar{v} > \alpha_{\max,\mathcal{I}_k}$ must be true because $\alpha_{\max,\mathcal{I}_k} \geq \alpha_{\min,\mathcal{I}_k}$, which implies that the threshold is strictly increasing in β, γ, d, and $\bar{v}$. The threshold of (1.20) is less than the threshold of (1.24) if $\bar{v}$ is sufficiently large, that is, $\bar{v} > (\alpha_{\min,\mathcal{I}_k} + \alpha_{\max,\mathcal{I}_k})/\beta$. To summarize, both the left and the right inequality of (1.17) entail a restriction with regard to the range of $\{\alpha_i \mid i \in \mathcal{I}_k\}$, where the restriction imposed by the left inequality is stronger than that imposed by the right inequality if $\bar{v}$ is sufficiently large. ◇

It has been argued that the two inequalities (1.17) impose restrictions on the range of the players' idiosyncrasies for every maximal subset of weakly connected players of Γ. These restrictions are of a local nature if G has more than one weakly connected component. Taken all together, the local restrictions amount to a global restriction, which is given by $\beta f(0) < \alpha_{\min} \leq \alpha_{\max} < \beta f(\bar{v})$. This is stated formally in Lemma 1.22.

Lemma 1.22 *If Conditions 1.15.1 and 1.15.2 and the two inequalities (1.17) are satisfied, then $\beta f(0)\mathbf{1}_n <_c \boldsymbol{\alpha} <_c \beta f(\bar{v})\mathbf{1}_n$, which in turn is equivalent to $\beta f(0) < \alpha_{\min}$ (Condition 1.14.3)* and *$\alpha_{\max} < \beta f(\bar{v})$ (Condition 1.14.4).*

Lemma 1.22 shows that the conditions of Remark 1.20 are sufficient for $\beta f(0) < \alpha_{\min} \leq \alpha_{\max} < \beta f(\bar{v})$, and so are the conditions of Proposition 1.14. Both sets of conditions impose therefore the same restriction on the players' idiosyncrasies. In comparison, the conditions of Proposition 1.15 impose a stronger restriction on their idiosyncrasies, namely,

$$\beta f(0) + \gamma\big(f(0) - f(\bar{v})\big) < \alpha_{\min} \leq \alpha_{\max} < \beta f(\bar{v}) + \gamma\big(f(\bar{v}) - f(0)\big),$$

where $\beta f(0) < \beta f(0) + \gamma(f(0) - f(\bar{v}))$ and $\beta f(\bar{v}) + \gamma(f(\bar{v}) - f(0)) < \beta f(\bar{v})$. It follows that the players of Γ must be more similar in terms of their idiosyncrasies if γ is negative than if γ is positive in order that a unique and interior NE exists.[42]

1.3.3.4 Discussion of Proposition 1.17

Suppose $\mathcal{Y} = \mathbb{R}_+$. The NALA game Γ has a unique interior NE if Conditions 1.17.1 to 1.17.4 and Condition 1.17.6 are satisfied.[43] The above conditions ensure that all players play a positive action. In this respect, Condition 1.17.3 (respectively, Condition 1.17.4) is critical for players with an empty (respectively, a nonempty) out-neighborhood. Condition 1.17.4 is equivalent to Condition 1.15.3 with

$$\frac{1}{\beta}\left(\alpha_{\max} + \frac{|\gamma|}{\beta + 2\gamma}(\alpha_{\max} - \alpha_{\min})\right)$$

42. It should be borne in mind, however, that the conditions of Propositions 1.14 and 1.15 are sufficient but not necessary for the existence of a unique and interior NE of Γ.

43. See the remark in Footnote 37.

substituted for $f(\bar{v})$ (cf. Result 1.31.1 and Remark 1.36 in Section 1.3.4.2). Conditions 1.17.3 and 1.17.4 imply that

$$\alpha_{\max} - \alpha_{\min} < \frac{\beta+\gamma}{|\gamma|}(\alpha_{\min} - \beta f(0))$$

and therefore entail a restriction with regard to the range of the players' idiosyncrasies, provided that Condition 1.17.2 is satisfied. Condition 1.17.3 ensures that $\mathbf{0}_n$ is not a boundary NE of Γ. Conditions 1.17.3 and 1.17.5 rule out the possibility that Γ has a boundary NE where some players play zero and the remaining players play a positive action. Condition 1.17.5 is equivalent to

$$\beta(\beta+2\gamma)f(0) < (\beta+\gamma)\alpha_{\min} + \gamma\alpha_{\max} + \gamma\left(\alpha_{\min} - \beta f(0) + \frac{\gamma}{\beta}(\alpha_{\max} - \beta f(0))\right),$$

from which it follows that Condition 1.17.5 is stronger than Condition 1.17.4 if $\alpha_{\min} - \beta f(0) \geq (|\gamma|/\beta)(\alpha_{\max} - \beta f(0))$, where $|\gamma|/\beta < 1/2$, provided that Conditions 1.17.1 and 1.17.2 are satisfied. It is important to note that Condition 1.17.5 cannot be met if $|\gamma|(\beta+\gamma) \geq \beta(\beta+2\gamma)$ or, equivalently, $-\beta/2 < \gamma \leq (\sqrt{5}-3)\beta/2 \approx -0.38\beta$, provided that Conditions 1.17.1, 1.17.2, and 1.17.3 are satisfied.

1.3.4 Properties of interior Nash equilibria

For this section, let $\Gamma(G) := (\mathcal{I}, G, \mathcal{Y}, \{(\alpha_i, \beta, \gamma)\}_{i\in\mathcal{I}}, f)$ be a generic NALA game, and let $\boldsymbol{\alpha}$ be defined as in Section 1.3.2. Suppose $\beta > 0$ and $\gamma > -\beta/2$, so that $\beta + \gamma > 0$ and $|\gamma/(\beta+\gamma)| < 1$. In addition, suppose $\Gamma(G)$ has a unique and interior NE $\boldsymbol{y}^\star := (y_1^\star, \ldots, y_n^\star)$, which is given by (1.7).

Hereinafter, $\boldsymbol{y}^\star$ is written as $\boldsymbol{y}^\star(\boldsymbol{\alpha}, \beta, \gamma, f, G)$ in order to emphasize that it is a mapping of $\boldsymbol{\alpha}$, β, γ, f, and G. The same applies to the components of $\boldsymbol{y}^\star$. Depending on the context and situation, some or all of the arguments of $\boldsymbol{y}^\star$ and of its components may be omitted. In all discussions to follow, actions are to be understood as equilibrium actions.

The remainder of this section is structured as follows. Section 1.3.4.1 studies how players' actions respond to changes in their preference parameters and to changes in the digraph by which they are connected. Section 1.3.4.2 focuses on the actions of weakly connected players.

1.3.4.1 Comparative statics

First, I discuss the effects of changes in the players' preference parameters. The discussion rests upon the following result.

Proposition 1.23 *Let $(i,j) \in \mathcal{I}^2$, and let θ be α_j, β, or γ. The partial derivatives of $y_i^\star(\boldsymbol{\alpha}, \beta, \gamma, f, G)$ and $y_i^\star(\boldsymbol{\alpha}, \beta, \gamma, \mathrm{id}_{\mathcal{Y}}, G)$ with respect to θ have the same sign. The partial derivatives of $\boldsymbol{y}^\star(\boldsymbol{\alpha}, \beta, \gamma, \mathrm{id}_{\mathcal{Y}}, G)$ with respect to $\boldsymbol{\alpha}$, β, and γ are given by*

$$\boldsymbol{J}(\beta,\gamma,G) := \frac{\partial \boldsymbol{y}^\star(\boldsymbol{\alpha}, \beta, \gamma, \mathrm{id}_{\mathcal{Y}}, G)}{\partial \boldsymbol{\alpha}} = \frac{1}{\beta+\gamma}\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1}, \qquad (1.25)$$

$$\frac{\partial \boldsymbol{y}^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_{\mathcal{Y}},G)}{\partial \beta} = -\boldsymbol{J}(\beta,\gamma,G)\boldsymbol{y}^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_{\mathcal{Y}},G), \tag{1.26}$$

and

$$\frac{\partial \boldsymbol{y}^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_{\mathcal{Y}},G)}{\partial \gamma} = \boldsymbol{J}(\beta,\gamma,G)\big(\bar{\boldsymbol{A}}(G)-\boldsymbol{I}_n\big)\boldsymbol{y}^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_{\mathcal{Y}},G), \tag{1.27}$$

respectively. The Jacobian (matrix) $\boldsymbol{J}(\beta,\gamma,G)$ has the following properties:

(*1.23.1*) *If $-\beta/2 < \gamma < 0$, then*

$$\frac{1}{\beta+\gamma}\boldsymbol{I}_n - \frac{|\gamma|}{\beta(\beta+2\gamma)}\mathbf{1}_n\mathbf{1}_n^\top \leq_c \boldsymbol{J}(\beta,\gamma,G) \leq_c \frac{\beta+\gamma}{\beta(\beta+2\gamma)}\mathbf{1}_n\mathbf{1}_n^\top$$

and for all $i \in \mathcal{I}$, $[\boldsymbol{J}(\beta,\gamma,G)]_{i,i} > 0$.[44]

(*1.23.2*) *If $\gamma = 0$, then $\boldsymbol{J}(\beta,\gamma,G) = (1/\beta)\boldsymbol{I}_n$.*

(*1.23.3*) *If $\gamma > 0$, then $1/(\beta+\gamma)\boldsymbol{I}_n \leq_c \boldsymbol{J}(\beta,\gamma,G) \leq_c (1/\beta)\mathbf{1}_n\mathbf{1}_n^\top$.*

(*1.23.4*) *If G is strongly connected and $\gamma > 0$, then $\boldsymbol{O}_n <_c \boldsymbol{J}(\beta,\gamma,G)$.*

(*1.23.5*) *Suppose $\gamma > 0$. For all $(i,j) \in \mathcal{I}^2$ with $i \neq j$, $[\boldsymbol{J}(\beta,\gamma,G)]_{i,j} > 0$ if and only if there exists a walk in G from i to j.*

(*1.23.6*) *If $\beta > \gamma \geq 0$, then, for all $(i,j,k) \in \mathcal{I}^3$ with $(i,j) \neq (k,k)$, $[\boldsymbol{J}(\beta,\gamma,G)]_{i,j} < 1/(\beta+\gamma) \leq [\boldsymbol{J}(\beta,\gamma,G)]_{k,k}$.*[45]

Proposition 1.23 gives the means to a first-order approximation of the response of a player's action to a change in one of his or another player's preference parameters.[46] It also forms the basis for results on the monotonicity of the players' actions with respect to their preference parameters. The presence or absence of a monotonic dependence is thereby independent of the form of f because $\partial f > 0$ (Assumption F).

Some of the monotonicity results given hereinafter are true only locally, that is, they are true only in a neighborhood of the players' preference parameters for which for each of its points the associated NALA game has a unique and interior NE of the form (1.7). The localness of the results is thereby largely determined by the type of the action space.

Let $(i,j) \in \mathcal{I}^2$ with $i \neq j$. In general, player i's action is (i) strictly increasing in α_i (see partial derivative (1.25) and Results 1.23.1, 1.23.2, and 1.23.3); (ii) increasing in α_j if $\gamma \geq 0$ (see partial derivative (1.25) and Results 1.23.2 and 1.23.3); (iii) strictly increasing in α_j if $\gamma > 0$ and player j is an out-neighbor or a higher-order out-neighbor of player i in G, so that there exists a walk in G from i to j (see partial derivative (1.25) and Result 1.23.5);[47] (iv) strictly decreasing in β over some interval

44. Note that $1/(\beta+\gamma) < |\gamma|/(\beta(\beta+2\gamma))$ if $-\beta/2 < \gamma < (\sqrt{5}-3)\beta/2$.
45. If $\beta > \gamma \geq 0$, then $\gamma/(\beta+\gamma) < 1/2$.
46. This follows from the fact that for all $i \in \mathcal{I}$, $y_i^\star(\boldsymbol{\alpha},\beta,\gamma,f,G) = f^{-1}(y_i^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_{\mathcal{Y}},G))$.
47. This result underscores the importance of one implication of Result 1.23.5, namely, for all $(i,j) \in \mathcal{I}^2$ with $i \neq j$, if there exists a walk in G from i to j, then $[\boldsymbol{J}(\beta,\gamma,G)]_{i,j} > 0$.

Figure 1.9. A star-shaped digraph (Examples 1.24, 1.49, and 1.60) or digraph component (Example 1.32) of order 3

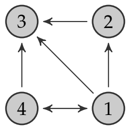


Figure 1.10. A digraph of order 4 (Example 1.25)

$I \subset \mathbb{R}_{++}$ if $\gamma \geq 0$, where I is such that for all $b \in I$, $\boldsymbol{y}^\star(\boldsymbol{\alpha}, b, \gamma, \mathrm{id}_{\mathcal{Y}}, G) \in \mathbb{R}^n_{++}$ (see partial derivative (1.26) and Results 1.23.2 and 1.23.3);[48] and (v) not monotonic in γ even if $\gamma \geq 0$ (see partial derivative (1.27) and Example 1.24). Player i's action may be strictly decreasing in α_j, independent of α_j, or strictly increasing in α_j if $-\beta/2 < \gamma < 0$ and player j is an out-neighbor or a higher-order out-neighbor of player i in G (see partial derivative (1.25) and Result 1.23.1 and Example 1.25).

Example 1.24 Suppose $\mathcal{I} = \{1, 2, 3\}$ and $\mathcal{A}(G) = \{(1,2), (1,3), (2,1), (3,1)\}$. See Figure 1.9 for an illustration of G. The endogenous effects matrix of G is given by

$$\bar{\boldsymbol{A}}(G) = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}. \tag{1.28}$$

In addition, suppose $\mathcal{Y} = \mathbb{R}_+$, $\alpha_1 = 3/4$, $\alpha_2 = 5/7$, $\alpha_3 = 4/7$, $\beta = 1$, $\gamma > 0$, and $f = \mathrm{id}_{\mathbb{R}_+}$. Let $\boldsymbol{y}^\star(\gamma) = (y_1^\star(\gamma), y_2^\star(\gamma), y_3^\star(\gamma))$ denote the unique and interior NE of $\Gamma(G)$. Straightforward calculations yield

$$\boldsymbol{y}^\star(\gamma) = \boldsymbol{\alpha} + \frac{1}{28}\frac{\gamma}{(\gamma+1)(2\gamma+1)}\begin{pmatrix} -3\gamma - 3 \\ -\gamma + 1 \\ 7\gamma + 5 \end{pmatrix},$$

which implies that

$$\frac{\partial y_2^\star(\gamma)}{\partial \gamma} = -\frac{1}{28}\frac{5\gamma^2 + 2\gamma - 1}{(\gamma+1)^2(2\gamma+1)^2}.$$

It follows that player 2's action is strictly increasing on $(0, (\sqrt{6}-1)/5)$ and strictly decreasing on $((\sqrt{6}-1)/5, +\infty)$. ◇

Example 1.25 Suppose $\mathcal{I} = \{1, 2, 3, 4\}$ and $\mathcal{A}(G) = \{(1,2), (1,3), (1,4), (2,3), (4,1), (4,3)\}$. See Figure 1.10 for an illustration of G. In addition, suppose $\gamma =$

48. For example, $I = \mathbb{R}_{++}$ if $\mathcal{Y} = \mathbb{R}_+$, $\gamma \geq 0$, and $\alpha_{\min} > 0$ (see Proposition 1.16), and $I = (\alpha_{\max}/\bar{v}, +\infty)$ if $\mathcal{Y} = [0, \bar{v}]$, $\gamma \geq 0$, and $\alpha_{\min} > 0$ (see Proposition 1.14).

$-(2/5)\beta$. Note that player 3 is an out-neighbor of player 1 in G, so that there exists a walk in G from player 1 to player 3. Straightforward calculations yield

$$J(\beta,\gamma,G) = \frac{1}{15\beta}\begin{pmatrix} 27 & -6 & 0 & -6 \\ 0 & 25 & -10 & 0 \\ 0 & 0 & 15 & 0 \\ -9 & 2 & -5 & 27 \end{pmatrix},$$

from which it follows that $\partial y_1^\star(\boldsymbol{\alpha},\beta,\gamma,f,G)/\partial\alpha_3 = 0$, $\partial y_4^\star(\boldsymbol{\alpha},\beta,\gamma,f,G)/\partial\alpha_1 < 0$, and $\partial y_4^\star(\boldsymbol{\alpha},\beta,\gamma,f,G)/\partial\alpha_2 > 0$.[49] To sum up, player 1's action is independent of player 3's idiosyncrasy although player 3 is an out-neighbor of player 1 in G, and player 4's action is strictly decreasing in player 1's idiosyncrasy and strictly increasing in player 2's idiosyncrasy. ◇

Apart from the foregoing results, Proposition 1.23 is also instrumental in discussing the effects of a change in a player's action on other players' actions. As to that, it is important to bear in mind that at the NE of $\Gamma(G)$ no player has an incentive to unilaterally deviate from his action. The question "How does a change in a player's action affect other players' actions?" can thus not be answered unless one is willing to admit of exogenous shocks in the players' preference parameters. A player's idiosyncrasy is the only preference parameter that is not common to all players. It is therefore the only atom of $\Gamma(G)$ that can bring about an isolated change in a player's action. An exogenous shock to a player's idiosyncrasy always causes a change in the player's action that is of the same sign as the shock (see partial derivative (1.25) and Results 1.23.1, 1.23.2, and 1.23.3). Apart from this direct effect, the shock may also have an indirect effect on the player's in-neighbors as it may affect their social norms and in turn their actions. The indirect effect on a player's in-neighbor is different from zero and of the same sign as the shock if $\gamma > 0$ (see partial derivative (1.25) and Result 1.23.5); it may, however, be zero if $-\beta/2 < \gamma < 0$ (see Example 1.25). The indirect effect on other players' actions propagates through G along the inverses of walks if $\gamma > 0$ (see partial derivative (1.25) and Result 1.23.5), where the magnitude of the indirect effect is smaller than that of the direct effect if $\beta > \gamma$, that is, the common private cost parameter exceeds the common social cost parameter, and $f = \mathrm{id}_{\mathcal{Y}}$ (see partial derivative (1.25) and Result 1.23.6).

Second, I discuss the effects of changes in the digraph connecting the players. I confine the analysis to a particular change in the topology of the digraph: the addition of a single arc. I analyze whether and to what extent the players' actions respond to such a change.

Some notational groundwork is needed to state the main result (Proposition 1.27). Assume that there is no arc from player x to player y in G. Adding the arc (x,y) to G yields the digraph $G \boxplus (x,y) := (\mathcal{I}, \mathcal{A}(G) \cup \{(x,y)\})$, which induces the generic

49. Note that

$$\forall (i,j) \in \mathcal{I}^2 \quad \frac{\partial y_i^\star(\boldsymbol{\alpha},\beta,\gamma,f,G)}{\partial\alpha_j} = \frac{1}{\partial f(y_i^\star(\boldsymbol{\alpha},\beta,\gamma,f,G))}\,\frac{\partial y_i^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_{\mathcal{Y}},G)}{\partial\alpha_j},$$

where $\partial f(y_i^\star(\boldsymbol{\alpha},\beta,\gamma,f,G)) > 0$ and $\partial y_i^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_{\mathcal{Y}},G)/\partial\alpha_j = [J(\beta,\gamma,G)]_{i,j}$.

NALA game $\Gamma(G \boxplus (x,y)) := (\mathcal{I}, G \boxplus (x,y), \mathcal{Y}, \{(\alpha_i, \beta, \gamma)\}_{i \in \mathcal{I}}, f)$. The unique and interior NE of $\Gamma(G \boxplus (x,y))$, provided that it exists, is denoted by $\boldsymbol{y}^\star(G \boxplus (x,y))$ or $(y_1^\star(G \boxplus (x,y)), \dots, y_n^\star(G \boxplus (x,y)))$.[50] In order to conform to this notation, the NE of $\Gamma(G)$ is written as $\boldsymbol{y}^\star(G)$ or $(y_1^\star(G), \dots, y_n^\star(G))$ in what follows. The two digraphs G and $G \boxplus (x,y)$ have different arc sets, which implies that their endogenous effects matrices are different. The two matrices are closely related to each other though. In particular, adding the arc (x,y) to G translates into a change of the xth row of $\bar{\boldsymbol{A}}(G)$, whereas all other rows remain unaffected. Let $\boldsymbol{\Delta}(x,y,G) \in \mathbb{R}^n$ be defined by

$$\forall i \in \mathcal{I} \quad [\boldsymbol{\Delta}(x,y,G)]_i := \begin{cases} \delta_{i,y} - \delta_{i,x} & \text{if } \deg_G^+(x) = 0, \\ \dfrac{\mathbb{1}_{\mathcal{N}_G^+(x) \cup \{y\}}(i)}{\deg_G^+(x) + 1} - \dfrac{\mathbb{1}_{\mathcal{N}_G^+(x)}(i)}{\deg_G^+(x)} & \text{if } \deg_G^+(x) > 0. \end{cases}$$

The endogenous effects matrices of G an $G \boxplus (x,y)$ are related to each other through $\boldsymbol{\Delta}(x,y,G)$ by

$$\bar{\boldsymbol{A}}\big(G \boxplus (x,y)\big) = \bar{\boldsymbol{A}}(G) + \boldsymbol{e}_x \boldsymbol{\Delta}(x,y,G)^\mathsf{T}, \tag{1.29}$$

where $\boldsymbol{e}_x = (\delta_{1,x}, \dots, \delta_{n,x})$. The Jacobian (matrix) $\boldsymbol{J}(\beta,\gamma,G)$ plays a key role in the characterization of the effects on players' actions arising from adding (x,y) to G. To this end, let $\boldsymbol{j}_x(\beta,\gamma,G)$ denote the xth column of $\boldsymbol{J}(\beta,\gamma,G)$, that is,

$$\boldsymbol{j}_x(\beta,\gamma,G) := \boldsymbol{J}(\beta,\gamma,G)\boldsymbol{e}_x = \partial\big(\alpha_x \mapsto \boldsymbol{y}^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_{\mathcal{Y}},G)\big). \tag{1.30}$$

Some basic properties of $\boldsymbol{j}_x(\beta,\gamma,G)$ are summarized in the following corollary to Proposition 1.23.

Corollary 1.26 *The vector $\boldsymbol{j}_x(\beta,\gamma,G)$ has the following properties:*

(1.26.1) *If $-\beta/2 < \gamma < 0$, then*

$$\frac{1}{\beta+\gamma}\boldsymbol{e}_x - \frac{|\gamma|}{\beta(\beta+2\gamma)}\mathbf{1}_n \leq_c \boldsymbol{j}_x(\beta,\gamma,G) \leq_c \frac{\beta+\gamma}{\beta(\beta+2\gamma)}\mathbf{1}_n$$

and $[\boldsymbol{j}_x(\beta,\gamma,G)]_x > 0$.

(1.26.2) *If $\gamma = 0$, then $\boldsymbol{j}_x(\beta,\gamma,G) = (1/\beta)\boldsymbol{e}_x$.*

(1.26.3) *If $\gamma > 0$, then $1/(\beta+\gamma)\boldsymbol{e}_x \leq_c \boldsymbol{j}_x(\beta,\gamma,G) \leq_c (1/\beta)\mathbf{1}_n$.*

(1.26.4) *If G is strongly connected and $\gamma > 0$, then $\mathbf{0}_n <_c \boldsymbol{j}_x(\beta,\gamma,G)$.*

(1.26.5) *Suppose $\gamma > 0$. For all $i \in \mathcal{I} \setminus \{x\}$, $[\boldsymbol{j}_x(\beta,\gamma,G)]_i > 0$ if and only if there exists a walk in G from i to x.*

50. Under the maintained assumptions that $\beta > 0$, $\gamma > -\beta/2$, and $\Gamma(G)$ has a unique and interior NE, $\Gamma(G \boxplus (x,y))$ does not necessarily have a unique and interior NE. If, in addition to $\beta > 0$ and $\gamma > -\beta/2$, $\Gamma(G)$ satisfies the matching sufficient conditions for the existence of a unique and interior NE (see Propositions 1.13 to 1.17 in Section 1.3.3), then $\Gamma(G \boxplus (x,y))$ has a unique and interior NE, which is of the form (1.7).

(1.26.6) *If* $\beta > \gamma \geq 0$, *then, for all* $i \in \mathcal{I} \setminus \{x\}$, $[\boldsymbol{j}_x(\beta,\gamma,G)]_i < 1/(\beta+\gamma) \leq [\boldsymbol{j}_x(\beta,\gamma,G)]_x$.

(1.26.7) *If* $\deg^+_G(x) = 0$, *then* $[\boldsymbol{j}_x(\beta,\gamma,G)]_x = 1/\beta > 0$.

The previous definitions and considerations lead to the following result.

Proposition 1.27 *If* $\Gamma(G \boxplus (x,y))$ *has a unique and interior NE, then*

$$\boldsymbol{f}\big(\boldsymbol{y}^\star(G \boxplus (x,y))\big) - \boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big) = \frac{\gamma\big\langle \boldsymbol{\Delta}(x,y,G), \boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big)\big\rangle}{1-\gamma\langle \boldsymbol{\Delta}(x,y,G), \boldsymbol{j}_x(\beta,\gamma,G)\rangle}\boldsymbol{j}_x(\beta,\gamma,G) \quad (1.31)$$

and

$$\begin{aligned}\boldsymbol{f}\big(\boldsymbol{y}^\star(G \boxplus (x,y))\big) - \bar{\boldsymbol{A}}\big(G \boxplus (x,y)\big)\boldsymbol{f}\big(\boldsymbol{y}^\star(G \boxplus (x,y))\big) = \boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big) \\ - \bar{\boldsymbol{A}}(G)\boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big) - \frac{\beta\big\langle \boldsymbol{\Delta}(x,y,G), \boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big)\big\rangle}{1-\gamma\langle \boldsymbol{\Delta}(x,y,G), \boldsymbol{j}_x(\beta,\gamma,G)\rangle}\boldsymbol{j}_x(\beta,\gamma,G), \quad (1.32)\end{aligned}$$

where $1-\gamma\langle \boldsymbol{\Delta}(x,y,G), \boldsymbol{j}_x(\beta,\gamma,G)\rangle > 0$.[51,52]

Within the setup of Proposition 1.27, if $f = \mathrm{id}_{\mathcal{Y}}$, then (1.31) describes the adjustment in $\boldsymbol{y}^\star(G)$ that is caused by the extra arc from player x to player y, otherwise it describes the adjustment in $\boldsymbol{f}(\boldsymbol{y}^\star(G))$.[53]

For the following discussion suppose $\gamma > 0$ and $f = \mathrm{id}_{\mathcal{Y}}$, so that $\boldsymbol{f}(\boldsymbol{y}^\star(G)) = \boldsymbol{y}^\star(G)$. According to (1.31), the difference $\boldsymbol{y}^\star(G \boxplus (x,y)) - \boldsymbol{y}^\star(G)$ is a scalar multiple of $\boldsymbol{j}_x(\beta,\gamma,G)$, which is a measure of the sensitivity of the players' actions with respect to a change in player x's marginal private benefit, α_x. A necessary condition for player $i \in \mathcal{I} \setminus \{x\}$ to be affected by the extra arc from player x to player y is that $[\boldsymbol{j}_x(\beta,\gamma,G)]_i > 0$, which is true if and only if there exists a walk in G from player i to player x (Result 1.26.5), that is, player x lies in the out-neighborhood or a higher-order out-neighborhood in G of player i. Since $\boldsymbol{0}_n \leq_c \boldsymbol{j}_x(\beta,\gamma,G)$ if $\gamma > 0$ (Result 1.26.3), the sign of the change in action cannot differ across the players who are affected by the extra arc. According to (1.31), the sign of the change is equal to the sign of

$$\frac{\gamma\langle \boldsymbol{\Delta}(x,y,G), \boldsymbol{y}^\star(G)\rangle}{1-\gamma\langle \boldsymbol{\Delta}(x,y,G), \boldsymbol{j}_x(\beta,\gamma,G)\rangle}. \quad (1.33)$$

The denominator of (1.33) is positive. As regards its numerator, if $\deg^+_G(x) = 0$, then $\langle \boldsymbol{\Delta}(x,y,G), \boldsymbol{y}^\star(G)\rangle = y^\star_y(G) - \alpha_x/\beta$, and if $\deg^+_G(x) > 0$, then

$$\langle \boldsymbol{\Delta}(x,y,G), \boldsymbol{y}^\star(G)\rangle = \frac{\sum_{i\in\mathcal{N}^+_G(x)\cup\{y\}} y^\star_i(G)}{\deg^+_G(x)+1} - \frac{\sum_{i\in\mathcal{N}^+_G(x)} y^\star_i(G)}{\deg^+_G(x)}$$

51. The inequalities $\beta > 0$ and $\gamma > -\beta/2$ are sufficient for $1-\gamma\langle \boldsymbol{\Delta}(x,y,G), \boldsymbol{j}_x(\beta,\gamma,G)\rangle > 0$.

52. If $\deg^+_G(x) = 0$, then $\langle \boldsymbol{\Delta}(x,y,G), \boldsymbol{j}_x(\beta,\gamma,G)\rangle = [\boldsymbol{j}_x(\beta,\gamma,G)]_y - 1/\beta$, and if $\deg^+_G(x) > 0$, then

$$\langle \boldsymbol{\Delta}(x,y,G), \boldsymbol{j}_x(\beta,\gamma,G)\rangle = \frac{1}{\deg^+_G(x)+1}\left([\boldsymbol{j}_x(\beta,\gamma,G)]_y - \frac{\sum_{i\in\mathcal{N}^+_G(x)}[\boldsymbol{j}_x(\beta,\gamma,G)]_i}{\deg^+_G(x)}\right).$$

53. If $\mathcal{Y} = \mathbb{R}_+$ and $f \approx \log$, then, for all $i \in \mathcal{I}$, $f(y^\star_i(G \boxplus (x,y))) - f(y^\star_i(G))$ is a first-order approximation of the relative change in $y^\star_i(G)$, $y^\star_i(G \boxplus (x,y))/y^\star_i(G) - 1$.

$$= \frac{1}{\deg_G^+(x)+1}\left(y_y^\star(G) - \frac{\sum_{i\in\mathcal{N}_G^+(x)} y_i^\star(G)}{\deg_G^+(x)}\right).$$

Thus, $\langle \boldsymbol{\Delta}(x,y,G), \boldsymbol{y}^\star(G)\rangle$ is equal to the difference between player y's and player x's action if $\deg_G^+(x) = 0$ and equal to the change in the average action of player x's out-neighbors if $\deg_G^+(x) > 0$. Under the maintained assumption that $\gamma > 0$,

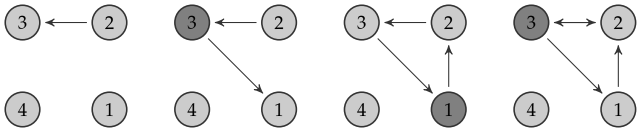
$$\operatorname{sgn}((1.33)) = \begin{cases} \operatorname{sgn}\left(y_y^\star(G) - \dfrac{\alpha_x}{\beta}\right) & \text{if } \deg_G^+(x) = 0, \\ \operatorname{sgn}\left(y_y^\star(G) - \dfrac{\sum_{i\in\mathcal{N}_G^+(x)} y_i^\star(G)}{\deg_G^+(x)}\right) & \text{if } \deg_G^+(x) > 0. \end{cases}$$

This result allows to predict the direction of the change in player x's action that results from forming a new arc to player y. First, consider the case where player x has no out-neighbors in G. If his action is less (respectively, greater) than player y's action, then he will increase (respectively, decrease) his action. Second, consider the case where player x has at least one out-neighbor in G. If the average action of his current out-neighbors, that is, his out-neighbors in G, is less (respectively, greater) than the action of his new out-neighbor y, then he will increase (respectively, decrease) his action. In both cases, all players who are affected by the extra arc will adjust their actions in the direction of player x's adjustment; the magnitudes of their adjustments are thereby smaller than the magnitude of player x's adjustment if the common social cost parameter is less than the common private cost parameter (Result 1.26.6). As regards the second case, the adjustment in player x's action does not necessarily imply that the social distance between player x and his out-neighbors, that is, the distance between his action and the average action of his out-neighbors, decreases. This can be seen from (1.32), which describes the change in the difference between a player's action and the average action of his out-neighbors, provided that his out-neighborhood in G is not empty.[54] Suppose player $i \in \mathcal{I}$ is affected by the extra arc from player x to player y and player i's out-neighborhood in G is not empty. According to (1.32), if the difference between player i's action and the average action of his out-neighbors in G is positive (respectively, negative) and the extra arc causes player x to decrease (respectively, increase) his action, then the

54. If a player has an empty out-neighborhood in G, then the corresponding equation of the system (1.32) does not describe the change in the difference between his action and the average action of his out-neighbors that results from adding an extra arc to G. This can be seen as follows. Suppose $\gamma > 0$ and $f = \mathrm{id}_{\mathbb{R}_+}$, and let $i \in \mathcal{I}$ with $\mathcal{N}_G^+(i) = \varnothing$. If $i = x$, then $[\bar{\boldsymbol{A}}(G)\boldsymbol{y}^\star(G)]_i = y_x^\star(G)$, $[\bar{\boldsymbol{A}}(G \boxplus (x,y))\boldsymbol{y}^\star(G \boxplus (x,y))]_i = y_y^\star(G \boxplus (x,y))$ (because $\mathcal{N}_G^+(x) = \varnothing$ implies that $\mathcal{N}_{G\boxplus(x,y)}^+(x) = \{y\}$), $[\boldsymbol{j}_x(\beta,\gamma,G)]_i = 1/\beta$ (Result 1.26.7), $\langle \boldsymbol{\Delta}(x,y,G), \boldsymbol{j}_x(\beta,\gamma,G)\rangle = [\boldsymbol{j}_x(\beta,\gamma,G)]_y - 1/\beta$, and $\langle \boldsymbol{\Delta}(x,y,G), \boldsymbol{y}^\star(G)\rangle = y_y^\star(G) - y_x^\star(G)$, so that the ith equation of the system (1.32) is

$$y_x^\star(G \boxplus (x,y)) - y_y^\star(G \boxplus (x,y)) = \frac{1}{1-\gamma\left([\boldsymbol{j}_x(\beta,\gamma,G)]_y - 1/\beta\right)}\left(y_x^\star(G) - y_y^\star(G)\right).$$

If $i \neq x$, then $[\bar{\boldsymbol{A}}(G)\boldsymbol{y}^\star(G)]_i = y_i^\star(G)$, $[\bar{\boldsymbol{A}}(G \boxplus (x,y))\boldsymbol{y}^\star(G \boxplus (x,y))]_i = y_i^\star(G \boxplus (x,y))$ (because $i \neq x$ and $\mathcal{N}_G^+(i) = \varnothing$ imply that $\mathcal{N}_{G\boxplus(x,y)}^+(i) = \varnothing$), and $[\boldsymbol{j}_x(\beta,\gamma,G)]_i = 0$ (Results 1.26.3 and 1.26.5), so that the ith equation of the system (1.32) is $0 = 0$, a tautology.



(a) G_1 (b) G_2 (c) G_3 (d) G_4

(e) G_5 (f) G_6 (g) G_7 (h) G_8

Figure 1.11. A finite sequence of digraphs of order 4 (Example 1.28)

social distance between player i and his out-neighbors increases. If the difference between player i's action and the average action of his out-neighbors in G is positive (respectively, negative) and the extra arc causes player x to increase (respectively, decrease) his action, then the social distance between player i and his out-neighbors may decrease (see Example 1.28, in particular, the discussion of (1.34) and (1.35) that is related to player 3) or increase (see Example 1.28, in particular, the discussion of (1.34) and (1.35) that is related to player 1). To sum up, adding an arc to the digraph by which the players are connected does not necessarily imply that the affected players conform more to the average behavior of their out-neighbors—in the sense that the social distances between the players and their out-neighbors decrease—than before the addition of the arc.

Proposition 1.27 and the subsequent discussion are illustrated in Example 1.28. Besides this, Example 1.28 demonstrates that the NE of a (generic) NALA game is devoid of the following three comparative statics properties: in general, a player's action, the social distance between a player and his out-neighbors, and the players' aggregate action are monotonic in the number of arcs or, equivalently, in the density of the digraph by which the players are connected.[55]

Example 1.28 Suppose $\mathcal{I} = \{1, 2, 3, 4\}$, $\mathcal{A}(G) = \{(2, 3)\}$, $\mathcal{Y} = \mathbb{R}_+$, $\beta = 1$, $\gamma = 1/3$, and $f = \mathrm{id}_{\mathbb{R}_+}$. Let $(\bar{\alpha}, \epsilon) \in \mathbb{R}_{++} \times \mathbb{R} \setminus \{0\}$ be such that $\bar{\alpha} > \max\{-3\epsilon, \epsilon\}$. In addition, suppose $\alpha_1 = \alpha_3 = \bar{\alpha}$, $\alpha_2 = \bar{\alpha} - \epsilon$, and $\alpha_4 = \bar{\alpha} + 3\epsilon$. Evidently, $\{\alpha_i \mid i \in \mathcal{I}\} \subset \mathbb{R}_{++}$. Let $G_1 := G$, $G_2 := G_1 \boxplus (3, 1)$, $G_3 := G_2 \boxplus (1, 2)$, $G_4 := G_3 \boxplus (3, 2)$, $G_5 := G_4 \boxplus (1, 3)$, $G_6 := G_5 \boxplus (2, 1)$, $G_7 := G_6 \boxplus (1, 4)$, and $G_8 := G_7 \boxplus (4, 1)$. Note that, for all $g \in \{1, \ldots, 8\}$, $|\mathcal{A}(G_g)| = g$, in other words, the subscript g in G_g is equal to the size of G_g, that is, the number of arcs of G_g. The finite sequence of digraphs $(G_1, \ldots, G_8)$ may be interpreted as the evolution of the digraph G over eight periods of time: the digraph G is given by G_1 in period 1; in period 2, player 3 forms an arc to player 1, which results in the digraph G_2; in

55. The density of a digraph of order $N > 1$ is defined as the ratio of the number of its arcs to the maximum number of its arcs, $N(N-1)$.

period 3, player 1 forms an arc to player 2, which results in the digraph G_3; in period 4, player 3 forms an arc to player 2, which results in the digraph G_4; finally, in period 8, player 4 forms an arc to player 1, which results in the digraph G_8. See Figure 1.11 for an illustration of the digraphs G_1 to G_8, wherein the players who form an arc are depicted with a disk in dark gray. For all $g \in \{1, \dots, 8\}$, let $\boldsymbol{y}^\star(G_g)$ denote the unique and interior NE of the NALA game $\Gamma(G_g) := (\mathcal{I}, G_g, \mathcal{Y}, \{(\alpha_i, \beta, \gamma)\}_{i \in \mathcal{I}}, f)$. Straightforward calculations yield

$$\boldsymbol{y}^\star(G_1) = \boldsymbol{y}^\star(G_2) = \bar{\alpha}\mathbf{1}_4 + \frac{1}{4}\epsilon \begin{pmatrix} 0 \\ -3 \\ 0 \\ 12 \end{pmatrix}, \quad \boldsymbol{y}^\star(G_3) = \boldsymbol{y}^\star(G_2) - \frac{1}{84}\epsilon \begin{pmatrix} 16 \\ 1 \\ 4 \\ 0 \end{pmatrix},$$

$$\boldsymbol{y}^\star(G_4) = \boldsymbol{y}^\star(G_3) - \frac{4}{861}\epsilon \begin{pmatrix} 1 \\ 4 \\ 16 \\ 0 \end{pmatrix}, \quad \boldsymbol{y}^\star(G_5) = \boldsymbol{y}^\star(G_4) + \frac{1}{369}\epsilon \begin{pmatrix} 31 \\ 1 \\ 4 \\ 0 \end{pmatrix},$$

$$\boldsymbol{y}^\star(G_6) = \boldsymbol{y}^\star(G_5), \quad \boldsymbol{y}^\star(G_7) = \boldsymbol{y}^\star(G_6) + \frac{31}{738}\epsilon \begin{pmatrix} 7 \\ 1 \\ 1 \\ 0 \end{pmatrix},$$

and

$$\boldsymbol{y}^\star(G_8) = \boldsymbol{y}^\star(G_7) - \frac{77}{8{,}774}\epsilon \begin{pmatrix} 7 \\ 1 \\ 1 \\ 82 \end{pmatrix}.$$

Thus, if $\epsilon > 0$, then $\boldsymbol{y}^\star(G_1) = \boldsymbol{y}^\star(G_2) \geq_c \boldsymbol{y}^\star(G_3) \geq_c \boldsymbol{y}^\star(G_4) \leq_c \boldsymbol{y}^\star(G_5) = \boldsymbol{y}^\star(G_6) \leq_c \boldsymbol{y}^\star(G_7) >_c \boldsymbol{y}^\star(G_8)$, so that the mapping $g \mapsto \boldsymbol{y}^\star(G_g)$ is not monotonic. See Figure 1.12 for an illustration, wherein the actions of the players who form an arc are depicted with a marker in dark gray. In addition, we find

$$\boldsymbol{y}^\star(G_3) - \bar{\boldsymbol{A}}(G_3)\boldsymbol{y}^\star(G_3) = \frac{1}{287}\epsilon \begin{pmatrix} 164 \\ -205 \\ 41 \\ 0 \end{pmatrix}$$

and

$$\boldsymbol{y}^\star(G_4) - \bar{\boldsymbol{A}}(G_4)\boldsymbol{y}^\star(G_4) = \frac{1}{287}\epsilon \begin{pmatrix} 168 \\ -189 \\ 105 \\ 0 \end{pmatrix}.$$

The difference between player 3's action and the average action of his out-neighbors is equal to $(41/287)\epsilon$ in $\Gamma(G_3)$, and therefore positive (respectively, negative) if

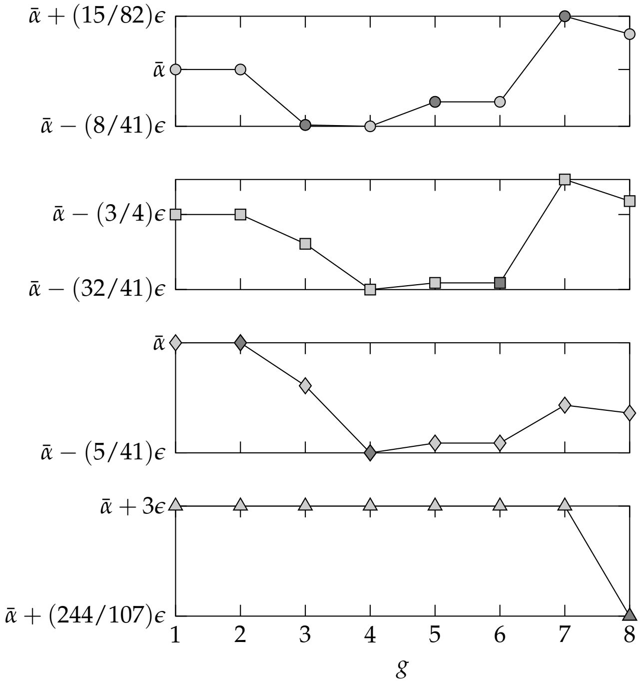


Figure 1.12. Actions of players 1 (—○—), 2 (—□—), 3 (—◇—), and 4 (—△—) in $\Gamma(G_g)$ for $g \in \{1, \ldots, 8\}$ if $\epsilon > 0$ (Example 1.28)

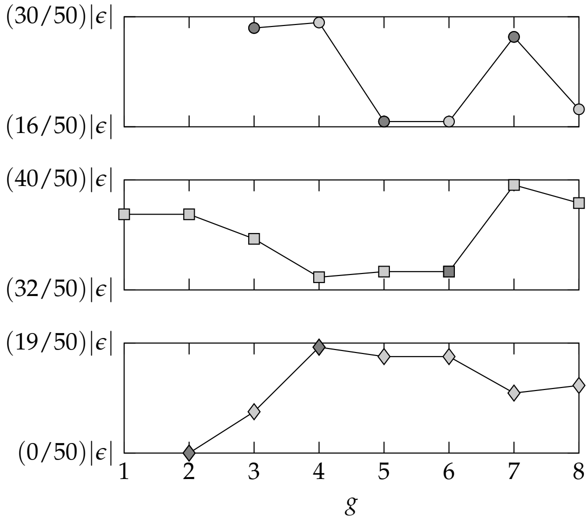


Figure 1.13. Social distances of players 1 (—○—), 2 (—□—), and 3 (—◇—) to their out-neighbors in $\Gamma(G_g)$ for $g \in \{1, \ldots, 8\}$ (Example 1.28)

$\epsilon > 0$ (respectively, $\epsilon < 0$). Adding $(3,2)$ to G_3 causes player 3 to decrease (respectively, increase) his action by $(64/861)\epsilon$ if $\epsilon > 0$ (respectively, $\epsilon < 0$) because

$$\operatorname{sgn}\left(y_2^\star(G_3) - \frac{\sum_{i\in\mathcal{N}^+_{G_3}(3)} y_i^\star(G_3)}{\deg^+_{G_3}(3)}\right) = \operatorname{sgn}\big(y_2^\star(G_3) - y_1^\star(G_3)\big) = \operatorname{sgn}(-\epsilon).$$

Consequently, the social distance between player 3 and his out-neighbors increases from $(41/287)|\epsilon|$ to $(105/287)|\epsilon|$, which is in accordance with (1.32). As regards players 1 and 2, adding $(3,2)$ to G_3 increases the social distance between player 1 and his out-neighbors, whereas the corresponding distance decreases for player 2. Finally, we find

$$\boldsymbol{y}^\star(G_6) - \bar{\boldsymbol{A}}(G_6)\boldsymbol{y}^\star(G_6) = \frac{1}{246}\epsilon\begin{pmatrix} 82 \\ -164 \\ 82 \\ 0 \end{pmatrix} \tag{1.34}$$

and

$$\boldsymbol{y}^\star(G_7) - \bar{\boldsymbol{A}}(G_7)\boldsymbol{y}^\star(G_7) = \frac{1}{246}\epsilon\begin{pmatrix} -135 \\ -195 \\ 51 \\ 0 \end{pmatrix}. \tag{1.35}$$

The difference between player 1's action and the average action of his out-neighbors is equal to $(82/246)\epsilon$ in $\Gamma(G_6)$, and therefore positive (respectively, negative) if $\epsilon > 0$ (respectively, $\epsilon < 0$). Adding $(1,4)$ to G_6 causes player 1 to increase (respectively, decrease) his action by $(217/738)\epsilon$ if $\epsilon > 0$ (respectively, $\epsilon < 0$). As a consequence, the difference between player 1's action and the average action of his out-neighbors *decreases* (respectively, increases) from $(82/246)\epsilon$ to $-(135/246)\epsilon$ if $\epsilon > 0$ (respectively, $\epsilon < 0$), so that the social distance between player 1 and his out-neighbors *increases* from $(82/246)|\epsilon|$ to $(135/246)|\epsilon|$. As regards player 3, adding $(1,4)$ to G_6 decreases his social distance to his out-neighbors from $(82/246)|\epsilon|$ to $(51/246)|\epsilon|$. See Figure 1.13 for an illustration of the social distances between players 1, 2, and 3 and their out-neighbors in $\Gamma(G_g)$ for $g \in \{1,\ldots,8\}$, wherein the distances of the players who form an arc are depicted with a marker in dark gray.$\diamond$

1.3.4.2 Weakly connected players

The focus of this section is on the actions of weakly connected players. A discussion of their properties requires an additional assumption and some extra notation, which I introduce next.

In what follows, suppose Condition G is satisfied. Let G_κ be a weakly connected component of G of order $n_\kappa > 1$, and let $\mathcal{I}_\kappa := \mathcal{V}(G_\kappa)$, that is, $\mathcal{I}_\kappa$ represents the set of weakly connected players.[56] The component G_κ induces the generic NALA

56. The existence of a weakly connected component of G of order larger than one follows from Condition G.

game $\Gamma(G_\kappa) := (\mathcal{I}_\kappa, G_\kappa, \mathcal{Y}, \{(\alpha_i, \beta, \gamma)\}_{i\in\mathcal{I}_\kappa}, f)$. Let $\alpha_{\min,\mathcal{I}_\kappa} := \min\{\alpha_i \mid i \in \mathcal{I}_\kappa\}$ and $\alpha_{\max,\mathcal{I}_\kappa} := \max\{\alpha_i \mid i \in \mathcal{I}_\kappa\}$. Let $h\colon \mathcal{I}_\kappa \to \{1,\dots,n_\kappa\}$ be the unique order isomorphism. Finally, let $\boldsymbol{\alpha}_{\mathcal{I}_\kappa} := (\alpha_{h^{-1}(1)},\dots,\alpha_{h^{-1}(n_\kappa)})$ and $\boldsymbol{y}^\star_{\mathcal{I}_\kappa} := (y^\star_{h^{-1}(1)},\dots,y^\star_{h^{-1}(n_\kappa)})$, that is, $\boldsymbol{\alpha}_{\mathcal{I}_\kappa}$ (respectively, $\boldsymbol{y}^\star_{\mathcal{I}_\kappa}$) is the vector that lies in the rows of $\boldsymbol{\alpha}$ (respectively, $\boldsymbol{y}^\star$, the NE of $\Gamma(G)$) indexed by $\mathcal{I}_\kappa = \{h^{-1}(1),\dots,h^{-1}(n_\kappa)\}$.

Most results of this section involve equalities or inequalities that are stated in terms of $\boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa})$. Since f is surjective and strictly increasing (Assumption F), they can be written equivalently in terms of $\boldsymbol{y}^\star_{\mathcal{I}_\kappa}$. For example, if $\mathfrak{R}$ is one of the binary relations $=$, $\leq$, $<$, $\geq$, or $>$ and $\boldsymbol{\mathfrak{R}}$ is the corresponding binary relation on $\mathcal{M}(n_\kappa, 1, \mathbb{R}) \cong \mathbb{R}^{n_\kappa}$, $c \in f(\operatorname{int}(\mathcal{Y}))$, and $\boldsymbol{c} := (c_1,\dots,c_{n_\kappa}) \in f(\operatorname{int}(\mathcal{Y}))^{n_\kappa}$, then $\boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa})\ \boldsymbol{\mathfrak{R}}\ c\mathbf{1}_{n_\kappa}$ is equivalent to $\boldsymbol{y}^\star_{\mathcal{I}_\kappa}\ \boldsymbol{\mathfrak{R}}\ f^{-1}(c)\mathbf{1}_{n_\kappa}$ and $\boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa})\ \boldsymbol{\mathfrak{R}}\ \boldsymbol{c}$ is equivalent to, for all $i \in \{1,\dots,n_\kappa\}$, $y^\star_{h^{-1}(i)}\ \mathfrak{R}\ f^{-1}(c_i)$.

I discuss five results for $\boldsymbol{y}^\star_{\mathcal{I}_\kappa}$. The first result (Proposition 1.29) relates $\boldsymbol{y}^\star_{\mathcal{I}_\kappa}$ to $\Gamma(G_\kappa)$, thereby providing a characterization of $\boldsymbol{y}^\star_{\mathcal{I}_\kappa}$. The second result (Proposition 1.31) establishes a lower and an upper bound on $\boldsymbol{y}^\star_{\mathcal{I}_\kappa}$ in terms of the players' idiosyncrasies and shows that similar players play similar actions. The third result (Proposition 1.33) states some additional lower bounds for $\boldsymbol{y}^\star_{\mathcal{I}_\kappa}$ in terms of $\boldsymbol{\alpha}_{\mathcal{I}_\kappa}$. The fourth result (Proposition 1.34) provides a necessary and a sufficient condition for a symmetric $\boldsymbol{y}^\star_{\mathcal{I}_\kappa}$.[57] The fifth result (Proposition 1.35) is about the actions of weakly ex ante homogeneous players.

Proposition 1.29 *The action profile $\boldsymbol{y}^\star_{\mathcal{I}_\kappa}$ is the unique and interior NE of $\Gamma(G_\kappa)$ and satisfies*

$$\boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}) = \frac{1}{\beta+\gamma}\left(\boldsymbol{I}_{n_\kappa} - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G_\kappa)\right)^{-1}\boldsymbol{\alpha}_{\mathcal{I}_\kappa}. \tag{1.36}$$

A comparison of (1.36) and (1.7) reveals the common structural form of the systems of equations that govern the actions of weakly connected players and the actions of the entire set of players. Besides this, Proposition 1.29 shows that a player's action is functionally independent of the action played by any player who is not located in the same weakly connected component.

A statement similar to Proposition 1.29 is in general not true for a strongly connected component of G. This is illustrated in the following example.

Example 1.30 Let $\bar{\alpha} > 0$ and $\epsilon > 0$. Suppose $\mathcal{I} = \{1,2,3,4\}$, $\mathcal{A}(G) = \{(1,2),(1,4),(2,3),(3,1)\}$, $\mathcal{Y} = \mathbb{R}_+$, $\alpha_1 = \alpha_2 = \alpha_3 = \bar{\alpha}$, $\alpha_4 = \bar{\alpha}+\epsilon$, $\beta = 1$, $\gamma = 1/2$, and $f = \mathrm{id}_{\mathbb{R}_+}$. The digraph G has one weakly connected component, namely, G, and two strongly connected components, namely, $G_s := (\mathcal{I}_s, \mathcal{A}_s)$ and $(\{4\},\emptyset)$, where $\mathcal{I}_s := \{1,2,3\}$ and $\mathcal{A}_s := \mathcal{A}(G) \setminus \{(1,4)\}$. See Figure 1.14 (a) for an illustration of G and Figure 1.14 (b) for an illustration of G_s. The component G_s induces the NALA game $\Gamma(G_s) := (\mathcal{I}_s, G_s, \mathcal{Y}, \{(\alpha_i,\beta,\gamma)\}_{i\in\mathcal{I}_s}, f)$. Both NALA games $\Gamma(G)$ and $\Gamma(G_s)$ satisfy the sufficient conditions for the existence of a unique and interior NE (Proposition 1.16). Let $\boldsymbol{y}^\star(G) = (y^\star_1(G), y^\star_2(G), y^\star_3(G), y^\star_4(G))$ and $\boldsymbol{y}^\star(G_s)$ denote

57. The action profile $\boldsymbol{y}^\star_{\mathcal{I}_\kappa}$ is called symmetric if $\boldsymbol{y}^\star_{\mathcal{I}_\kappa} = \bar{y}\mathbf{1}_{n_\kappa}$ for some $\bar{y} \in \operatorname{int}(\mathcal{Y})$.

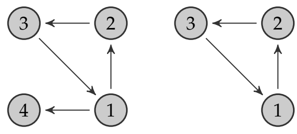


Figure 1.14. A digraph of order 4 (panel (a)) with a strongly connected component of order 3 (panel (b)) (Example 1.30)

the NEs of $\Gamma(G)$ and $\Gamma(G_s)$, respectively. In accordance with the definition of $\boldsymbol{y}^\star_{\mathcal{I}_\kappa}$, let $\boldsymbol{y}^\star_{\mathcal{I}_s} := (y^\star_1(G), y^\star_2(G), y^\star_3(G))$. Straightforward calculations yield

$$\boldsymbol{y}^\star(G) = \bar{\alpha}\mathbf{1}_4 + \frac{1}{53}\epsilon \begin{pmatrix} 9 \\ 1 \\ 3 \\ 53 \end{pmatrix} \quad \text{and} \quad \boldsymbol{y}^\star(G_s) = \bar{\alpha}\mathbf{1}_3,$$

from which

$$\boldsymbol{y}^\star_{\mathcal{I}_s} = \bar{\alpha}\mathbf{1}_3 + \frac{1}{53}\epsilon \begin{pmatrix} 9 \\ 1 \\ 3 \end{pmatrix} \neq \bar{\alpha}\mathbf{1}_3 = \boldsymbol{y}^\star(G_s).$$

follows. This shows that Proposition 1.29 is not true for G_s. ◇

Proposition 1.31 *(1.31.1) If $-\beta/2 < \gamma < 0$, then*

$$\frac{1}{\beta}\left(\alpha_{\min,\mathcal{I}_\kappa} - \frac{|\gamma|}{\beta+2\gamma}(\alpha_{\max,\mathcal{I}_\kappa} - \alpha_{\min,\mathcal{I}_\kappa})\right)\mathbf{1}_{n_\kappa} \leq_c \boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa})$$
$$\leq_c \frac{1}{\beta}\left(\alpha_{\max,\mathcal{I}_\kappa} + \frac{|\gamma|}{\beta+2\gamma}(\alpha_{\max,\mathcal{I}_\kappa} - \alpha_{\min,\mathcal{I}_\kappa})\right)\mathbf{1}_{n_\kappa}. \quad (1.37)$$

If Conditions 1.15.1 to 1.15.4 are satisfied, then the lower bound of $\boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa})$ is greater than $f(0)\mathbf{1}_{n_\kappa}$ and the upper bound of $\boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa})$ is less than $f(\bar{v})\mathbf{1}_{n_\kappa}$.

(1.31.2) If $\gamma = 0$, then $\boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}) = (1/\beta)\boldsymbol{\alpha}_{\mathcal{I}_\kappa}$.

(1.31.3) If $\gamma > 0$, then $(\alpha_{\min,\mathcal{I}_\kappa}/\beta)\mathbf{1}_{n_\kappa} \leq_c \boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}) \leq_c (\alpha_{\max,\mathcal{I}_\kappa}/\beta)\mathbf{1}_{n_\kappa}$.

Proposition 1.31 establishes a lower and an upper bound on the players' actions in terms of their idiosyncrasies. The two bounds coincide if $\gamma = 0$ (Result 1.31.2). A positive γ entails a higher lower bound and a lower upper bound on the players' actions than does a negative γ, provided that the players of $\Gamma(G_\kappa)$ are ex ante heterogeneous, in which case $\alpha_{\min,\mathcal{I}_\kappa} < \alpha_{\max,\mathcal{I}_\kappa}$.[58] The maximum possible range

58. Note that the sufficient conditions for the existence of a unique and interior NE of $\Gamma(G_\kappa)$ are stronger for a negative γ than for a positive γ with regard to the range of the players' idiosyncrasies in case $\mathcal{Y} = [0, \bar{v}]$; specifically, $\alpha_{\max} - \alpha_{\min} < (\beta + 2\gamma)(f(\bar{v}) - f(0)) < \beta(f(\bar{v}) - f(0))$ if $\gamma < 0$ and $\alpha_{\max} - \alpha_{\min} < \beta(f(\bar{v}) - f(0))$ if $\gamma \geq 0$.

(which is a measure of potential variation) of the players' actions for a positive γ is therefore smaller than for a negative γ. This result is in accordance with Assumption U. A positive γ corresponds to the case of conformist behavior where the players have an incentive to conform to their out-neighbors' actions, whereas a negative γ corresponds to the case of anti-conformist behavior where the players have an incentive to differentiate themselves from their out-neighbors' actions. It is therefore to be expected that a negative γ entails a greater variation in the players' actions than does a positive γ. Both Results 1.31.1 and 1.31.3 imply that similar players behave similarly. Strictly speaking, if weakly connected players are alike in that the range of their idiosyncrasies is small, then they play similar actions, where the range of their actions is deceasing in the range of their idiosyncrasies.

If all players of $\Gamma(G_\kappa)$ had no out-neighbors in G_κ, a hypothesis that is in conflict with the assumption that G_κ is a weakly connected component of G of order $n_\kappa > 1$, then $\boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}) = (1/\beta)\boldsymbol{\alpha}_{\mathcal{I}_\kappa}$, irrespective of the value of γ.[59] It follows that the players of $\Gamma(G_\kappa)$ act as if they had no out-neighbors if $\gamma = 0$ (Result 1.31.2). If $\gamma \neq 0$, then neither $(1/\beta)\boldsymbol{\alpha}_{\mathcal{I}_\kappa} \leq_c \boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa})$ nor $\boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}) \leq_c (1/\beta)\boldsymbol{\alpha}_{\mathcal{I}_\kappa}$ is true in general; specifically, not all players do necessarily play an action that is larger than the action they would play in isolation or in case of an empty out-neighborhood. This is illustrated in Example 1.32.

Example 1.32 Suppose G_κ is of order $n_\kappa = 3$ with vertex set $\mathcal{I}_\kappa = \{1,2,3\}$ and arc set $\mathcal{A}(G_\kappa) = \{(1,2),(1,3),(2,1),(3,1)\}$. See Figure 1.9 for an illustration of G_κ. In addition, suppose $\mathcal{Y} = \mathbb{R}_+$, $\alpha_1 = 3/4$, $\alpha_2 = 5/7$, $\alpha_3 = 4/7$, $\beta = 1$, $\gamma > 0$, and $f = \mathrm{id}_{\mathbb{R}_+}$. Straightforward calculations yield (see also Example 1.24)

$$\boldsymbol{y}^\star_{\mathcal{I}_\kappa} = \boldsymbol{\alpha}_{\mathcal{I}_\kappa} + \frac{1}{28}\frac{\gamma}{(\gamma+1)(2\gamma+1)}\begin{pmatrix} -3\gamma-3 \\ -\gamma+1 \\ 7\gamma+5 \end{pmatrix},$$

from which it follows that the inequality $\boldsymbol{\alpha}_{\mathcal{I}_\kappa} \leq_c \boldsymbol{y}^\star_{\mathcal{I}_\kappa}$ is not true. ◇

A few inequalities involving $\boldsymbol{\alpha}_{\mathcal{I}_\kappa}$ and $\boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa})$, including $(1/\beta)\boldsymbol{\alpha}_{\mathcal{I}_\kappa} \leq_c \boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa})$, are true under certain conditions, which are stated in the following result.

Proposition 1.33 *(1.33.1) If $\gamma > 0$, then $\boldsymbol{0}_{n_\kappa} \leq_c \boldsymbol{\alpha}_{\mathcal{I}_\kappa}$ is sufficient for $1/(\beta+\gamma)\boldsymbol{\alpha}_{\mathcal{I}_\kappa} \leq_c \boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa})$ and $\boldsymbol{0}_{n_\kappa} <_c \boldsymbol{\alpha}_{\mathcal{I}_\kappa}$ is sufficient for $1/(\beta+\gamma)\boldsymbol{\alpha}_{\mathcal{I}_\kappa} <_c \boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa})$.*[60]

(1.33.2) If $\gamma \neq 0$, then $\mathrm{sgn}(\gamma)\boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}) \leq_c \mathrm{sgn}(\gamma)\bar{\boldsymbol{A}}(G_\kappa)\boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa})$ is necessary and sufficient for $(1/\beta)\boldsymbol{\alpha}_{\mathcal{I}_\kappa} \leq_c \boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa})$ and $\mathrm{sgn}(\gamma)\boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}) <_c \mathrm{sgn}(\gamma)\bar{\boldsymbol{A}}(G_\kappa)\boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa})$ is necessary and sufficient for $(1/\beta)\boldsymbol{\alpha}_{\mathcal{I}_\kappa} <_c \boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa})$.

Proposition 1.34 *(1.34.1) If $\boldsymbol{\alpha}_{\mathcal{I}_\kappa} = \bar{\alpha}\boldsymbol{1}_{n_\kappa}$ for some $\bar{\alpha} \in \mathbb{R}$, then $\boldsymbol{y}^\star_{\mathcal{I}_\kappa} = f^{-1}(\bar{\alpha}/\beta)\boldsymbol{1}_{n_\kappa}$.*

(1.34.2) If $\boldsymbol{y}^\star_{\mathcal{I}_\kappa} = \bar{y}\boldsymbol{1}_{n_\kappa}$ for some $\bar{y} \in \mathrm{int}(\mathcal{Y})$, then $\boldsymbol{\alpha}_{\mathcal{I}_\kappa} = \beta f(\bar{y})\boldsymbol{1}_{n_\kappa}$.

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59. If all players of $\Gamma(G_\kappa)$ had no out-neighbors in G_κ, then $\bar{\boldsymbol{A}}(G_\kappa) = \boldsymbol{I}_{n_\kappa}$, so that $\boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}) = (1/\beta)\boldsymbol{\alpha}_{\mathcal{I}_\kappa}$ according to (1.36).

60. Note that $1/(\beta+\gamma)\boldsymbol{\alpha}_{\mathcal{I}_\kappa} <_c (1/\beta)\boldsymbol{\alpha}_{\mathcal{I}_\kappa}$ if $\beta > 0$ and $\gamma > 0$.

Result 1.34.1 gives a sufficient condition for a symmetric NE: if the players of $\Gamma(G_\kappa)$ are strongly ex ante homogeneous, then they play the same action.[61] Remarkably enough, this result is independent of the topology of G_κ. In particular, different positions within G_κ, for example, in terms of in-degree and out-degree, do not imply different actions, provided that the players of $\Gamma(G_\kappa)$ are strongly ex ante homogeneous. Result 1.34.2 gives a necessary condition for a symmetric NE, thereby complementing Result 1.34.1: if the players of $\Gamma(G_\kappa)$ play the same action, then they must be strongly ex ante homogeneous. Taken together, Results 1.34.1 and 1.34.2 show that the variation in the players' idiosyncrasies is the primary source of the variation in the players' actions. It is for this fact that the notion of ex ante homogeneity has been defined with respect to the players' idiosyncrasies only, thereby ignoring the players' positions within the digraph.

Proposition 1.35 *(1.35.1) If $\boldsymbol{\alpha}_{\mathcal{I}_\kappa} = \bar{\boldsymbol{A}}(G_\kappa)\boldsymbol{\alpha}_{\mathcal{I}_\kappa}$, then $\boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}) = (1/\beta)\boldsymbol{\alpha}_{\mathcal{I}_\kappa}$.*

(1.35.2) If $\gamma \neq 0$ and $\boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}) = (1/\beta)\boldsymbol{\alpha}_{\mathcal{I}_\kappa}$, then $\boldsymbol{\alpha}_{\mathcal{I}_\kappa} = \bar{\boldsymbol{A}}(G_\kappa)\boldsymbol{\alpha}_{\mathcal{I}_\kappa}$.

If the players of $\Gamma(G_\kappa)$ are weakly ex ante homogeneous, then they act as if they had no out-neighbors (Result 1.35.1). In this case, the preference for conformist or anti-conformist behavior has no bearing on equilibrium actions. If the players of $\Gamma(G_\kappa)$ have a preference for either conformist or anti-conformist behavior, that is, $\gamma \neq 0$, and they act as if they had no out-neighbors, then they must be weakly ex ante homogeneous (Result 1.35.2). In summary, weakly connected players act as if they had no out-neighbors if and only if they are weakly ex ante homogeneous, provided that $\gamma \neq 0$.

An algebraic interpretation of Result 1.35.1 reads as follows: Suppose $f = \mathrm{id}_{\mathcal{Y}}$ and the players of $\Gamma(G_\kappa)$ are weakly ex ante homogeneous. It follows that

$$\forall i \in \mathcal{I}_\kappa \quad y_i^\star = \frac{1}{\beta}\alpha_i = \frac{1}{\beta}\frac{\sum_{j\in\mathcal{N}_G^+(i)}\alpha_j}{\deg_G^+(i)} = \frac{\sum_{j\in\mathcal{N}_G^+(i)} y_j^\star}{\deg_G^+(i)},$$

that is, each player's action is equal to the average action of his out-neighbors.[62] In summary, if weakly connected players are similar to their out-neighbors in that their marginal private benefits are equal to the average of their out-neighbors' marginal private benefits, then they act like the average of their out-neighbors.

Remark 1.36 Propositions 1.31, 1.33, 1.34, and 1.35 are true also for $\boldsymbol{y}^\star$, with the obvious changes in notation.

1.3.5 Welfare analysis

In this section, I adopt the perspective of a benevolent social planner who acts on behalf of the players of a NALA game. The planner is benevolent in that his

61. An analogous result has been established for Patacchini and Zenou's (2012) local average game (see Proposition 1).

62. Note that, for all $i \in \mathcal{I}_\kappa$, $\mathcal{N}^+_{G_\kappa}(i) = \mathcal{N}^+_G(i)$ and $\deg^+_{G_\kappa}(i) = \deg^+_G(i)$, because G_κ is a weakly connected component of G.

sole objective is to maximize social welfare defined as players' total utility. This assumption necessitates that every single aspect of the NALA game in question is known to the planner. In particular, the planner must have information on the players' preference parameters, the function f, and the digraph by which the players are connected.

The subsequent analysis calls for some additional notation and terminology. Let $\Gamma := (\mathcal{I}, G, \mathcal{Y}, \{(\alpha_i, \beta, \gamma)\}_{i\in\mathcal{I}}, f)$ be a generic NALA game, $\boldsymbol{\alpha}$ be defined as in Section 1.3.2, and

$$\boldsymbol{S}(G) := \big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)^{\mathsf{T}}\big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big). \tag{1.38}$$

Some elementary properties of $\boldsymbol{S}(G)$ are given in Lemma 1.37. The planner's social welfare function $w\colon \mathcal{Y}^n \to \mathbb{R}$ is defined by $w(\boldsymbol{y}) := \sum_{i\in\mathcal{I}} u_i(\boldsymbol{y})$ and satisfies $w(\boldsymbol{y}) = \langle \boldsymbol{\alpha}, \boldsymbol{f}(\boldsymbol{y})\rangle - (1/2)\langle \boldsymbol{f}(\boldsymbol{y}), (\beta\boldsymbol{I}_n + \gamma\boldsymbol{S}(G))\boldsymbol{f}(\boldsymbol{y})\rangle$. A global maximum point of w is called a *social optimum* of Γ. The players' actions that maximize w are called *socially optimal* actions.

Lemma 1.37 *The matrix $\boldsymbol{S}(G)$ has the following properties:*

(1.37.1) *If Condition G is satisfied, then $\boldsymbol{S}(G)$ is different from $\boldsymbol{O}_n$, symmetric, and non-negative definite with $0 < \rho(\boldsymbol{S}(G)) \leq 2M(G)$, where*[63]

$$M(G) := \max\left\{ \mathbb{1}_{\mathcal{I}\setminus\mathcal{I}_0^+(G)}(j) + \frac{\deg_G^-(j)}{\min\{\deg_G^+(i) \mid i \in \mathcal{I}\setminus\mathcal{I}_0^+(G)\}} \,\middle|\, j \in \mathcal{I} \right\}.$$

(1.37.2) *If Condition G is not satisfied, then $\boldsymbol{S}(G) = \boldsymbol{O}_n$, so that $\rho(\boldsymbol{S}(G)) = 0$.*

The welfare analysis presented hereinafter is confined to the case where Γ has a unique and interior social optimum for it admits of a closed-form characterization of socially optimal actions, which is the subject of the following result.

Proposition 1.38 *If Γ has a unique and interior social optimum $\boldsymbol{y}_S^\star$, then it is given by*

$$\boldsymbol{f}(\boldsymbol{y}_S^\star) = \big(\beta\boldsymbol{I}_n + \gamma\boldsymbol{S}(G)\big)^{-1}\boldsymbol{\alpha}. \tag{1.39}$$

The remainder of this section is structured as follows. The analysis leads off with the statement of sufficient conditions for the existence of a unique and interior social optimum of Γ, followed by a discussion (Section 1.3.5.1). It continues with an exposition of basic properties of socially optimal actions (Section 1.3.5.2). Following this, socially optimal actions are compared to NE actions (Section 1.3.5.3). The analysis concludes with a result that parallels the second fundamental theorem of welfare economics, namely, a social optimum can be decentralized as a NE (Section 1.3.5.4).

63. Condition G implies that $\min\{\deg_G^+(i) \mid i \in \mathcal{I}\setminus\mathcal{I}_0^+(G)\} > 0$.

1.3.5.1 Existence and uniqueness of interior social optima

The focus of this section is on sufficient conditions for the existence of a unique and interior social optimum of Γ. A compact statement of the conditions requires some extra notation: Let $\bar{\alpha}_{\mathcal{I}} := (1/n)\sum_{i\in\mathcal{I}}\alpha_i$ and

$$s(\beta,\gamma,G) := \begin{cases} 1 - \dfrac{|\gamma|\rho\big(\boldsymbol{S}(G)\big)}{\beta} & \text{if } \beta > 0 \text{ and } \gamma < 0, \\ 1 & \text{else.} \end{cases} \tag{1.40}$$

As regards $s(\beta,\gamma,G)$, if Condition G is satisfied, $\beta > 0$, and $\gamma < 0$ is such that $\gamma\rho(\boldsymbol{S}(G)) > -\beta$, then $0 < s(\beta,\gamma,G) < 1$.

The sufficient conditions depend critically on the topological properties of $\mathcal{Y}$. It is for this reason that they are stated separately for each type of action space.

Proposition 1.39 *Suppose $\mathcal{Y} = \mathbb{R}$. The NALA game Γ has a unique social optimum if three conditions are satisfied: (1.39.1) $\beta > 0$, (1.39.2) $\gamma\rho(\boldsymbol{S}(G)) > -\beta$, and (1.39.3) f is not bounded below and above.*

Proposition 1.40 *Suppose $\mathcal{Y} = \mathbb{R}_+$. The NALA game Γ has a unique and interior social optimum if four conditions are satisfied: (1.40.1) $\beta > 0$, (1.40.2) $\gamma\rho(\boldsymbol{S}(G)) > -\beta$, (1.40.3) $\sum_{i\in\mathcal{I}}|\alpha_i - \bar{\alpha}_{\mathcal{I}}| < s(\beta,\gamma,G)(\bar{\alpha}_{\mathcal{I}} - \beta f(0))$, and (1.40.4) f is not bounded above.*

Proposition 1.41 *Suppose $\mathcal{Y} = [0,\bar{v}]$. The NALA game Γ has a unique and interior social optimum if three conditions are satisfied: (1.41.1) $\beta > 0$, (1.41.2) $\gamma\rho(\boldsymbol{S}(G)) > -\beta$, and (1.41.3) $\sum_{i\in\mathcal{I}}|\alpha_i - \bar{\alpha}_{\mathcal{I}}| < s(\beta,\gamma,G)\min\{\bar{\alpha}_{\mathcal{I}} - \beta f(0), \beta f(\bar{v}) - \bar{\alpha}_{\mathcal{I}}\}$.*

Central to the discussion of Propositions 1.39, 1.40, and 1.41—and also central to their proofs—is an auxiliary function $\widetilde{w}\colon f(\mathcal{Y})^n \to \mathbb{R}$ that is defined by $\widetilde{w} := w \circ (f^{-1},\dots,f^{-1})$ and satisfies $\widetilde{w}(\boldsymbol{z}) = \langle\boldsymbol{\alpha},\boldsymbol{z}\rangle - (1/2)\langle\boldsymbol{z},(\beta\boldsymbol{I}_n + \gamma\boldsymbol{S}(G))\boldsymbol{z}\rangle$. The definition of $\widetilde{w}$ suggests that a global maximum point of $\widetilde{w}$ is related to a global maximum point of w. Indeed, $(y_1,\dots,y_n) \in \mathcal{Y}^n$ is a global maximum point of w if and only if $(f(y_1),\dots,f(y_n)) \in f(\mathcal{Y})^n$ is a global maximum point of $\widetilde{w}$. This result is essential to conceive the logic underlying this discussion.

As will be shown below, the domain of $\widetilde{w}$, $f(\mathcal{Y})^n$, is convex for each type of action space. All three sets of sufficient conditions, Conditions 1.39.1, 1.39.2, and 1.39.3, Conditions 1.40.1 to 1.40.4, and Conditions 1.41.1, 1.41.2, and 1.41.3, have the first two conditions in common: $\beta > 0$ and $\gamma\rho(\boldsymbol{S}(G)) > -\beta$.[64] Taken together, the two inequalities imply that the symmetric matrix $\beta\boldsymbol{I}_n + \gamma\boldsymbol{S}(G)$ is positive definite (and therefore nonsingular), which in turn implies that $\widetilde{w}$ is strictly concave. If $\beta > 0$ and $\gamma\rho(\boldsymbol{S}(G)) > -\beta$, then the function $\boldsymbol{z} \mapsto \widetilde{w}(\boldsymbol{z})$ with domain $\mathbb{R}^n$ (sic) has a unique global maximum point $\boldsymbol{z}^\star \in \mathbb{R}^n$, which is given by $\boldsymbol{z}^\star = \boldsymbol{Q}(\beta,\gamma,G)\boldsymbol{\alpha}$, where

$$\boldsymbol{Q}(\beta,\gamma,G) := \big(\beta\boldsymbol{I}_n + \gamma\boldsymbol{S}(G)\big)^{-1}. \tag{1.41}$$

64. If Condition G is not satisfied, then $\rho(\boldsymbol{S}(G)) = 0$ (Result 1.37.2), so that $\gamma\rho(\boldsymbol{S}(G)) > -\beta$ is equivalent to $\beta > 0$.

It should be noted that $z^\star$ is not necessarily a global maximum point of $\widetilde{w}$ because merely $z^\star \in \mathbb{R}^n$ but not necessarily $z^\star \in \operatorname{int}(f(\mathcal{Y})^n)$. In general, the matrix $Q(\beta,\gamma,G)$ is neither positive nor nonnegative (see Example 1.42), which is relevant for establishing $z^\star \in \operatorname{int}(f(\mathcal{Y})^n)$ if $\mathcal{Y}$ is equal to $\mathbb{R}_+$ or $[0,\bar{v}]$. The situation is therefore different from that of the NE of Γ. In particular, if $\gamma > 0$, then the matrix $\beta I_n + \gamma S(G) = (\beta+\gamma) I_n - \gamma(\bar{A}(G)^\top + \bar{A}(G) - \bar{A}(G)^\top \bar{A}(G))$ is generally not an M-matrix because $\bar{A}(G)^\top + \bar{A}(G) - \bar{A}(G)^\top \bar{A}(G)$ is generally not nonnegative, so that the inverse of $\beta I_n + \gamma S(G)$ is generally not nonnegative. The matrix $Q(\beta,\gamma,G)$ has, however, properties that can be exploited to find sufficient conditions for $z^\star \in \operatorname{int}(f(\mathcal{Y})^n)$. A summary of relevant properties of $Q(\beta,\gamma,G)$ is given in Lemma 1.43.

Example 1.42 Suppose $\mathcal{I} = \{1,2,3,4\}$, $\mathcal{A}(G) = \{(1,2),(1,3),(1,4),(2,1),(2,3),(3,1),(3,2)\}$, $\beta = 1$, and $\gamma = 1/3$. See Figure 1.11 (g) for an illustration of G. Straightforward calculations yield

$$Q(\beta,\gamma,G) = \frac{1}{553}\begin{pmatrix} 388 & 64 & 64 & 37 \\ 64 & 408 & 92 & -11 \\ 64 & 92 & 408 & -11 \\ 37 & -11 & -11 & 538 \end{pmatrix}.$$

Evidently, $Q(\beta,\gamma,G)$ is neither positive nor nonnegative. ◇

Lemma 1.43 *Suppose $\beta > 0$ and $\gamma\rho(S(G)) > -\beta$. The matrix $Q(\beta,\gamma,G)$ has the following properties:*

(1.43.1) *$Q(\beta,\gamma,G)$ is symmetric and positive definite.*

(1.43.2) *If $\gamma < 0$, then $\max\sigma(Q(\beta,\gamma,G)) = 1/(\beta - |\gamma|\rho(S(G))) = 1/(\beta s(\beta,\gamma,G))$, and $\max\sigma(Q(\beta,\gamma,G)) = 1/\beta = 1/(\beta s(\beta,\gamma,G))$ else.*

(1.43.3) *For all $i \in \mathcal{I}$, $\min\sigma(Q(\beta,\gamma,G)) \le [Q(\beta,\gamma,G)]_{i,i} \le \max\sigma(Q(\beta,\gamma,G))$.*

(1.43.4) *For all $(i,j) \in \mathcal{I}^2$ with $i \neq j$, $|[Q(\beta,\gamma,G)]_{i,j}| < \max\sigma(Q(\beta,\gamma,G))$.*

(1.43.5) *$Q(\beta,\gamma,G)\mathbf{1}_n = (1/\beta)\mathbf{1}_n$.*

For all $i \in \mathcal{I}$, let $y^\star_{S,i} := f^{-1}(z^\star_i)$, where $z^\star_i$ denotes the ith component of $z^\star$.[65] If $z^\star \in \operatorname{int}(f(\mathcal{Y})^n)$, then $z^\star = (f(y^\star_{S,1}),\dots,f(y^\star_{S,n}))$ is an interior global maximum point of $\widetilde{w}$, which implies that $y^\star_S := (y^\star_{S,1},\dots,y^\star_{S,n})$ is an interior global maximum point of w. Using the notation introduced in Section 1.3.1, $z^\star = f(y^\star_S)$ and $y^\star_S = f^{-1}(z^\star)$. The conditions specified in Propositions 1.39, 1.40, and 1.41 other than $\beta > 0$ and $\gamma\rho(S(G)) > -\beta$ ensure that $z^\star \in \operatorname{int}(f(\mathcal{Y})^n)$ or, equivalently, $y^\star_S \in \operatorname{int}(\mathcal{Y}^n)$. Two of these conditions, namely, Conditions 1.40.3 and 1.41.3, are based on Results 1.43.1 to 1.43.4.

65. Since Assumption F is an integral part of the definition of a NALA game, f is bijective (Lemma 1.1).

In the remainder of this discussion, I show that $\boldsymbol{y}_S^\star \in \operatorname{int}(\mathcal{Y}^n)$. To this end, I suppose $\beta > 0$ and $\gamma\rho(\boldsymbol{S}(G)) > -\beta$. First, consider the case $\mathcal{Y} = \mathbb{R}$. Assumption F and Condition 1.39.3 together imply that $\operatorname{int}(f(\mathbb{R})^n) = \mathbb{R}^n$, which is a convex set. Consequently, $\boldsymbol{z}^\star \in \operatorname{int}(f(\mathbb{R})^n)$ and $\boldsymbol{y}_S^\star \in \mathbb{R}^n$.

Second, consider the case $\mathcal{Y} = \mathbb{R}_+$. Assumption F and Condition 1.40.4 together imply that $\operatorname{int}(f(\mathbb{R}_+)^n) = (f(0), +\infty)^n$, which is a convex set. If the players of Γ are strongly ex ante homogeneous, so that $\boldsymbol{\alpha} = \bar{\alpha}\mathbf{1}_n$ for some $\bar{\alpha} \in \mathbb{R}$, then $\boldsymbol{z}^\star = (\bar{\alpha}/\beta)\mathbf{1}_n$ because $\boldsymbol{Q}(\beta, \gamma, G)\mathbf{1}_n = (1/\beta)\mathbf{1}_n$ (Result 1.43.5), which implies that $\boldsymbol{z}^\star \in \operatorname{int}(f(\mathbb{R}_+)^n)$ and $\boldsymbol{y}_S^\star \in \operatorname{int}(\mathbb{R}_+^n)$ if $\bar{\alpha}/\beta > f(0)$. This suggests that $\boldsymbol{y}_S^\star \in \operatorname{int}(\mathbb{R}_+^n)$ is true even if the players of Γ are ex ante heterogeneous, provided that they are sufficiently similar in terms of their idiosyncrasies and $\bar{\alpha}_{\mathcal{I}}/\beta > f(0)$. Proposition 1.40 confirms this hypothesis. If the players of Γ are strongly ex ante homogeneous, then $\sum_{i\in\mathcal{I}}|\alpha_i - \bar{\alpha}_{\mathcal{I}}| = 0$, so that Condition 1.40.3 is equivalent to $\bar{\alpha}_{\mathcal{I}}/\beta > f(0)$. If the players of Γ are ex ante heterogeneous, then $\sum_{i\in\mathcal{I}}|\alpha_i - \bar{\alpha}_{\mathcal{I}}| > 0$, so that Condition 1.40.3 implies that $\bar{\alpha}_{\mathcal{I}}/\beta > f(0)$. In this case, Condition 1.40.3 entails a restriction regarding the dissimilarity of the players' idiosyncrasies. Indeed, the inequality $\sum_{i\in\mathcal{I}}|\alpha_i - \bar{\alpha}_{\mathcal{I}}| < s(\beta, \gamma, G)(\bar{\alpha}_{\mathcal{I}} - \beta f(0))$ is true if the idiosyncrasies are sufficiently similar. For example, if for all $j \in \mathcal{I}$, $|\alpha_j - \bar{\alpha}_{\mathcal{I}}| < s(\beta, \gamma, G)(\bar{\alpha}_{\mathcal{I}} - \beta f(0))/n$, then $\sum_{i\in\mathcal{I}}|\alpha_i - \bar{\alpha}_{\mathcal{I}}| < s(\beta, \gamma, G)(\bar{\alpha}_{\mathcal{I}} - \beta f(0))$. The restriction regarding the dissimilarity of the players' idiosyncrasies for a negative γ is stronger than for a positive γ, provided that Condition G is satisfied. Indeed, if $\gamma \geq 0$, then $s(\beta, \gamma, G) = 1$, and if Condition G is satisfied and $\gamma < 0$ is such that $\gamma\rho(\boldsymbol{S}(G)) > -\beta$, then $0 < s(\beta, \gamma, G) < 1$ because $\rho(\boldsymbol{S}(G)) > 0$ (Result 1.37.1).

Third, consider the case $\mathcal{Y} = [0, \bar{v}]$. Since $[0, \bar{v}]$ is compact, Assumption F by itself implies that $\operatorname{int}(f([0, \bar{v}])^n) = (f(0), f(\bar{v}))^n$, which is a convex set. No assumption is therefore needed about the boundedness of f. If the players of Γ are strongly ex ante homogeneous, then Condition 1.41.3 is equivalent to $f(0) < \bar{\alpha}_{\mathcal{I}}/\beta < f(\bar{v})$. In this case, $\boldsymbol{z}^\star = (\bar{\alpha}_{\mathcal{I}}/\beta)\mathbf{1}_n \in \operatorname{int}(f([0, \bar{v}])^n)$ and $\boldsymbol{y}_S^\star \in \operatorname{int}([0, \bar{v}]^n)$ if $f(0) < \bar{\alpha}_{\mathcal{I}}/\beta < f(\bar{v})$. If the players of Γ are ex ante heterogeneous, Condition 1.41.3 ensures too that $\boldsymbol{y}_S^\star \in \operatorname{int}([0, \bar{v}]^n)$. Similar to Condition 1.40.3, Condition 1.41.3 entails a restriction regarding the dissimilarity of the players' idiosyncrasies. Finally, note that Condition 1.41.3 is stronger than Condition 1.40.3.

1.3.5.2 Properties of interior social optima

For this section, suppose $\beta > 0$. In addition, suppose Γ has a unique and interior social optimum $\boldsymbol{y}_S^\star$, which is given by (1.39). Some basic properties of $\boldsymbol{y}_S^\star$ are given by the following result.

Proposition 1.44 *(1.44.1) If $\gamma = 0$, then $\boldsymbol{f}(\boldsymbol{y}_S^\star) = (1/\beta)\boldsymbol{\alpha}$.*

(1.44.2) If $\boldsymbol{\alpha} = \bar{\alpha}\mathbf{1}_n$ for some $\bar{\alpha} \in \mathbb{R}$, then $\boldsymbol{y}_S^\star = f^{-1}(\bar{\alpha}/\beta)\mathbf{1}_n$.

(1.44.3) If $\boldsymbol{y}_S^\star = \bar{y}\mathbf{1}_n$ for some $\bar{y} \in \operatorname{int}(\mathcal{Y})$, then $\boldsymbol{\alpha} = \beta f(\bar{y})\mathbf{1}_n$.

(1.44.4) If $\boldsymbol{\alpha} = \bar{\boldsymbol{A}}(G)\boldsymbol{\alpha}$, then $\boldsymbol{f}(\boldsymbol{y}_S^\star) = (1/\beta)\boldsymbol{\alpha}$.

Results 1.44.1 to 1.44.4 are analogous to those of the interior NE of $\mathit{\Gamma}$ (Results 1.31.2, 1.34.1, 1.34.2, and 1.35.1 and Remark 1.36), and so are their interpretations.

A representation of $\boldsymbol{f}(\boldsymbol{y}_S^\star)$ that is equivalent to (1.39) if $\beta+\gamma>0$ is

$$\boldsymbol{f}(\boldsymbol{y}_S^\star) = \frac{1}{\beta+\gamma}\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G) - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)^\top\big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)\right)^{-1}\boldsymbol{\alpha}, \tag{1.42}$$

where $\bar{\boldsymbol{A}}(G)^\top(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)) \neq \boldsymbol{O}_n$ if Condition G is satisfied (Lemma 1.45). The representation (1.42) unveils the algebraic similarity between the social optimum and the NE of $\mathit{\Gamma}$, which will be elaborated on in Sections 1.3.5.3 and 1.3.5.4.

Lemma 1.45 *If Condition G is satisfied, then* $\bar{\boldsymbol{A}}(G)^\top\bar{\boldsymbol{A}}(G) \neq \bar{\boldsymbol{A}}(G)^\top$.

1.3.5.3 Efficiency of interior Nash equilibria

For this section, suppose $\beta>0$ and $\gamma>-\beta/2$. In addition, suppose $\mathit{\Gamma}$ has a unique and interior NE $\boldsymbol{y}_N^\star := (y_{N,1}^\star,\dots,y_{N,n}^\star)$, which is given by (1.7), and a unique and interior social optimum $\boldsymbol{y}_S^\star := (y_{S,1}^\star,\dots,y_{S,n}^\star)$, which is given by (1.39). The NE of $\mathit{\Gamma}$ is called *efficient* if $\boldsymbol{y}_N^\star = \boldsymbol{y}_S^\star$, that is, for all $i\in\mathcal{I}$, $y_{N,i}^\star = y_{S,i}^\star$; otherwise it is called *inefficient*.

NE actions, equilibrium actions for short, have much in common with socially optimal actions, but there are notable differences. This is apparent from a comparison of (1.7) to (1.42). Despite its merits, matrix notation might not fully disclose the nature of the difference between the two types of actions. A close inspection of the systems of equations that govern equilibrium actions and socially optimal actions proves useful in this respect. According to (1.7), for all $i\in\mathcal{I}$, if $\deg_G^+(i)=0$, then $f(y_{N,i}^\star) = (1/\beta)\alpha_i$, and if $\deg_G^+(i)>0$, then

$$f(y_{N,i}^\star) = \frac{\alpha_i}{\beta+\gamma} + \frac{\gamma}{\beta+\gamma}\frac{1}{\deg_G^+(i)}\sum_{j\in\mathcal{N}_G^+(i)} f(y_{N,j}^\star). \tag{1.43}$$

According to (1.42), for all $i\in\mathcal{I}$, if $\deg_G^+(i)=0$, then[66]

$$f(y_{S,i}^\star) = \frac{\alpha_i}{\beta} + \frac{\gamma}{\beta}\sum_{j\in\mathcal{N}_G^-(i)}\frac{1}{\deg_G^+(j)}\left(f(y_{S,j}^\star) - \frac{1}{\deg_G^+(j)}\sum_{k\in\mathcal{N}_G^+(j)} f(y_{S,k}^\star)\right), \tag{1.44}$$

and if $\deg_G^+(i)>0$, then

$$\begin{aligned} f(y_{S,i}^\star) = {} & \frac{\alpha_i}{\beta+\gamma} + \frac{\gamma}{\beta+\gamma}\frac{1}{\deg_G^+(i)}\sum_{j\in\mathcal{N}_G^+(i)} f(y_{S,j}^\star) \\ & + \frac{\gamma}{\beta+\gamma}\sum_{j\in\mathcal{N}_G^-(i)}\frac{1}{\deg_G^+(j)}\left(f(y_{S,j}^\star) - \frac{1}{\deg_G^+(j)}\sum_{k\in\mathcal{N}_G^+(j)} f(y_{S,k}^\star)\right). \end{aligned} \tag{1.45}$$

66. For all $(i,j)\in\mathcal{I}^2$, $j\in\mathcal{N}_G^-(i)$ if and only if $i\in\mathcal{N}_G^+(j)$, which implies that $\deg_G^+(j)>0$ is necessary for $j\in\mathcal{N}_G^-(i)$.

A player's action enters the welfare function through two different channels: directly through his utility function and indirectly through the social component of the utility functions of the players to whom he is an out-neighbor, that is, the utility functions of his in-neighbors. As a consequence, a player's socially optimal action depends not only on his out-neighbors' actions (via the direct channel by means of the social component) but also on his in-neighbors' actions (via the indirect channel), which in turn depend on their out-neighbors' and in-neighbors' actions, and so forth. This fact is reflected in (1.45) and also partly in (1.44). In contrast, a player's equilibrium action depends exclusively on his out-neighbors' actions, if at all, which in turn depend on their out-neighbors' actions. This is reflected in (1.43). The foregoing considerations highlight a fundamental difference between equilibrium actions and socially optimal actions: both an agent's in-neighborhood and out-neighborhood shape his socially optimal action, whereas it is an agent's out-neighborhood only that matters for his equilibrium action.

In general, equilibrium actions are inefficient. The reason for this is obvious. In the NE, the players disregard the influence emanating from their in-neighbors' actions, whereas this influence is accounted for by the social optimum. As a result, equilibrium actions are in general inefficient. There exist, however, a few cases where equilibrium actions are efficient. An exhaustive list of these cases is given in the corollary to the following result.

Proposition 1.46 *The NE of Γ is efficient if and only if $\boldsymbol{f}(\boldsymbol{y}_N^\star) - (1/\beta)\boldsymbol{\alpha} \in \ker(\bar{\boldsymbol{A}}(G)^\mathsf{T})$, where $\boldsymbol{f}(\boldsymbol{y}_N^\star) - (1/\beta)\boldsymbol{\alpha} = -(\gamma/\beta)(\boldsymbol{f}(\boldsymbol{y}_N^\star) - \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}_N^\star))$.*

Corollary 1.47 *The NE of Γ is efficient if and only if at least one of the following three conditions is satisfied: (1.47.1) $\gamma = 0$, (1.47.2) $\boldsymbol{\alpha} = \bar{\boldsymbol{A}}(G)\boldsymbol{\alpha}$, or (1.47.3) $\boldsymbol{0}_n \neq \boldsymbol{f}(\boldsymbol{y}_N^\star) - (1/\beta)\boldsymbol{\alpha}$ and $\boldsymbol{f}(\boldsymbol{y}_N^\star) - (1/\beta)\boldsymbol{\alpha} \in \ker(\bar{\boldsymbol{A}}(G)^\mathsf{T})$.*

If the players of Γ have a preference for nonconformist behavior (Condition 1.47.1), then $\boldsymbol{f}(\boldsymbol{y}_N^\star) = (1/\beta)\boldsymbol{\alpha}$ (Result 1.31.2 and Remark 1.36) and $\boldsymbol{f}(\boldsymbol{y}_S^\star) = (1/\beta)\boldsymbol{\alpha}$ (Result 1.44.1), from which it follows that equilibrium actions are efficient. If the players of Γ are weakly ex ante homogeneous (Condition 1.47.2), then $\boldsymbol{f}(\boldsymbol{y}_N^\star) = (1/\beta)\boldsymbol{\alpha}$ (Result 1.35.1 and Remark 1.36) and $\boldsymbol{f}(\boldsymbol{y}_S^\star) = (1/\beta)\boldsymbol{\alpha}$ (Result 1.44.4). In this case, too, equilibrium actions are efficient. The same applies in case of strongly ex ante homogeneous players. Conditions 1.47.1 and 1.47.3 are mutually exclusive for $\boldsymbol{f}(\boldsymbol{y}_N^\star) = (1/\beta)\boldsymbol{\alpha}$ if $\gamma = 0$. If the players of Γ have a preference for conformist or anti-conformist behavior, that is, $\gamma \neq 0$, then Conditions 1.47.2 and 1.47.3 are mutually exclusive because $\boldsymbol{\alpha} = \bar{\boldsymbol{A}}(G)\boldsymbol{\alpha}$ is both necessary and sufficient for $\boldsymbol{f}(\boldsymbol{y}_N^\star) = (1/\beta)\boldsymbol{\alpha}$ if $\gamma \neq 0$ (Results 1.35.1 and 1.35.2 and Remark 1.36). If $0 \notin \sigma(\bar{\boldsymbol{A}}(G))$, then $\ker(\bar{\boldsymbol{A}}(G)^\mathsf{T}) = \{\boldsymbol{0}_n\}$.[67] This is, for example, the case if G is complete (Example 1.48). It follows that $0 \in \sigma(\bar{\boldsymbol{A}}(G))$ is necessary for Condition 1.47.3.

Example 1.48 If G is complete, then $\sigma(\bar{\boldsymbol{A}}(G)) = \{1/(1-n), 1\}$. ◇

For the remainder of this discussion, suppose Condition 1.47.3 is satisfied, which implies that Conditions 1.47.1 and 1.47.2 are not satisfied. Condition 1.47.3 may

67. The eigenspace of $\bar{A}(G)^\mathsf{T}$ associated to the eigenvalue 0 is identical to $\ker(\bar{A}(G)^\mathsf{T})$.

be interpreted in a number of ways: algebraically, geometrically, and in terms of players' in- and out-neighborhoods.

First, the algebraic interpretation. Condition 1.47.3 implies that

$$\bar{\boldsymbol{A}}(G)^{\mathsf{T}}\big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)\boldsymbol{f}(\boldsymbol{y}_N^\star) = \boldsymbol{0}_n, \tag{1.46}$$

which in turn implies that

$$\boldsymbol{f}(\boldsymbol{y}_N^\star) = \frac{1}{\beta+\gamma}\boldsymbol{\alpha} + \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}_N^\star) + \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)^{\mathsf{T}}\big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)\boldsymbol{f}(\boldsymbol{y}_N^\star).$$

Thus, $\boldsymbol{f}(\boldsymbol{y}_N^\star)$ satisfies the same system of equations as does $\boldsymbol{f}(\boldsymbol{y}_S^\star)$ (see, for example, (1.42)). It follows that equilibrium actions are efficient.

Second, the geometric interpretation. The vector $\boldsymbol{f}(\boldsymbol{y}_N^\star)$ may be decomposed into two parts, $(1/\beta)\boldsymbol{\alpha}$ and $\boldsymbol{f}(\boldsymbol{y}_N^\star) - (1/\beta)\boldsymbol{\alpha}$. If $\gamma = 0$, then $\boldsymbol{f}(\boldsymbol{y}_N^\star) = (1/\beta)\boldsymbol{\alpha}$.[68] Hence, $\boldsymbol{f}(\boldsymbol{y}_N^\star) - (1/\beta)\boldsymbol{\alpha}$ represents the part of $\boldsymbol{f}(\boldsymbol{y}_N^\star)$ that is attributable to the players' preference for conformist or anti-conformist behavior and their connections in G. A first geometric result is directly related to $\boldsymbol{f}(\boldsymbol{y}_N^\star) - (1/\beta)\boldsymbol{\alpha}$. Since $\boldsymbol{f}(\boldsymbol{y}_N^\star) - (1/\beta)\boldsymbol{\alpha}$ lies in the kernel of $\bar{\boldsymbol{A}}(G)^{\mathsf{T}}$, it is orthogonal to all columns of $\bar{\boldsymbol{A}}(G)$ and therefore orthogonal to the linear space spanned by the columns of $\bar{\boldsymbol{A}}(G)$, which is denoted by c-sp$(\bar{\boldsymbol{A}}(G))$. A second geometric result is indirectly related to $\boldsymbol{f}(\boldsymbol{y}_N^\star) - (1/\beta)\boldsymbol{\alpha}$. Since $\ker(\bar{\boldsymbol{A}}(G)^{\mathsf{T}})$ is a linear subspace of $\mathbb{R}^n$, $\boldsymbol{f}(\boldsymbol{y}_N^\star) - (1/\beta)\boldsymbol{\alpha} \in \ker(\bar{\boldsymbol{A}}(G)^{\mathsf{T}})$ and $\boldsymbol{f}(\boldsymbol{y}_N^\star) - (1/\beta)\boldsymbol{\alpha} = -(\gamma/\beta)(\boldsymbol{f}(\boldsymbol{y}_N^\star) - \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}_N^\star))$ imply that $\boldsymbol{f}(\boldsymbol{y}_N^\star) - \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}_N^\star) \in \ker(\bar{\boldsymbol{A}}(G)^{\mathsf{T}})$, that is, $\boldsymbol{f}(\boldsymbol{y}_N^\star) - \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}_N^\star)$ is orthogonal to c-sp$(\bar{\boldsymbol{A}}(G))$. In other words, $\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}_N^\star)$ is the orthogonal projection of $\boldsymbol{f}(\boldsymbol{y}_N^\star)$ onto c-sp$(\bar{\boldsymbol{A}}(G))$. Specifically, if $f = \mathrm{id}_{\mathcal{Y}}$, the vector of social differences in equilibrium, that is, the vector of differences between the players' actions and their out-neighbors' average actions, is orthogonal to the vector of out-neighbors' average actions. The interpretation of $\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}_N^\star)$ as the image of an orthogonal projection is also reflected in (1.46). The orthogonal projection onto c-sp$(\bar{\boldsymbol{A}}(G))$ may be represented by a square matrix $\boldsymbol{P}_{\text{c-sp}(\bar{\boldsymbol{A}}(G))}$ of order n. Using this notation, $\boldsymbol{P}_{\text{c-sp}(\bar{\boldsymbol{A}}(G))}\boldsymbol{f}(\boldsymbol{y}_N^\star) = \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}_N^\star)$ and $(\boldsymbol{I}_n - \boldsymbol{P}_{\text{c-sp}(\bar{\boldsymbol{A}}(G))})\boldsymbol{f}(\boldsymbol{y}_N^\star) = (\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G))\boldsymbol{f}(\boldsymbol{y}_N^\star)$, where $\boldsymbol{I}_n - \boldsymbol{P}_{\text{c-sp}(\bar{\boldsymbol{A}}(G))}$ is the matrix representing the orthogonal projection onto the orthogonal complement to c-sp$(\bar{\boldsymbol{A}}(G))$, which is denoted by c-sp$(\bar{\boldsymbol{A}}(G))^{\perp}$. Since $(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G))\boldsymbol{f}(\boldsymbol{y}_N^\star)$ lies in c-sp$(\bar{\boldsymbol{A}}(G))^{\perp}$, it is orthogonal to every column of $\bar{\boldsymbol{A}}(G)$, which is exactly what is expressed by (1.46). The foregoing geometric interpretation of Condition 1.47.3 is illustrated by Example 1.49.

Third, the interpretation in terms of players' in- and out-neighborhoods. Condition 1.47.3 provides a restriction on the column space of $\bar{\boldsymbol{A}}(G)$, c-sp$(\bar{\boldsymbol{A}}(G))$, for $(\boldsymbol{f}(\boldsymbol{y}_N^\star) - (1/\beta)\boldsymbol{\alpha})^{\mathsf{T}}\bar{\boldsymbol{A}}(G) = \boldsymbol{0}_n^{\mathsf{T}}$ shows that some linear combination of the components of any column of $\bar{\boldsymbol{A}}(G)$ must be zero.[69] The restrictions obeyed by the columns

68. If Condition G is not satisfied, then $\boldsymbol{f}(\boldsymbol{y}_N^\star) = (1/\beta)\boldsymbol{\alpha}$ too.

69. The system of equations $(\boldsymbol{f}(\boldsymbol{y}_N^\star) - (1/\beta)\boldsymbol{\alpha})^{\mathsf{T}}\bar{\boldsymbol{A}}(G) = \boldsymbol{0}_n^{\mathsf{T}}$ is equivalent to

$$\forall i \in \mathcal{I} \quad \sum_{j \in \mathcal{N}_G^-(i)} \frac{1}{\deg_G^+(j)} f(y_{N,j}^\star) = \sum_{j \in \mathcal{N}_G^-(i)} \frac{1}{\deg_G^+(j)} \frac{\alpha_j}{\beta}.$$

of $\bar{A}(G)$—and therefore by every vector in c-sp$(\bar{A}(G))$—depend on $\boldsymbol{\alpha}$ and $\bar{A}(G)$ because $\boldsymbol{f}(\boldsymbol{y}_N^\star) - (1/\beta)\boldsymbol{\alpha}$ is a mapping of $\boldsymbol{\alpha}$ and $\bar{A}(G)$. Each column (respectively, row) of $\bar{A}(G)$ encodes information about a player's in-neighborhood (respectively, out-neighborhood) in G. Condition 1.47.3 entails therefore restrictions on the players' in-neighborhoods, which are related to the players' idiosyncracies and their out-neighborhoods. The restrictions are thereby stronger than the restrictions implied by the canonical relation between the players' in- and out-neighborhoods in G, namely, for all $(i,j) \in \mathcal{I}^2$, $j \in \mathcal{N}_G^+(i)$ if and only if $i \in \mathcal{N}_G^-(j)$.

Example 1.49 Suppose $\mathcal{I} = \{1,2,3\}$, $\mathcal{A}(G) = \{(1,2),(1,3),(2,1),(3,1)\}$, $\mathcal{Y} = \mathbb{R}_+$, $\alpha_1 = 8/3$, $\alpha_2 = 1/3$, $\alpha_3 = 5$, $\beta = 1$, $\gamma = 3/4$, and $f = \mathrm{id}_{\mathbb{R}_+}$. See Figure 1.9 for an illustration of G. The endogenous effects matrix of G is given by (1.28). Note that $\sigma(\bar{A}(G)) = \{-1,0,1\}$. The kernel of $\bar{A}(G)^\top$ and the column space of $\bar{A}(G)$ are given by

$$\ker\bigl(\bar{A}(G)^\top\bigr) = \left\{ s \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \middle| \; s \in \mathbb{R} \right\}$$

and

$$\text{c-sp}(\bar{A}(G)) = \left\{ s \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \middle| \; (s,t) \in \mathbb{R}^2 \right\},$$

where $\ker(\bar{A}(G)^\top)$ may be represented by a line and c-sp$(\bar{A}(G))$ by a plane in Euclidean space $\mathbb{R}^3$. Straightforward calculations yield

$$\boldsymbol{y}_N^\star = \frac{1}{3} \begin{pmatrix} 8 \\ 4 \\ 12 \end{pmatrix}$$

and

$$\boldsymbol{y}_N^\star - \frac{1}{\beta}\boldsymbol{\alpha} = -\frac{\gamma}{\beta}(\boldsymbol{y}_N^\star - \bar{A}(G)\boldsymbol{y}_N^\star) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Since $\boldsymbol{y}_N^\star - (1/\beta)\boldsymbol{\alpha} \in \ker(\bar{A}(G)^\top)$, $\boldsymbol{y}_S^\star = \boldsymbol{y}_N^\star$ according to Proposition 1.46. This result is illustrated in Figure 1.15, where the origin of the Euclidian space $\mathbb{R}^3$ is marked by O and the axes represent players' actions. For all pairs (X,Y) of distinct points in $\{A,N,O,P,Q\}$, $\overrightarrow{XY}$ denotes the Euclidean vector with initial point X and terminal point Y. Using this notation, the points A, N, P, and Q are such that $\overrightarrow{OA} = (1/\beta)\boldsymbol{\alpha}$, $\overrightarrow{ON} = \boldsymbol{y}_N^\star$, $\overrightarrow{OP} = \bar{A}(G)\boldsymbol{y}_N^\star$, $\overrightarrow{OQ} = \gamma/(\beta+\gamma)\bar{A}(G)\boldsymbol{y}_N^\star$, and $\overrightarrow{QN} = 1/(\beta+\gamma)\boldsymbol{\alpha}$. Evidently, both $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ lie in the plane c-sp$(\bar{A}(G))$, which is depicted in gray. According to (1.7), $\boldsymbol{y}_N^\star = 1/(\beta+\gamma)\boldsymbol{\alpha} + \gamma/(\beta+\gamma)\bar{A}(G)\boldsymbol{y}_N^\star$, which translates into $\overrightarrow{ON} = \overrightarrow{OQ} + \overrightarrow{QN}$. This is the canonical decomposition of $\boldsymbol{y}_N^\star$.

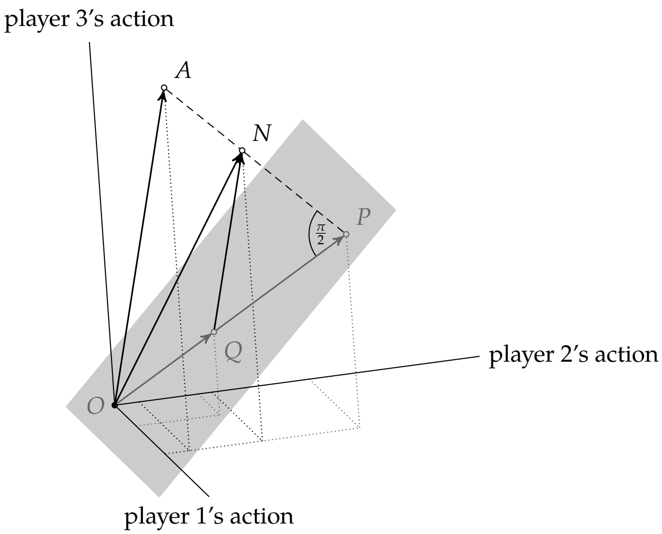

Figure 1.15. Illustration of Condition 1.47.3 of Corollary 1.47 in Euclidian space $\mathbb{R}^3$ (Example 1.49)

Alternatively, $\boldsymbol{y}_N^\star$ may be written as the sum of $\bar{\boldsymbol{A}}(G)\boldsymbol{y}_N^\star$ and $\boldsymbol{y}_N^\star - \bar{\boldsymbol{A}}(G)\boldsymbol{y}_N^\star$, in Euclidean vector notation, $\overrightarrow{ON} = \overrightarrow{OP} + \overrightarrow{PN}$. Since $\boldsymbol{y}_N^\star - \bar{\boldsymbol{A}}(G)\boldsymbol{y}_N^\star \in \ker(\bar{\boldsymbol{A}}(G)^\top)$, $\boldsymbol{y}_N^\star - \bar{\boldsymbol{A}}(G)\boldsymbol{y}_N^\star$ is orthogonal to $\bar{\boldsymbol{A}}(G)\boldsymbol{y}_N^\star$, that is, the angle (in radians) between $\overrightarrow{PO}$ and $\overrightarrow{PN}$ is equal to $\pi/2$.[70] Yet another decomposition of $\boldsymbol{y}_N^\star$ is given by $\boldsymbol{y}_N^\star = (1/\beta)\boldsymbol{\alpha} + (\boldsymbol{y}_N^\star - (1/\beta)\boldsymbol{\alpha})$, where $(1/\beta)\boldsymbol{\alpha}$ is the NE of Γ if $\gamma = 0$. This decomposition can be written as $\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN}$, where $\overrightarrow{AN} = \boldsymbol{y}_N^\star - (1/\beta)\boldsymbol{\alpha}$ is orthogonal to c-sp$(\bar{\boldsymbol{A}}(G))$. $\diamond$

In general, if $\boldsymbol{y}_N^\star$ is inefficient, neither $\boldsymbol{y}_S^\star \leq_c \boldsymbol{y}_N^\star$ nor $\boldsymbol{y}_N^\star \leq_c \boldsymbol{y}_S^\star$ is true. Even if equilibrium actions and socially optimal actions are different, they may share a common property, for example, aggregate action. This is, for example, the case if the digraph by which the players are connected is complete and $f = \mathrm{id}_{\mathcal{Y}}$, as demonstrated by the following result.

Proposition 1.50 *If G is complete, the players of Γ are ex ante heterogeneous, and $\gamma \neq 0$, then equilibrium actions and socially optimal actions have the following properties:*[71]

(1.50.1) *Equilibrium actions are inefficient, that is, $\boldsymbol{y}_N^\star \neq \boldsymbol{y}_S^\star$.*[72]

(1.50.2) $\langle \mathbf{1}_n, \boldsymbol{f}(\boldsymbol{y}_N^\star)\rangle = \langle \mathbf{1}_n, \boldsymbol{f}(\boldsymbol{y}_S^\star)\rangle = (1/\beta)\langle \mathbf{1}_n, \boldsymbol{\alpha}\rangle$.

(1.50.3) *If $\sum_{k=1}^n \alpha_k \in \mathcal{O}(n)$ as $n \to \infty$, then, for all $i \in \mathcal{I}$, $f(y_{N,i}^\star) - f(y_{S,i}^\star) \in o(1)$ as $n \to \infty$.*

70. In general, $\ker(\bar{\boldsymbol{A}}(G)^\top)$ is equal to the orthogonal complement to c-sp$(\bar{\boldsymbol{A}}(G))$.

71. The dependence of $y_{N,i}^\star$ and $y_{S,i}^\star$ on n is suppressed in all statements about the asymptotic behavior of $f(y_{N,i}^\star) - f(y_{S,i}^\star)$ and $y_{N,i}^\star - y_{S,i}^\star$ as $n \to \infty$.

72. This result follows directly from Corollaries 1.7 and 1.47 and Example 1.48.

(1.50.4) *If f^{-1} is Lipschitz continuous and $\sum_{k=1}^{n} \alpha_k \in \mathcal{O}(n)$ as $n \to \infty$, then, for all $i \in \mathcal{I}$, $y^{\star}_{N,i} - y^{\star}_{S,i} \in \mathcal{o}(1)$ as $n \to \infty$.*

(1.50.5) *If there exists an $\bar{\alpha} \in \mathbb{R}$ such that $(1/n)\sum_{k=1}^{n} \alpha_k - \bar{\alpha} \in \mathcal{o}(1)$ as $n \to \infty$, then, for all $i \in \mathcal{I}$, $y^{\star}_{N,i} - y^{\star}_{S,i} \in \mathcal{o}(1)$ as $n \to \infty$.*

Within the setup of Proposition 1.50, equilibrium actions are inefficient (Result 1.50.1), yet aggregate equilibrium action and aggregate socially optimal action are identical, provided that $f = \mathrm{id}_{\mathcal{Y}}$ (Result 1.50.2). Both aggregate actions are thereby equal to the aggregate action in a NALA game with an arbitrary digraph G and where the players have a preference for nonconformist behavior (cf. Result 1.31.2 and Remark 1.36) or the players are weakly ex ante homogeneous (cf. Result 1.35.1 and Remark 1.36). Even if equilibrium actions and socially optimal actions are different, the distance between the former and the latter is strictly decreasing in n, the number of players of Γ, provided that (i) $f = \mathrm{id}_{\mathcal{Y}}$ and the sum of the players' idiosyncracies, $\sum_{i=1}^{n} \alpha_i$, is at most of order n (Result 1.50.3); (ii) f^{-1} is Lipschitz continuous and $\sum_{i=1}^{n} \alpha_i$ is at most of order n (Result 1.50.4); or (iii) the average of the players' idiosyncracies, $(1/n)\sum_{i=1}^{n} \alpha_i$, converges as $n \to \infty$ (Result 1.50.5).

1.3.5.4 Decentralization of social optima

Suppose Condition G is satisfied and Γ has a unique and interior social optimum $\boldsymbol{y}^{\star}_{S}$, which is given by (1.39).[73] The social optimum of Γ may be decentralized as a NE, as demonstrated by the following result.

Proposition 1.51 *Suppose $\beta > 0$ and $\gamma > -\beta/2$. The social optimum of Γ, $\boldsymbol{y}^{\star}_{S}$, can be decentralized as the unique and interior NE of the NALA game $(\mathcal{I}, G, \mathcal{Y}, \{(\alpha_{S,i}, \beta, \gamma)\}_{i\in\mathcal{I}}, f)$, where the profile of the players' idiosyncrasies, $\boldsymbol{\alpha}_S := (\alpha_{S,1}, \ldots, \alpha_{S,n})$, is given by $\boldsymbol{\alpha}_S = \boldsymbol{\alpha} + \gamma \bar{\boldsymbol{A}}(G)^{\mathsf{T}}(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G))\boldsymbol{f}(\boldsymbol{y}^{\star}_{S})$. If*

$$\rho\Big(\gamma \bar{\boldsymbol{A}}(G)^{\mathsf{T}}(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G))\boldsymbol{J}(\beta,\gamma,G)\Big) < 1, \tag{1.47}$$

then $\boldsymbol{\alpha}_S = (\boldsymbol{I}_n - \gamma \bar{\boldsymbol{A}}(G)^{\mathsf{T}}(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G))\boldsymbol{J}(\beta,\gamma,G))^{-1}\boldsymbol{\alpha}$. A sufficient condition for (1.47) is $-\beta/(2+2N(G)) < \gamma < \beta/(2N(G))$, where[74]

$$N(G) := \max\left\{ \mathbb{1}_{\mathcal{I}_0^+(G)}(j) + \frac{\deg^-_G(j)}{\min\{\deg^+_G(i) \mid i \in \mathcal{I} \setminus \mathcal{I}_0^+(G)\}} \;\middle|\; j \in \mathcal{I} \right\}.$$

In general, $\boldsymbol{\alpha}_S - \boldsymbol{\alpha}$ is neither nonnegative nor nonpositive, but $\boldsymbol{\alpha}_S = \boldsymbol{\alpha}$ if and only if at least one of the Conditions 1.47.1, 1.47.2, or 1.47.3 is satisfied.

There are—at least theoretically—two ways to decentralize the social optimum of Γ. First, the social planner may target the players' idiosyncracies directly. A suitable policy measure entails altering each player's idiosyncrasy from α_i to $\alpha_{S,i}$. Subsequently, the players play the generic NALA game $(\mathcal{I}, G, \mathcal{Y}, \{(\alpha_{S,i}, \beta, \gamma)\}_{i\in\mathcal{I}}, f)$. Second, the social planner may target the players' idiosyncracies indirectly. Specifically,

73. The question of decentralization does not arise if Condition G is not satisfied for in this case the interior social optimum of Γ is equal to its interior NE.

74. Condition G implies that $\min\{\deg^+_G(i) \mid i \in \mathcal{I} \setminus \mathcal{I}_0^+(G)\} > 0$.

the planner may subsidize or tax the players' idiosyncracies within a two-stage game. In the first stage of the game, the planner announces to subsidize or tax a player's private benefit per unit of action (or per unit of transformed action if $f \neq \mathrm{id}_{\mathcal{Y}}$) by $\varsigma_i := \alpha_{S,i} - \alpha_i$. In the second stage of the game, the players play a game that is strategically equivalent to the generic NALA game $(\mathcal{I}, G, \mathcal{Y}, \{(\alpha_{S,i}, \beta, \gamma)\}_{i\in\mathcal{I}}, f)$, where for all $i \in \mathcal{I}$, player i's utility function satisfies (cf. (1.4))

$$u_i(y_1, \ldots, y_n) = \varsigma_i f(y_i) + \alpha_i f(y_i) - \frac{\beta}{2} f(y_i)^2 - \frac{\gamma}{2}\left(f(y_i) - \sum_{j\in\mathcal{I}} \bar{a}_{i,j}(G) f(y_j)\right)^2.$$

Both direct and indirect policy measures are individually tailored to each player's idiosyncrasy. Needless to say, it depends largely on the context, that is, on the phenomenon under consideration, which of the two policies, if any, is feasible.

1.3.6 Policy analysis

In this section, I discuss NALA games from the perspective of a policy maker whose sole objective is to decrease or increase to a degree aggregate equilibrium action, aggregate action for short, depending on whether action has a negative or a positive connotation. The analysis rests on the implicit assumption that the policy maker has measures at his disposal to alter, at least to some extent, the players' preference parameters and the topology of the digraph by which the players are connected.

As usual, let $\Gamma := (\mathcal{I}, G, \mathcal{Y}, \{(\alpha_i, \beta, \gamma)\}_{i\in\mathcal{I}}, f)$ be a generic NALA game, and let $\boldsymbol{\alpha}$ be defined as in Section 1.3.2. Suppose $\beta > 0$ and $\gamma > -\beta/2$, so that $\beta + \gamma > 0$ and $|\gamma/(\beta+\gamma)| < 1$. In addition, suppose Γ has a unique and interior NE $\boldsymbol{y}^\star := (y_1^\star, \ldots, y_n^\star)$, which is given by (1.7).

Hereinafter, $\boldsymbol{y}^\star$ is written as $\boldsymbol{y}^\star(\boldsymbol{\alpha}, \beta, \gamma, f, G)$ in order to emphasize that it is a mapping of $\boldsymbol{\alpha}$, β, γ, f, and G. The same applies to the components of $\boldsymbol{y}^\star$. Depending on the context and situation, some or all of the arguments of $\boldsymbol{y}^\star$ and of its components may be omitted. The function of aggregate action is written as $\langle \mathbf{1}_n, \boldsymbol{y}^\star\rangle \colon \mathcal{Y}^n \to n\mathcal{Y}$.[75]

1.3.6.1 Targeting the players' preference parameters

The policy maker may consider targeting some or all of the players' preference parameters in order to achieve his objective of changing aggregate action in one direction or the other.

Suppose the policy maker's *target* for aggregate action is $\tau \in \mathbb{R}$. The target τ is called *attainable* if there exist $\tilde{\boldsymbol{\alpha}} := (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n) \in \mathbb{R}^n$, $\tilde{\beta} > 0$, and $\tilde{\gamma} > -\tilde{\beta}/2$ such that the generic NALA game $(\mathcal{I}, G, \mathcal{Y}, \{(\tilde{\alpha}_i, \tilde{\beta}, \tilde{\gamma})\}_{i\in\mathcal{I}}, f)$ has a unique and interior NE $\boldsymbol{y}^\star(\tilde{\boldsymbol{\alpha}}, \tilde{\beta}, \tilde{\gamma}, f, G)$ that satisfies $\tau = \langle \mathbf{1}_n, \boldsymbol{y}^\star(\tilde{\boldsymbol{\alpha}}, \tilde{\beta}, \tilde{\gamma}, f, G)\rangle$, in which case τ is called *attainable by* $\{(\tilde{\alpha}_i, \tilde{\beta}, \tilde{\gamma})\}_{i\in\mathcal{I}}$. The set of attainable targets for aggregate action is denoted by $\mathcal{T}^\star(\mathcal{Y})$. The target τ is called *implementable* if it is attainable by some $\{(\tilde{\alpha}_i, \tilde{\beta}, \tilde{\gamma})\}_{i\in\mathcal{I}}$ and there exists a policy measure to alter the players' preference parameters from $\{(\alpha_i, \beta, \gamma)\}_{i\in\mathcal{I}}$ to $\{(\tilde{\alpha}_i, \tilde{\beta}, \tilde{\gamma})\}_{i\in\mathcal{I}}$. A characterization of $\mathcal{T}^\star(\mathcal{Y})$ is given in the following result.

75. The set $n\mathcal{Y}$ is equal to $\{ny \mid y \in \mathcal{Y}\}$; for example, $n\mathbb{R} = \mathbb{R}$, $n\mathbb{R}_+ = \mathbb{R}_+$, and $n[0, \bar{v}] = [0, n\bar{v}]$.

Proposition 1.52 *A target τ for aggregate action is attainable by $\{(\beta f(\tau/n), \beta, 0)\}_{i\in\mathcal{I}}$ if $\tau \in \operatorname{int}(n\mathcal{Y})$. It follows that $\mathcal{T}^\star(\mathcal{Y}) = \operatorname{int}(n\mathcal{Y})$.*

A target τ in $\mathcal{T}^\star(\mathcal{Y})$ is implementable if the policy maker has measures at his disposal to alter each player's idiosyncrasy from α_i to $\beta f(\tau/n)$ and the players' common social cost or benefit parameter from γ to 0. Assuming that the players of Γ are ex ante heterogeneous, the corresponding policy is necessarily player tailored and may be particularly difficult to implement if the difference between $\beta f(\tau/n)$ and α_i is negative for some players and positive for others. The difficulty may be alleviated if τ is attainable by some $\{(\tilde{\alpha}_i, \tilde{\beta}, \tilde{\gamma})\}_{i\in\mathcal{I}}$ that is less challenging to implement than $\{(\beta f(\tau/n), \beta, 0)\}_{i\in\mathcal{I}}$. But what is to be done if a target is attainable but not implementable? This may be the case if at least one of two conditions is satisfied: First, the policy maker is able to alter some but not all of the players' preference parameters. Second, the players' preference parameters can be altered only to a limited degree. In either case, the policy maker is advised to switch the original target for an implementable target that decreases the distance to the original target most. In order to formulate a policy that decreases this distance, it is important to know whether aggregate action is monotone in some of the players' preference parameters. The presence or absence of such a monotonic dependence is the subject of the following result.

Proposition 1.53 *The partial derivatives of $\langle \mathbf{1}_n, \boldsymbol{y}^\star\rangle$ with respect to $\boldsymbol{\alpha}$, β, and γ are given by*

$$\frac{\partial\langle \mathbf{1}_n, \boldsymbol{y}^\star\rangle}{\partial \boldsymbol{\alpha}} = \boldsymbol{w}(f, \boldsymbol{y}^\star)^\top \boldsymbol{J}(\beta, \gamma, G), \tag{1.48}$$

$$\frac{\partial\langle \mathbf{1}_n, \boldsymbol{y}^\star\rangle}{\partial \beta} = -\langle \boldsymbol{w}(f, \boldsymbol{y}^\star), \boldsymbol{J}(\beta, \gamma, G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle, \tag{1.49}$$

and

$$\frac{\partial\langle \mathbf{1}_n, \boldsymbol{y}^\star\rangle}{\partial \gamma} = \big\langle \boldsymbol{w}(f, \boldsymbol{y}^\star), \boldsymbol{J}(\beta, \gamma, G)\big(\bar{\boldsymbol{A}}(G) - \boldsymbol{I}_n\big)\boldsymbol{f}(\boldsymbol{y}^\star)\big\rangle, \tag{1.50}$$

where $\boldsymbol{w}(f, \boldsymbol{y}^\star) := (1/\partial f(y_1^\star), \ldots, 1/\partial f(y_n^\star)) \in \mathbb{R}^n_{++}$ with $\boldsymbol{w}(\mathrm{id}_{\mathcal{Y}}, \boldsymbol{y}^\star) = \mathbf{1}_n$.

(1.53.1) If $\gamma \geq 0$, then $\partial\langle \mathbf{1}_n, \boldsymbol{y}^\star\rangle/\partial\boldsymbol{\alpha} >_c \mathbf{0}_n^\top$.

(1.53.2) If $\gamma \geq 0$ and $\boldsymbol{f}(\boldsymbol{y}^\star) \in \mathbb{R}^n_+ \setminus \{\mathbf{0}_n\}$, then $\partial\langle \mathbf{1}_n, \boldsymbol{y}^\star\rangle/\partial\beta < 0$.

(1.53.3) $\partial\langle \mathbf{1}_n, \boldsymbol{y}^\star\rangle/\partial\gamma|_{\gamma=0} = (1/\beta^2)\langle \boldsymbol{w}(f, (1/\beta)\boldsymbol{\alpha}), \bar{\boldsymbol{A}}(G)\boldsymbol{\alpha} - \boldsymbol{\alpha}\rangle$.

If the players' common social cost or benefit parameter is nonnegative, then aggregate action is strictly increasing in the players' idiosyncracies (Result 1.53.1) and strictly decreasing in the players' common private cost parameter (Result 1.53.2), where the latter result is true if, for example, equilibrium actions exceed a certain critical threshold. Aggregate equilibrium action is in general not monotonic in the players' common social cost or benefit parameter. This is illustrated by Example 1.54.

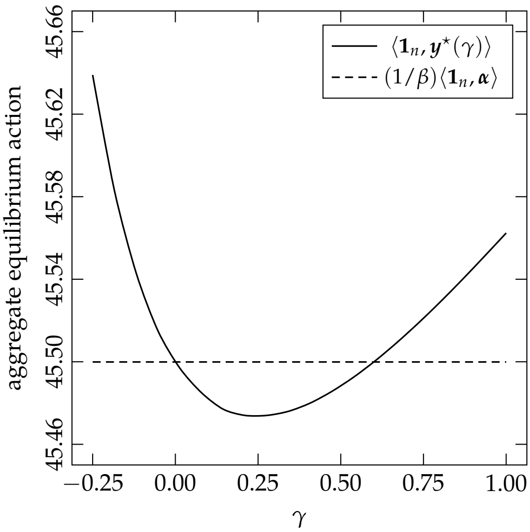

Figure 1.16. Aggregate equilibrium action as a function of γ (Example 1.54)

Example 1.54 Suppose $\mathcal{I} = \{1,2,3,4\}$, $\mathcal{A}(G) = \{(1,2),(1,3),(1,4),(2,1),(2,3),(3,1),(3,2)\}$, $\mathcal{Y} = [0,20]$, $\alpha_1 = 8$, $\alpha_2 = \alpha_4 = 12$, $\alpha_3 = 27/2$, $\beta = 1$, $\gamma > -1/4$, and $f = \mathrm{id}_{[0,20]}$. See Figure 1.11 (g) for an illustration of G. Let $\boldsymbol{y}^\star(\gamma)$ denote the unique and interior NE of Γ. Straightforward calculations yield

$$\boldsymbol{y}^\star(\gamma) = \frac{1}{3\gamma^3 + 29\gamma^2 + 36\gamma + 12}\begin{pmatrix} 36\gamma^3 + 321\gamma^2 + 342\gamma + 96 \\ 36\gamma^3 + 324\gamma^2 + 417\gamma + 144 \\ 36\gamma^3 + 327\gamma^2 + 444\gamma + 162 \\ 36\gamma^3 + 348\gamma^2 + 432\gamma + 144 \end{pmatrix},$$

from which it follows that

$$\langle \mathbf{1}_n, \boldsymbol{y}^\star(\gamma)\rangle = \frac{48\gamma^2 + 408\gamma + 273}{\gamma^2 + 9\gamma + 6} \quad \text{and} \quad \frac{\partial\langle \mathbf{1}_n, \boldsymbol{y}^\star(\gamma)\rangle}{\partial\gamma} = \frac{24\gamma^2 + 30\gamma - 9}{(\gamma^2 + 9\gamma + 6)^2},$$

where both polynomials $3\gamma^3 + 29\gamma^2 + 36\gamma + 12$ and $\gamma^2 + 9\gamma + 6$ are positive on $(-1/4, +\infty)$. The polynomial $24\gamma^2 + 30\gamma - 9$ has one positive root in $(-1/4, +\infty)$, which is equal to $1/4$. It follows that aggregate action is strictly decreasing on $(-1/4, 1/4)$ and strictly increasing on $(1/4, +\infty)$. This is illustrated in Figure 1.16, where the solid line represents the graph of aggregate action as a function of γ. $\diamond$

Of particular interest to the policy maker may be the behavior of aggregate action as a function of the players' common social cost parameter in a right neighborhood of zero. To illustrate this point, consider the following scenario of an egocentric society. Suppose the members of a (human) population have a preference for nonconformist behavior, even though they are connected to each other in some way. In addition, suppose the policy maker has the power to formulate and implement a policy aiming at changing the members' behavior to the effect that they conform with social norms. In the context of NALA games, this policy corresponds to an increase in γ from zero to some positive value $\tilde{\gamma}$. What will be the effect of such a policy on behavior and in particular on aggregate action? The answer depends on the magnitude of $\tilde{\gamma}$ and the characteristics of the underlying NALA game other than γ (which is zero), specifically, the players' preference parameters and the connections among the players. In general, aggregate action may decrease, increase, or not change at all. This is reflected by Result 1.53.3. For example, if $f = \mathrm{id}_{\mathcal{Y}}$, then $\partial\langle \mathbf{1}_n, \boldsymbol{y}^\star\rangle/\partial\gamma|_{\gamma=0} = (1/\beta^2)\langle \mathbf{1}_n, \bar{\boldsymbol{A}}(G)\boldsymbol{\alpha} - \boldsymbol{\alpha}\rangle$ may be negative, zero, or positive. Since Result 1.53.3 is relevant only for small changes in γ, the difference $\langle \mathbf{1}_n, \boldsymbol{y}^\star(\tilde{\gamma})\rangle - \langle \mathbf{1}_n, \boldsymbol{y}^\star(0)\rangle = \langle \mathbf{1}_n, \boldsymbol{y}^\star(\tilde{\gamma})\rangle - (1/\beta)\langle \mathbf{1}_n, \boldsymbol{\alpha}\rangle$ is more appropriate to predict the impact of the foregoing policy on aggregate action if $\tilde{\gamma}$ is not small and $\gamma \mapsto \langle \mathbf{1}_n, \boldsymbol{y}^\star(\gamma)\rangle$ is not accurately enough approximated by an affine function. It is hardly surprising that $\langle \mathbf{1}_n, \boldsymbol{y}^\star(\tilde{\gamma})\rangle - (1/\beta)\langle \mathbf{1}_n, \boldsymbol{\alpha}\rangle$ may be negative, zero, or positive. This is illustrated in the continuation of Example 1.54.

Example 1.54 (cont'd) Figure 1.16 shows that $\gamma \mapsto \langle \mathbf{1}_n, \boldsymbol{y}^\star(\gamma)\rangle$ has a negative slope at zero, in particular, $\partial\langle \mathbf{1}_n, \boldsymbol{y}^\star\rangle/\partial\gamma|_{\gamma=0} = -1/4$. Simple algebra yields

$$\langle \mathbf{1}_n, \boldsymbol{y}^\star(\gamma)\rangle - (1/\beta)\langle \mathbf{1}_n, \boldsymbol{\alpha}\rangle = \frac{5\gamma^2 - 3\gamma}{2(\gamma^2 + 9\gamma + 6)}.$$

As mentioned before, the polynomial $\gamma^2 + 9\gamma + 6$ is positive on $(-1/4, +\infty)$. The zero set of the polynomial $5\gamma^2 - 3\gamma$ is equal to $\{0, 3/5\}$. It follows that the difference between $\langle \mathbf{1}_n, \boldsymbol{y}^\star(\gamma)\rangle$ and $(1/\beta)\langle \mathbf{1}_n, \boldsymbol{\alpha}\rangle$ is negative on $(0, 3/5)$ and positive on $(-1/4, 0)$ and $(3/5, +\infty)$. In Figure 1.16, $\langle \mathbf{1}_n, \boldsymbol{y}^\star(\gamma)\rangle - (1/\beta)\langle \mathbf{1}_n, \boldsymbol{\alpha}\rangle$ is equal to the vertical difference between the solid line and the dashed line, where the latter represents the graph of the constant function $\gamma \mapsto (1/\beta)\langle \mathbf{1}_n, \boldsymbol{\alpha}\rangle$. $\diamond$

Proposition 1.53 and the subsequent discussion can be summarized as follows. Regardless of the structure of Γ, which is determined by G, $\mathcal{Y}$, $\boldsymbol{\alpha}$, β, γ, and f, aggregate action is strictly increasing in $\boldsymbol{\alpha}$ if $\gamma \geq 0$. It is strictly decreasing in β if $\gamma \geq 0$ and, for example, $\mathcal{Y} \in \{\mathbb{R}_+, [0, \bar{v}]\}$ and $f = \mathrm{id}_{\mathcal{Y}}$. In general, it is not monotonic in γ. These results suggest that the policy maker should not target the players' common social cost or benefit parameter, that is, the players' common preference for conformist or anti-conformist behavior, in order to alter aggregate action in one direction or the other, especially if only part of the structure of Γ is known to him. Instead, he is advised to target the players' idiosyncrasies. The players' common private cost parameter may also represent a suitable target.

1.3.6.2 Targeting the players' connections

Besides the players' preference parameters, the policy maker may also consider targeting the players' connections.

Aggregate equilibrium action is in general not monotonic in the density of the digraph by which the players are connected. This is illustrated in the continuation

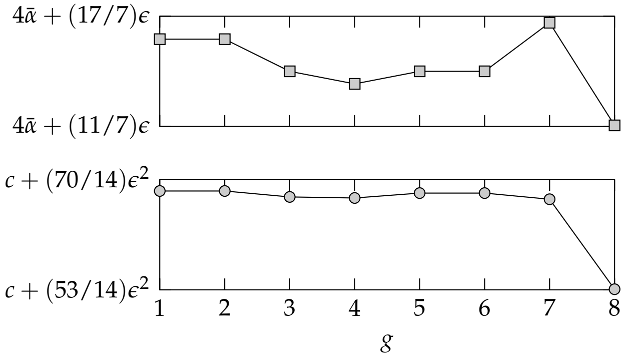

Figure 1.17. Aggregate equilibrium action (—□—) and equilibrium welfare (—○—) in $\Gamma(G_g)$ for $g \in \{1, \ldots, 8\}$ if $\epsilon > 0$, where $c := 2(\bar{\alpha} + (1/2)\epsilon)^2 > 0$ (Example 1.28)

of Example 1.28. A policy aimed at changing the digraph's density is therefore in general not expedient to change aggregate action in one direction or the other. A policy that decreases the digraph's density could, for example, consist in isolating a player (or a group of players) from all other players by cutting all his (or their) connections to all other players.

Example 1.28 (cont'd) Simple calculations show that if $\epsilon > 0$, then $\langle \mathbf{1}_n, \boldsymbol{y}^\star(G_1)\rangle = \langle \mathbf{1}_n, \boldsymbol{y}^\star(G_2)\rangle > \langle \mathbf{1}_n, \boldsymbol{y}^\star(G_3)\rangle > \langle \mathbf{1}_n, \boldsymbol{y}^\star(G_4)\rangle < \langle \mathbf{1}_n, \boldsymbol{y}^\star(G_5)\rangle = \langle \mathbf{1}_n, \boldsymbol{y}^\star(G_6)\rangle < \langle \mathbf{1}_n, \boldsymbol{y}^\star(G_7)\rangle > \langle \mathbf{1}_n, \boldsymbol{y}^\star(G_8)\rangle$, that is, the function $g \mapsto \langle \mathbf{1}_n, \boldsymbol{y}^\star(G_g)\rangle$ is not monotonic.[76] See Figure 1.17 for an illustration. Let G_0 and G_{12} denote the empty and complete digraph on $\mathcal{I}$, respectively. The NEs of the corresponding NALA games satisfy $\langle \mathbf{1}_n, \boldsymbol{y}^\star(G_0)\rangle = \langle \mathbf{1}_n, \boldsymbol{y}^\star(G_{12})\rangle = \langle \mathbf{1}_n, \boldsymbol{\alpha}\rangle$ (see (1.7) and Result 1.50.2). Since $\langle \mathbf{1}_n, \boldsymbol{y}^\star(G_3)\rangle = \langle \mathbf{1}_n, \boldsymbol{y}^\star(G_5)\rangle = \langle \mathbf{1}_n, \boldsymbol{y}^\star(G_6)\rangle = \langle \mathbf{1}_n, \boldsymbol{\alpha}\rangle$, the present example demonstrates that aggregate action can be lower or higher than the levels corresponding to an empty and a complete digraph (which are equal in the present example). ◇

1.3.6.3 Welfare effects of policy measures

A policy measure that alters aggregate action may also affect welfare. More specifically, if the policy maker alters some of the players' preference parameters or the topology of the digraph by which the players are connected, NE actions may change, aggregate action may change in turn, and welfare is likely to change as well.

Before discussing welfare effects of policy measures, a few words on notation and terminology are in order. Welfare at the NE $\boldsymbol{y}^\star(\boldsymbol{\alpha}, \beta, \gamma, f, G)$ of Γ, $w(\boldsymbol{y}^\star(\boldsymbol{\alpha}, \beta, \gamma, f, G))$, is called *equilibrium welfare* or, more precisely, *equilibrium welfare in Γ*. Depending on the context and situation, some or all of the arguments of $\boldsymbol{y}^\star$ in $w(\boldsymbol{y}^\star(\boldsymbol{\alpha}, \beta, \gamma, f, G))$ may be omitted.

76. Indeed, $\langle \mathbf{1}_n, \boldsymbol{y}^\star(G_1)\rangle = \langle \mathbf{1}_n, \boldsymbol{y}^\star(G_2)\rangle = 4\bar{\alpha} + (9/4)\epsilon$, $\langle \mathbf{1}_n, \boldsymbol{y}^\star(G_3)\rangle = \langle \mathbf{1}_n, \boldsymbol{y}^\star(G_5)\rangle = \langle \mathbf{1}_n, \boldsymbol{y}^\star(G_6)\rangle = \langle \mathbf{1}_n, \boldsymbol{\alpha}\rangle = 4\bar{\alpha} + 2\epsilon$, $\langle \mathbf{1}_n, \boldsymbol{y}^\star(G_4)\rangle = 4\bar{\alpha} + (78/41)\epsilon$, $\langle \mathbf{1}_n, \boldsymbol{y}^\star(G_7)\rangle = 4\bar{\alpha} + (195/82)\epsilon$, and $\langle \mathbf{1}_n, \boldsymbol{y}^\star(G_8)\rangle = 4\bar{\alpha} + (169/107)\epsilon$.

Suppose the policy maker's target τ for aggregate action is attainable by some family of preference parameters $\{(\tilde{\alpha}_i, \tilde{\beta}, \tilde{\gamma})\}_{i\in\mathcal{I}}$. If τ is implementable, equilibrium welfare changes by $w(\boldsymbol{y}^\star(\tilde{\boldsymbol{\alpha}}, \tilde{\beta}, \tilde{\gamma}, f, G)) - w(\boldsymbol{y}^\star(\boldsymbol{\alpha}, \beta, \gamma, f, G))$ once it is implemented. This difference may be negative, zero, or positive, depending on the structure of Γ, which encompasses $\{(\alpha_i, \beta, \gamma)\}_{i\in\mathcal{I}}$, and $\{(\tilde{\alpha}_i, \tilde{\beta}, \tilde{\gamma})\}_{i\in\mathcal{I}}$. The difference between $w(\boldsymbol{y}^\star(\tilde{\boldsymbol{\alpha}}, \tilde{\beta}, \tilde{\gamma}, f, G))$ and $w(\boldsymbol{y}^\star(\boldsymbol{\alpha}, \beta, \gamma, f, G))$ can be decomposed into three summands, namely,

$$\begin{aligned} &w\big(\boldsymbol{y}^\star(\tilde{\boldsymbol{\alpha}}, \tilde{\beta}, \tilde{\gamma}, f, G)\big) - w\big(\boldsymbol{y}^\star(\boldsymbol{\alpha}, \tilde{\beta}, \tilde{\gamma}, f, G)\big) \\ &\qquad\qquad + w\big(\boldsymbol{y}^\star(\boldsymbol{\alpha}, \tilde{\beta}, \tilde{\gamma}, f, G)\big) - w\big(\boldsymbol{y}^\star(\boldsymbol{\alpha}, \beta, \tilde{\gamma}, f, G)\big) \\ &\qquad\qquad\qquad\qquad + w\big(\boldsymbol{y}^\star(\boldsymbol{\alpha}, \beta, \tilde{\gamma}, f, G)\big) - w\big(\boldsymbol{y}^\star(\boldsymbol{\alpha}, \beta, \gamma, f, G)\big), \end{aligned}$$

which motivates the study of the presence or absence of monotonic relationships between the players' preference parameters and equilibrium welfare.

The following result characterizes equilibrium welfare in Γ.

Proposition 1.55 *The welfare function satisfies*

$$w(\boldsymbol{y}^\star) = \frac{\beta+\gamma}{2}\|\boldsymbol{f}(\boldsymbol{y}^\star)\|_2^2 - \frac{\gamma}{2}\|\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\|_2^2. \tag{1.51}$$

The partial derivatives of $w(\boldsymbol{y}^\star)$ with respect to $\boldsymbol{\alpha}$, β, and γ are given by

$$\frac{\partial w(\boldsymbol{y}^\star)}{\partial \boldsymbol{\alpha}} = \boldsymbol{\alpha}^\top \boldsymbol{J}(\beta,\gamma,G)^\top\big((\beta+\gamma)\boldsymbol{I}_n - \gamma\bar{\boldsymbol{A}}(G)^\top\bar{\boldsymbol{A}}(G)\big)\boldsymbol{J}(\beta,\gamma,G), \tag{1.52}$$

$$\frac{\partial w(\boldsymbol{y}^\star)}{\partial \beta} = -\frac{1}{2}\|\boldsymbol{f}(\boldsymbol{y}^\star)\|_2^2 - \gamma R(\boldsymbol{\alpha},\beta,\gamma,f,G), \tag{1.53}$$

and

$$\frac{\partial w(\boldsymbol{y}^\star)}{\partial \gamma} = -\frac{1}{2}\|\boldsymbol{f}(\boldsymbol{y}^\star)\|_2^2 + \frac{1}{2}\|\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\|_2^2 + \beta R(\boldsymbol{\alpha},\beta,\gamma,f,G), \tag{1.54}$$

where $R(\boldsymbol{\alpha},\beta,\gamma,f,G) := \langle(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G))\boldsymbol{f}(\boldsymbol{y}^\star), \boldsymbol{J}(\beta,\gamma,G)\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle$.

(1.55.1) If $\gamma = 0$, then $\partial w(\boldsymbol{y}^\star)/\partial\boldsymbol{\alpha} = (1/\beta)\boldsymbol{\alpha}^\top$.

(1.55.2) If $\gamma = 0$, then $\partial w(\boldsymbol{y}^\star)/\partial\beta = -\|\boldsymbol{\alpha}\|_2^2/(2\beta^2)$. If $\boldsymbol{\alpha} \neq \boldsymbol{0}_n$, $\gamma > 0$, and $\boldsymbol{0}_n \leq_c \boldsymbol{f}(\boldsymbol{y}^\star) \leq_c (1/\beta)\boldsymbol{\alpha}$, then $\partial w(\boldsymbol{y}^\star)/\partial\beta < 0$.

(1.55.3) $\partial w(\boldsymbol{y}^\star)/\partial\gamma|_{\gamma=0} = -\|\bar{\boldsymbol{A}}(G)\boldsymbol{\alpha} - \boldsymbol{\alpha}\|_2^2/(2\beta^2)$.

In general, equilibrium welfare is not monotonic in $\boldsymbol{\alpha}$ (see (1.52) and Example 1.56) and β (see (1.53) and Example 1.57). There are, however, several special cases in which a monotonic relationship exists. If $\gamma = 0$, then equilibrium welfare is strictly increasing in the players' idiosyncracies, provided that they are positive (Result 1.55.1). If $\gamma = 0$, then equilibrium welfare is strictly decreasing in the players' common private cost parameter β, provided that $\boldsymbol{\alpha} \neq \boldsymbol{0}_n$, which is equivalent to $\boldsymbol{f}(\boldsymbol{y}^\star) \neq \boldsymbol{0}_n$ (Result 1.55.2). It is also strictly decreasing in β if $\gamma > 0$, provided that $\boldsymbol{\alpha} \neq \mathbf{0}_n$ and the players' equilibrium actions are bounded below by $f^{-1}(0)$ and above by the actions they would play if they were weakly ex ante homogeneous (Result 1.35.1 and Remark 1.36 and Result 1.55.2).

Example 1.56 Suppose $\mathcal{A}(G) = \bigcup_{i \in \mathcal{I} \setminus \{1\}} \{(i,1)\}$, that is, G is star-shaped with arcs from every peripheral player $i \in \mathcal{I} \setminus \{1\}$ to the central player 1. In addition, suppose $\mathcal{Y} = \mathbb{R}_+$ and $f = \mathrm{id}_{\mathbb{R}_+}$. We have[77]

$$\forall j \in \mathcal{I} \quad \frac{\partial w(\boldsymbol{y}^\star)}{\partial \alpha_j} = \begin{cases} \dfrac{\gamma}{\beta(\beta+\gamma)} \displaystyle\sum_{i \in \mathcal{I} \setminus \{1\}} \alpha_i - \dfrac{(n-2)\gamma - \beta}{\beta(\beta+\gamma)} \alpha_j & \text{if } j = 1, \\ \dfrac{\gamma}{\beta(\beta+\gamma)} \alpha_1 + \dfrac{1}{\beta+\gamma} \alpha_j & \text{if } j \neq 1. \end{cases} \tag{1.55}$$

If $n > 2$, $\alpha_1 = \bar{\alpha} + \Delta\alpha$ and $\alpha_2 = \cdots = \alpha_n = \bar{\alpha}$ for some $(\bar{\alpha}, \Delta\alpha) \in \mathbb{R}_{++} \times \mathbb{R}$ with $\bar{\alpha} + \Delta\alpha > 0$, $\beta = 1$, and $\gamma > 1/(n-2)$, then

$$\frac{\partial w(\boldsymbol{y}^\star)}{\partial \alpha_1} < 0 \quad \Leftrightarrow \quad \Delta\alpha > \frac{1+\gamma}{(n-2)\gamma - 1} \bar{\alpha}. \tag{1.56}$$

In other words, provided that the number of players is large enough, the peripheral players have a common idiosyncrasy, the players' common private cost parameter is normalized to one, and the players' common social cost parameter exceeds a critical threshold, equilibrium welfare is strictly decreasing in the central player's idiosyncrasy if and only if the central player's idiosyncrasy is sufficiently different from the peripheral players' common idiosyncrasy. For example, if $n = 8$, $\alpha_1 = 1 + \Delta\alpha$ with $\Delta\alpha \in (-1, +\infty)$, $\alpha_2 = \cdots = \alpha_n = 1$, $\beta = 1$, and $\gamma = 3/4$, then $\partial w(\boldsymbol{y}^\star)/\partial \alpha_1 < 0$ if and only if $\Delta\alpha > 1/2$. In this example, equilibrium welfare is not monotonic in α_1, specifically, it is strictly increasing on $(0, 3/2)$ and strictly decreasing on $(3/2, +\infty)$. ◇

Example 1.57 Suppose $\mathcal{A}(G) = \bigcup_{i \in \mathcal{I} \setminus \{1\}} \{(i,1)\}$, $\mathcal{Y} = \mathbb{R}_+$, and $f = \mathrm{id}_{\mathbb{R}_+}$. We have[78]

$$\begin{aligned} \frac{\partial w(\boldsymbol{y}^\star)}{\partial \beta} = {} & \frac{(n-2)(2\beta+\gamma)\gamma - \beta^2}{2\beta^2(\beta+\gamma)^2} \alpha_1^2 \\ & - \frac{1}{2(\beta+\gamma)^2} \sum_{i \in \mathcal{I} \setminus \{1\}} \alpha_i^2 - \frac{(2\beta+\gamma)\gamma}{\beta^2(\beta+\gamma)^2} \sum_{i \in \mathcal{I} \setminus \{1\}} \alpha_1 \alpha_i. \end{aligned} \tag{1.57}$$

If $\alpha_1 = 1 + \Delta\alpha$ for some $\Delta\alpha \in (-1, +\infty)$ and $\alpha_2 = \cdots = \alpha_n = 1$, then

$$\left. \frac{\partial w(\boldsymbol{y}^\star)}{\partial \beta} \right|_{\beta=1} > 0 \quad \Leftrightarrow \quad \frac{(n-2)(2+\gamma)\gamma - 1}{2(1+\gamma)^2} (\Delta\alpha)^2 - \Delta\alpha - \frac{n}{2} > 0. \tag{1.58}$$

For example, if $n = 10$, $\alpha_1 = 1 + \Delta\alpha$ with $\Delta\alpha \in (-1, +\infty)$, $\alpha_2 = \cdots = \alpha_n = 1$, and $\gamma = 5/6$, then $\partial w(\boldsymbol{y}^\star)/\partial \beta|_{\beta=1} > 0$ if and if $\Delta\alpha > 11/7 \approx 1.57$. In this example, the

77. See Appendix D for the proofs of (1.55) and (1.56).
78. See Appendix D for the proofs of (1.57) and (1.58).

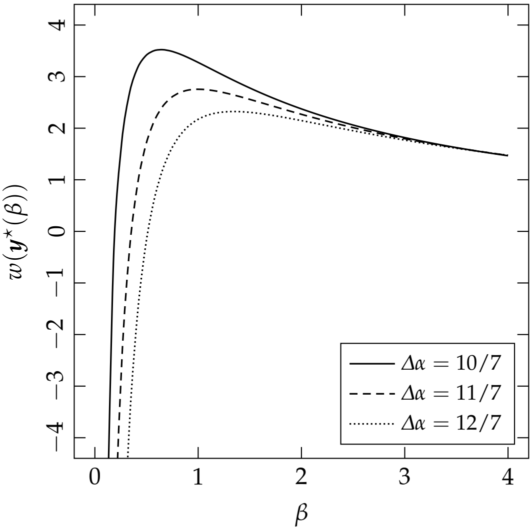

Figure 1.18. Equilibrium welfare as a function of β for different values of $\Delta\alpha$ (Example 1.57)

local behavior of $\beta \mapsto w(\boldsymbol{y}^\star(\beta))$ in a neighborhood of 1 depends critically on the value of $\Delta\alpha$. This is illustrated in Figure 1.18. It shows the graph of equilibrium welfare as a function of β if $n = 10$, $\alpha_1 = 1 + \Delta\alpha$, $\alpha_2 = \cdots = \alpha_n = 1$, and $\gamma = 5/6$ for three different values of $\Delta\alpha$: $\Delta\alpha \in \{10/7, 11/7, 12/7\}$. As can be seen from the figure, the slope of equilibrium welfare at $\beta = 1$ is negative for $\Delta\alpha = 10/7$, zero for $\Delta\alpha = 11/7$, and positive for $\Delta\alpha = 12/7$. Besides this, the figure shows that equilibrium welfare is not monotonic in β for all three values of $\Delta\alpha$. ◇

As regards equilibrium welfare as a function of γ, Result 1.55.3 characterizes its behavior in a neighborhood of zero; in particular, equilibrium welfare is constant if the players of Γ are weakly ex ante homogeneous, and it is strictly decreasing otherwise. A close inspection of (1.51) or (1.54) reveals that $\gamma \mapsto w(\boldsymbol{y}^\star(\gamma))$ is constant on its entire domain if the players of Γ are weakly ex ante homogeneous.[79] Apart from the preceding cases, more general results about the monotonic behavior of $\gamma \mapsto w(\boldsymbol{y}^\star(\gamma))$ are difficult to establish.[80] A related result about the slope of the secant line passing through zero and $\gamma \neq 0$ is, however, straightforward to obtain.[81]

Proposition 1.58 *If $\gamma \neq 0$, then*

$$w(\boldsymbol{y}^\star(\gamma)) - w(\boldsymbol{y}^\star(0)) = -\frac{\beta(\beta+\gamma)}{2\gamma}\left\| \boldsymbol{f}(\boldsymbol{y}^\star(\gamma)) - \frac{1}{\beta}\boldsymbol{\alpha} \right\|_2^2,$$

where $w(\boldsymbol{y}^\star(0)) = \|\boldsymbol{\alpha}\|_2^2/(2\beta)$.

79. Indeed, if $\boldsymbol{\alpha} = \bar{A}(G)\boldsymbol{\alpha}$, then $\boldsymbol{f}(\boldsymbol{y}^\star) = (1/\beta)\boldsymbol{\alpha} = (1/\beta)\bar{A}(G)\boldsymbol{\alpha} = \bar{A}(G)\boldsymbol{f}(\boldsymbol{y}^\star)$ (Result 1.35.1 and Remark 1.36), which implies that $w(\boldsymbol{y}^\star) = \|\boldsymbol{\alpha}\|_2^2/(2\beta)$, from which $\partial w(\boldsymbol{y}^\star)/\partial\gamma = 0$ follows.
80. See Section 1.3.8 for the discussion of an open problem.
81. A line connecting two points on a graph is called a secant line.

If the players of Γ are weakly ex ante homogeneous, then $w(\boldsymbol{y}^\star(\gamma)) = w(\boldsymbol{y}^\star(0))$ because $\boldsymbol{f}(\boldsymbol{y}^\star(\gamma)) = (1/\beta)\boldsymbol{\alpha}$ (Result 1.35.1 and Remark 1.36). If the players of Γ are not weakly ex ante homogeneous and have a preference for conformist (respectively, anti-conformist) behavior, then $w(\boldsymbol{y}^\star(\gamma)) < w(\boldsymbol{y}^\star(0))$ (respectively, $w(\boldsymbol{y}^\star(\gamma)) > w(\boldsymbol{y}^\star(0))$). Thus, in the context of the egocentric society described above, a policy that fosters conformist behavior will decrease welfare or leave it unchanged.

Apart from the foregoing literal interpretation, Proposition 1.58 also highlights the special role played by the empty digraph on $\mathcal{I}$ in equilibrium welfare because $w(\boldsymbol{y}^\star(\boldsymbol{\alpha}, \beta, \gamma, f, (\mathcal{I}, \emptyset))) = w(\boldsymbol{y}^\star(\boldsymbol{\alpha}, \beta, 0, f, G))$, that is, equilibrium welfare in a NALA game where all players are isolated is the same as equilibrium welfare in an otherwise identical NALA game where not all players are isolated and the players have a common preference for nonconformist behavior. In order to state the result about the special role of the empty digraph on $\mathcal{I}$ in equilibrium welfare, let $\mathcal{D}^\star$ denote the set of all digraphs D on $\mathcal{I}$ for which the generic NALA game $(\mathcal{I}, D, \mathcal{Y}, \{(\alpha_i, \beta, \gamma)\}_{i\in\mathcal{I}}, f)$ has a unique and interior NE $\boldsymbol{y}^\star(D) := \boldsymbol{y}^\star(\boldsymbol{\alpha}, \beta, \gamma, f, D)$.

Corollary 1.59 *Suppose $(1/\beta)\boldsymbol{\alpha} \in \operatorname{int}(\mathcal{Y}^n)$ or, equivalently, $(\mathcal{I}, \emptyset) \in \mathcal{D}^\star$. If the players of Γ have a preference for conformist (respectively, anti-conformist) behavior, then $(\mathcal{I}, \emptyset) \in \arg\max_{D\in\mathcal{D}^\star} w(\boldsymbol{y}^\star(D))$ (respectively, $(\mathcal{I}, \emptyset) \in \arg\min_{D\in\mathcal{D}^\star} w(\boldsymbol{y}^\star(D))$), that is, the empty digraph on $\mathcal{I}$ maximizes (respectively, minimizes) equilibrium welfare.*

The empty digraph on $\mathcal{I}$ provides an upper (respectively, lower) bound for equilibrium welfare if the players of Γ have a preference for conformist (respectively, anti-conformist) behavior; in contrast, the complete digraph on $\mathcal{I}$ does not provide a lower (respectively, upper) bound for equilibrium welfare, as demonstrated by Example 1.60. This asymmetry between the empty and complete digraph on $\mathcal{I}$ reflects that equilibrium welfare is in general not monotonic in the density of the digraph by which the players are connected, as further illustrated in the continuation of Example 1.28.

Example 1.60 Suppose $\mathcal{I} = \{1,2,3\}$, $\mathcal{Y} = [0, \bar{v}]$, $\beta = 1$, $\gamma \in \{-1/3, 1/3\}$, and $f = \mathrm{id}_{[0,\bar{v}]}$. Let $(\bar{\alpha}, \epsilon) \in \mathbb{R}^2_{++}$ be such that $(1/3)\bar{v} < \bar{\alpha}$ and $\bar{\alpha} + \epsilon < (2/3)\bar{v}$, and suppose $\alpha_1 = \bar{\alpha}$ and $\alpha_2 = \alpha_3 = \bar{\alpha} + \epsilon$. Let G_4 be the symmetric star-shaped digraph on $\mathcal{I}$ with center 1, that is, $\mathcal{A}(G_4) = \{(1,2), (1,3), (2,1), (3,1)\}$, and let G_6 be the complete digraph on $\mathcal{I}$. See Figure 1.9 for an illustration of G_4. For all $g \in \{4, 6\}$, let $\boldsymbol{y}^\star(G_g)$ denote the unique and interior NE of the NALA game $(\mathcal{I}, G_g, \mathcal{Y}, \{(\alpha_i, \beta, \gamma)\}_{i\in\mathcal{I}}, f)$. Simple calculations yield $w(\boldsymbol{y}^\star(G_4)) > w(\boldsymbol{y}^\star(G_6))$ if $\gamma = -1/3$ and $w(\boldsymbol{y}^\star(G_4)) < w(\boldsymbol{y}^\star(G_6))$ if $\gamma = 1/3$.[82] This shows that G_6 does not provide a lower (respectively, upper) bound for equilibrium welfare in case of conformist (respectively, anti-conformist) behavior. ◇

Example 1.28 (cont'd) Simple calculations show that $w(\boldsymbol{y}^\star(G_1)) = w(\boldsymbol{y}^\star(G_2)) > w(\boldsymbol{y}^\star(G_3)) > w(\boldsymbol{y}^\star(G_4)) < w(\boldsymbol{y}^\star(G_5)) = w(\boldsymbol{y}^\star(G_6)) > w(\boldsymbol{y}^\star(G_7)) > w(\boldsymbol{y}^\star(G_8))$

82. Indeed, if $\gamma = -1/3$, then $w(\boldsymbol{y}^\star(G_4)) = c + 4\epsilon^2$ and $w(\boldsymbol{y}^\star(G_6)) = c + (5/3)\epsilon^2$, and if $\gamma = 1/3$, then $w(\boldsymbol{y}^\star(G_4)) = c + (19/25)\epsilon^2$ and $w(\boldsymbol{y}^\star(G_6)) = c + (23/27)\epsilon^2$, where $c := (3/2)\bar{\alpha}^2 + 2\bar{\alpha}\epsilon$.

because $\epsilon \neq 0$, that is, the function $g \mapsto w(\boldsymbol{y}^\star(G_g))$ is not monotonic.[83] See Figure 1.17 for an illustration. $\diamond$

1.3.7 Extensions

In this section, I discuss three extensions of the generic NALA game introduced in Section 1.3.1. They concern the players' idiosyncrasies (Section 1.3.7.1), the players' common social cost or benefit parameter (Section 1.3.7.2), and the definition of the social component of the players' utility functions (Section 1.3.7.3).

1.3.7.1 Idiosyncrasies with local externalities

The generic NALA game $(\mathcal{I}, G, \mathcal{Y}, \{(\alpha_i, \beta, \gamma)\}_{i \in \mathcal{I}}, f)$ can be extended to the case where the players' idiosyncrasies are not constants but functions of more elementary idiosyncrasies, which may give rise to local externalities. In order to formalize this idea, let $\chi\colon \mathcal{I} \to \mathbb{R}$ be a function representing the players' *elementary idiosyncrasies*. A player's idiosyncrasy is assumed to depend on his own elementary idiosyncrasy and possibly on other players' elementary idiosyncrasies whose identities are given by the player's out-neighborhood in some digraph H on $\mathcal{I}$, which may or may not be equal to G. The function representing the players' idiosyncrasies is denoted by $\alpha(\chi, H)\colon \mathcal{I} \to \mathbb{R}$. The way in which $\alpha(\chi, H)$ depends on χ and H determines how a player's idiosyncrasy is affected by his out-neighbors' elementary idiosyncrasies. Of particular interest is the presence or absence of monotonic relationships. Definition LE introduces related terminology.[84]

Definition LE Player i's idiosyncrasy, $\alpha(\chi, H)(i)$, is said to exhibit *positive* (respectively, *negative*) *local externalities* if for all functions $\tilde{\chi}\colon \mathcal{I} \to \mathbb{R}$ with $\tilde{\chi}(i) = \chi(i)$ and $\{j \in \mathcal{N}_H^+(i) \mid \tilde{\chi}(j) \geq \chi(j)\} = \mathcal{N}_H^+(i)$, $\alpha(\tilde{\chi}, H)(i) \geq \alpha(\chi, H)(i)$ (respectively, $\alpha(\tilde{\chi}, H)(i) \leq \alpha(\chi, H)(i)$); and it is said to exhibit *strict positive* (respectively, *strict negative*) *local externalities* if for all functions $\tilde{\chi}\colon \mathcal{I} \to \mathbb{R}$ with $\tilde{\chi}(i) = \chi(i)$ and $\{j \in \mathcal{N}_H^+(i) \mid \tilde{\chi}(j) \geq \chi(j)\} = \mathcal{N}_H^+(i)$ and $\{j \in \mathcal{N}_H^+(i) \mid \tilde{\chi}(j) > \chi(j)\} \neq \emptyset$, $\alpha(\tilde{\chi}, H)(i) > \alpha(\chi, H)(i)$ (respectively, $\alpha(\tilde{\chi}, H)(i) < \alpha(\chi, H)(i)$).

In what follows, I give three examples of $\alpha(\chi, H)$, thereby assuming that H is not empty, that is, there exists a $k \in \mathcal{I}$ with $\deg_H^+(k) > 0$.

Example 1.61 Let $\alpha(\chi, H)\colon \mathcal{I} \to \mathbb{R}$ be defined by

$$\alpha(\chi, H)(i) := \begin{cases} \chi(i) & \text{if } \deg_H^+(i) = 0, \\ \chi(i) + \zeta_i \dfrac{\sum_{j \in \mathcal{N}_H^+(i)} \chi(j)}{\deg_H^+(i)} & \text{if } \deg_H^+(i) > 0, \end{cases} \tag{1.59}$$

83. Indeed, $w(\boldsymbol{y}^\star(G_1)) = w(\boldsymbol{y}^\star(G_2)) = c + (35/8)\epsilon^2$, $w(\boldsymbol{y}^\star(G_3)) = c + (181/42)\epsilon^2$, $w(\boldsymbol{y}^\star(G_4)) = c + (14{,}449/3{,}362)\epsilon^2$, $w(\boldsymbol{y}^\star(G_5)) = w(\boldsymbol{y}^\star(G_6)) = c + (235/54)\epsilon^2$, $w(\boldsymbol{y}^\star(G_7)) = c + (64{,}811/15{,}129)\epsilon^2$, and $w(\boldsymbol{y}^\star(G_8)) = c + (678{,}445/206{,}082)\epsilon^2$, where $c := 2(\bar{\alpha} + (1/2)\epsilon)^2 > 0$.

84. Galeotti et al. (2010) introduced a similar terminology to characterize a player's payoff function in a network game with incomplete information (see pp. 226–27).

where $\{\zeta_i\}_{i\in\mathcal{I}}$ is a family of real parameters. Player k's idiosyncrasy is independent of other players' elementary idiosyncrasies if $\zeta_k = 0$, and it exhibits strict positive (respectively, strict negative) local externalities if $\zeta_k > 0$ (respectively, $\zeta_k < 0$), where the magnitude of ζ_k is a measure of the strength of the local externalities. $\diamond$

Example 1.62 Let $\alpha(\chi, H)\colon \mathcal{I} \to \mathbb{R}$ be defined by $\alpha(\chi, H)(i) := \chi(i)$ if $\deg^+_H(i) = 0$ and $\alpha(\chi, H)(i) := \chi(i) \sum_{j\in\mathcal{N}^+_H(i)} \chi(j)$ if $\deg^+_H(i) > 0$. Player k's idiosyncrasy exhibits positive local externalities if $\chi \geq 0$ and strict positive local externalities if $\chi > 0$. $\diamond$

Example 1.63 Let $\alpha(\chi, H)\colon \mathcal{I} \to \mathbb{R}$ be defined by $\alpha(\chi, H)(i) := \chi(i)$ if $\deg^+_H(i) = 0$ and $\alpha(\chi, H)(i) := \chi(i) + \zeta \max\{\chi(j) \mid j \in \mathcal{N}^+_H(i)\}$ if $\deg^+_H(i) > 0$, where $\zeta \in \mathbb{R}$ is a parameter. Player k's idiosyncrasy exhibits positive (respectively, negative) local externalities if $\zeta > 0$ (respectively, $\zeta < 0$). $\diamond$

The function representing the players' idiosyncrasies may give rise to *exogenous effects*, wherein a player's NE action (action for short) varies directly with a statistic of the elementary idiosyncrasies of his out-neighbors in H.[85,86] This effect is to be contrasted with the dependence of a player's action on the elementary idiosyncrasies of his out-neighbors (and higher-order out-neighbors) in G. The dependence on other players' elementary idiosyncrasies is thereby a consequence of *endogenous effects*, wherein a player's action varies with the actions of his out-neighbors in G. Exogenous and endogenous effects represent two different channels through which a player's action is affected by other players' elementary idiosyncrasies, a direct and an indirect channel. The dependence of a player's action on his own elementary idiosyncrasy may give rise to *correlated effects*, wherein connected players play similar actions because they have similar elementary idiosyncrasies. The distinction between endogenous, exogenous, and correlated effects dates back to the original work of Manski (1993), who discusses identification of the three aforementioned effects—in particular, the non-identification of endogenous and exogenous effects—in the context of a statistical model that is commonly known as the *linear-in-means model*.

The theory of generic NALA games is based on the premise that the digraph G by which the players are connected is fixed. The same applies to the digraph H that admits of idiosyncrasies with local externalities. Both digraphs G and H may be seen as the result of a two-stage network formation game of complete information. At the outset, all players are isolated in G and H. In the first stage of the game, the players form arcs in G and H to other players. In the second stage of the game, the players choose their actions by playing a variant of a generic NALA game where the utility functions include a cost component for arc formation. The players' equilibrium utilities in this game represent the payoffs in the first stage of the game, that is, they define the players' value functions on the Cartesian square of the set of all digraphs on $\mathcal{I}$. The conception of a generic NALA game as the second stage of a

85. Exogenous effects are also referred to as *contextual effects*, especially but not exclusively in the sociological literature.

86. Examples of statistics include the mean (as in Example 1.61), the median, the mode, the sum (as in Example 1.62), the minimum, and the maximum (as in Example 1.63).

network formation game provides a rational for idiosyncrasies that admit of local externalities. They are necessary for a nonempty digraph to emerge if the first stage of the game involves only a single digraph, that is, in case G and H are identical, and the players have a preference for conformist behavior. The remainder of the section expands on this result.

Let $\mathcal{D}$ denote the set of all digraphs on $\mathcal{I}$. Suppose the function representing the players' elementary idiosyncrasies $\chi\colon \mathcal{I} \to \mathbb{R}$, the mapping $\alpha(\chi, \cdot)\colon \mathcal{D} \to \mathbb{R}^{\mathcal{I}}$, the parameters $\beta > 0$ and $\gamma > 0$, and the function $f \in \mathcal{F}(\mathcal{Y})$ are such that for all $D \in \mathcal{D}$, the generic NALA game $\Gamma(D) := (\mathcal{I}, D, \mathcal{Y}, \{(\alpha(\chi, D)(i), \beta, \gamma)\}_{i \in \mathcal{I}}, f)$ has a unique and interior NE $(y_1^\star(D), \ldots, y_n^\star(D))$.[87] For all $(i, D) \in \mathcal{I} \times \mathcal{D}$, let $u_i^\star(D)$ denote player i's *equilibrium utility* in $\Gamma(D)$, that is, his utility at the NE of $\Gamma(D)$, and let D_i denote the subdigraph of D in which player i has an empty out-neighborhood.

Proposition 1.64 *For all $D \in \mathcal{D}$ and for all $i \in \mathcal{I}$,*

$$u_i^\star(D) = \frac{\alpha(\chi, D)(i)^2}{2\beta} - \frac{\beta + \gamma}{2\beta\gamma}\Big(\alpha(\chi, D)(i) - \beta f\big(y_i^\star(D)\big)\Big)^2.$$

If the mapping $\alpha(\chi, \cdot)\colon \mathcal{D} \to \mathbb{R}^{\mathcal{I}}$ is constant, then, for all $D \in \mathcal{D}$ and for all $i \in \mathcal{I}$, $u_i^\star(D_i) \geq u_i^\star(D)$.

If the mapping $\alpha(\chi, \cdot)\colon \mathcal{D} \to \mathbb{R}^{\mathcal{I}}$ is constant, then an empty out-neighborhood is a weakly dominant strategy for all players. If in addition to the above condition the players' utility functions include a cost component for arc formation that is strictly increasing in a player's out-degree, then an empty out-neighborhood is a strictly dominant strategy for all players, which implies that the players remain isolated in G. Under the premise that the players have a preference for conformist behavior and arc formation is costly, a necessary condition for a nonempty digraph G to emerge from the network formation game is that the mapping $\alpha(\chi, \cdot)\colon \mathcal{D} \to \mathbb{R}^{\mathcal{I}}$ is not constant, which is necessary for idiosyncrasies to admit of strict positive or strict negative local externalities.[88]

1.3.7.2 Heterogeneous social cost or benefit parameters

In this section, I generalize the notion of a generic NALA game by relaxing the assumption that its players have a common social cost or benefit parameter γ. In particular, I assume that the players' preferences over $\mathcal{Y}^n$ can be represented by a family of utility functions $\{u_i\colon \mathcal{Y}^n \to \mathbb{R}\}_{i \in \mathcal{I}}$ that satisfies the following assumption.

Assumption U' For all $i \in \mathcal{I}$, player i's utility function $u_i\colon \mathcal{Y}^n \to \mathbb{R}$ is given by

$$u_i(y_1, \ldots, y_n) := p\big(f(y_i) \mid \alpha_i, \beta\big) + s_i\big(f(y_1), \ldots, f(y_n) \mid \gamma_i, G\big),$$

where $(\alpha_i, \beta, \gamma_i) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$ is a triple of parameters and $f \in \mathcal{F}(\mathcal{Y})$.[89]

87. The set of all functions with domain $\mathcal{I}$ and codomain $\mathbb{R}$ is denoted by $\mathbb{R}^{\mathcal{I}}$.

88. The players' idiosyncrasies do not admit of strict positive or strict negative local externalities if the mapping $\alpha(\chi, \cdot)\colon \mathcal{D} \to \mathbb{R}^{\mathcal{I}}$ is constant, that is, for all $D \in \mathcal{D}$, $\alpha(\chi, D) = \alpha(\chi, (\mathcal{I}, \varnothing))$.

89. See Assumption U for the definition of the private component function p and player i's social component function s_i.

Assumption U' leads to the notion of a (generic) NAHLA game.

Definition G' A *nonaffine heterogeneous local average game*, or *NAHLA game* for short, is a NALA game where the players' preferences over $\mathcal{Y}^n$ are represented by a family of utility functions $\{u_i\colon \mathcal{Y}^n \to \mathbb{R}\}_{i\in\mathcal{I}}$ that satisfies Assumption U'. A NAHLA game is denoted by the quintuple $(\mathcal{I}, G, \mathcal{Y}, \{(\alpha_i, \beta, \gamma_i)\}_{i\in\mathcal{I}}, f)$. A NAHLA game for which $\mathcal{Y}$ is left unspecified is referred to as a *generic NAHLA game*.

The remainder of this section is concerned with the existence of a unique and interior NE of the generic NAHLA game $\Gamma' := (\mathcal{I}, G, \mathcal{Y}, \{(\alpha_i, \beta, \gamma_i)\}_{i\in\mathcal{I}}, f)$. To this end, let $\alpha_{\min}$ and $\alpha_{\max}$ be defined as in Section 1.3.3, let $\gamma_{\min} := \min\{\gamma_i \,|\, i \in \mathcal{I}\}$ and $\gamma_{\max} := \max\{\gamma_i \,|\, i \in \mathcal{I}\}$, let $\boldsymbol{\alpha}$ be defined as in Section 1.3.2, and let $\boldsymbol{\gamma} := (\gamma_1, \dots, \gamma_n)$. In addition, let $\operatorname{diag}(\beta\mathbf{1}_n + \boldsymbol{\gamma})$ and $\operatorname{diag}(\boldsymbol{\gamma})$ denote the diagonal matrices of order n with the components in row i and column i equal to $\beta + \gamma_i$ and γ_i, respectively. Analogous to the case of NALA games, I state sufficient conditions for a unique and interior NE of Γ' separately for each type of action space: Proposition 1.65 covers the case $\mathcal{Y} = \mathbb{R}$ and admits of a negative and a nonnegative $\gamma_{\min}$; Proposition 1.66 covers the case $\mathcal{Y} = \mathbb{R}_+$ and is limited to a nonnegative $\gamma_{\min}$; and Proposition 1.67 deals with the case $\mathcal{Y} = [0, \bar{v}]$ and a nonnegative $\gamma_{\min}$.[90]

Proposition 1.65 *Suppose $\mathcal{Y} = \mathbb{R}$. The NAHLA game Γ' has a unique NE $\boldsymbol{y}^\star \in \mathbb{R}^n$, which is given by*

$$\boldsymbol{f}(\boldsymbol{y}^\star) = \big(\boldsymbol{I}_n - \operatorname{diag}(\beta\mathbf{1}_n + \boldsymbol{\gamma})^{-1}\operatorname{diag}(\boldsymbol{\gamma})\bar{\boldsymbol{A}}(G)\big)^{-1}\operatorname{diag}(\beta\mathbf{1}_n + \boldsymbol{\gamma})^{-1}\boldsymbol{\alpha}, \qquad (1.60)$$

if three conditions are satisfied: (1.65.1) $\beta > 0$, (1.65.2) $-\beta/2 < \gamma_{\min}$, and (1.65.3) f is not bounded below and above.

Proposition 1.66 *Suppose $\mathcal{Y} = \mathbb{R}_+$. The NAHLA game Γ' has a unique and interior NE $\boldsymbol{y}^\star \in \mathbb{R}^n_{++}$, which is given by (1.60), if four conditions are satisfied: (1.66.1) $\beta > 0$, (1.66.2) $\gamma_{\min} \geq 0$, (1.66.3) $\beta f(0) < \alpha_{\min}$, and (1.66.4) f is not bounded above.*

Proposition 1.67 *Suppose $\mathcal{Y} = [0, \bar{v}]$. The NAHLA game Γ' has a unique and interior NE $\boldsymbol{y}^\star \in (0, \bar{v})^n$, which is given by (1.60), if four conditions are satisfied: (1.67.1) $\beta > 0$, (1.67.2) $\gamma_{\min} \geq 0$, (1.67.3) $\beta f(0) < \alpha_{\min}$, and (1.67.4) $\alpha_{\max} < \beta f(\bar{v})$.*

The conditions of Proposition 1.65 are similar to those of Proposition 1.13: Conditions 1.13.1 and 1.65.1 and Conditions 1.13.4 and 1.65.3 are the same; Conditions 1.65.1 and 1.65.2 imply that $\beta + \gamma_{\min} > 0$, which is similar to Condition 1.13.2; and Conditions 1.65.1 and 1.65.2 imply that $\boldsymbol{I}_n - \operatorname{diag}(\beta\mathbf{1}_n + \boldsymbol{\gamma})^{-1}\operatorname{diag}(\boldsymbol{\gamma})\bar{\boldsymbol{A}}(G)$ is nonsingular, whereas Conditions 1.13.1, 1.13.2, and 1.13.3 imply that $\boldsymbol{I}_n - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G)$

90. The two cases where $\mathcal{Y}$ is equal to $\mathbb{R}_+$ or $[0, \bar{v}]$ and $\gamma_{\min}$ is negative are omitted from the present discussion. As regards the case $\mathcal{Y} = \mathbb{R}_+$ and $\gamma_{\min} \leq \gamma_{\max} < 0$, the NAHLA game Γ' has a unique interior NE if four conditions are satisfied: $\beta > 0$, $-\beta/2 < \gamma_{\min}$, an inequality that is similar in nature to the left inequality of (1.17), and f is not bounded above. As regards the case $\mathcal{Y} = [0, \bar{v}]$ and $\gamma_{\min} \leq \gamma_{\max} < 0$, the NAHLA game Γ' has a unique interior NE if three conditions are satisfied: $\beta > 0$, $-\beta/2 < \gamma_{\min}$, and two inequalities that are similar in nature to the inequalities (1.17).

is nonsingular. The conditions of Propositions 1.16 and 1.66 are almost identical. The same applies to the conditions of Propositions 1.14 and 1.67.

The system of equations governing the players' NE actions in a NAHLA game is similar in structure to that of a NALA game. This can be seen from a comparison of (1.60) and (1.7), where the former system, with both vector $\operatorname{diag}(\beta\mathbf{1}_n+\boldsymbol{\gamma})^{-1}\boldsymbol{\alpha}$ and diagonal matrix $\operatorname{diag}(\beta\mathbf{1}_n+\boldsymbol{\gamma})^{-1}\operatorname{diag}(\boldsymbol{\gamma})$ written out in full, reads as follows:

$$\boldsymbol{f}(\boldsymbol{y}^\star)=\left(\boldsymbol{I}_n-\begin{pmatrix}\frac{\gamma_1}{\beta+\gamma_1} & & 0\\ & \ddots & \\ 0 & & \frac{\gamma_n}{\beta+\gamma_n}\end{pmatrix}\bar{\boldsymbol{A}}(G)\right)^{-1}\begin{pmatrix}\frac{\alpha_1}{\beta+\gamma_1}\\ \vdots\\ \frac{\alpha_n}{\beta+\gamma_n}\end{pmatrix}.$$

An alternative representation of $\boldsymbol{f}(\boldsymbol{y}^\star)$, which is more compact than (1.60), is given by[91]

$$\boldsymbol{f}(\boldsymbol{y}^\star)=\Big(\beta\boldsymbol{I}_n-\operatorname{diag}(\boldsymbol{\gamma})\big(\bar{\boldsymbol{A}}(G)-\boldsymbol{I}_n\big)\Big)^{-1}\boldsymbol{\alpha}. \tag{1.61}$$

The generic NAHLA game Γ' may be generalized further by relaxing the assumption of a common private cost parameter. The resulting game, which is denoted by $(\mathcal{I},G,\mathcal{Y},\{(\alpha_i,\beta_i,\gamma_i)\}_{i\in\mathcal{I}},f)$, is strategically equivalent to the generic NAHLA game $(\mathcal{I},G,\mathcal{Y},\{(\alpha_i/\beta_i,1,\gamma_i/\beta_i)\}_{i\in\mathcal{I}},f)$ if $\{\beta_i \mid i\in\mathcal{I}\}\subset\mathbb{R}_{++}$.[92] The generic NAHLA game Γ' is strategically equivalent to the generic NAHLA game $(\mathcal{I},G,\mathcal{Y},\{(\alpha_i/\beta,1,\gamma_i/\beta)\}_{i\in\mathcal{I}},f)$ if $\beta>0$. The implicit normalization $\beta=1$ is, however, not without loss of generality from the perspective of a policy maker because it involves the loss of a policy variable.

1.3.7.3 Dichotomous neighborhoods

The extension discussed hereinafter is motivated by the works of social anthropologist A. L. Epstein and sociologist M. S. Granovetter.

91. A representation of $\boldsymbol{f}(\boldsymbol{y}^\star)$ by (1.60) in lieu of (1.61) has several merits, especially if Condition G is satisfied, which will be assumed hereinafter. First, (1.60) points to the central role played by the matrix $\boldsymbol{I}_n-\operatorname{diag}(\beta\mathbf{1}_n+\boldsymbol{\gamma})^{-1}\operatorname{diag}(\boldsymbol{\gamma})\bar{\boldsymbol{A}}(G)$, which is a nonsingular M-matrix with a nonnegative inverse that is bounded below by $\boldsymbol{I}_n$ if $\beta>0$ and $\gamma_{\min}\geq 0$, in establishing $\boldsymbol{y}^\star\in\operatorname{int}(\mathcal{Y}^n)$ if $\mathcal{Y}=\mathbb{R}_+$ or $\mathcal{Y}=[0,\bar{v}]$. The matrix $\beta\boldsymbol{I}_n-\operatorname{diag}(\boldsymbol{\gamma})(\bar{\boldsymbol{A}}(G)-\boldsymbol{I}_n)$ in the representation (1.61) is by contrast not an M-matrix if $\beta>0$ and $\gamma_{\min}\geq 0$ because $\bar{\boldsymbol{A}}(G)-\boldsymbol{I}_n$ is not nonnegative. Second, (1.61) may convey the false impression that $\gamma_{\max}$ must be bounded above for Γ' to have a unique and interior NE. As to that, Lemma B.3 implies that the matrix $\beta\boldsymbol{I}_n-\operatorname{diag}(\boldsymbol{\gamma})(\bar{\boldsymbol{A}}(G)-\boldsymbol{I}_n)$ is nonsingular if $\beta>0$ and $0\leq\gamma_{\min}\leq\gamma_{\max}<\beta/2$. Indeed, if $\beta>0$ and $0\leq\gamma_{\min}\leq\gamma_{\max}<\beta/2$, then $\rho((1/\beta)\operatorname{diag}(\boldsymbol{\gamma})(\bar{\boldsymbol{A}}(G)-\boldsymbol{I}_n))\leq(1/\beta)\|\operatorname{diag}(\boldsymbol{\gamma})\|_\infty\|\bar{\boldsymbol{A}}(G)-\boldsymbol{I}_n\|_\infty=2\gamma_{\max}/\beta<1$ (Lemma B.7 and (D.96)), which implies that the matrix $\beta(\boldsymbol{I}_n-(1/\beta)\operatorname{diag}(\boldsymbol{\gamma})(\bar{\boldsymbol{A}}(G)-\boldsymbol{I}_n))=\beta\boldsymbol{I}_n-\operatorname{diag}(\boldsymbol{\gamma})(\bar{\boldsymbol{A}}(G)-\boldsymbol{I}_n)$ is nonsingular (Lemma B.3). In contrast, the matrices $\operatorname{diag}(\beta\mathbf{1}_n+\boldsymbol{\gamma})$ and $\boldsymbol{I}_n-\operatorname{diag}(\beta\mathbf{1}_n+\boldsymbol{\gamma})^{-1}\operatorname{diag}(\boldsymbol{\gamma})\bar{\boldsymbol{A}}(G)$ are nonsingular if $\beta>0$ and $\gamma_{\min}\geq 0$, in which case $(\boldsymbol{I}_n-\operatorname{diag}(\beta\mathbf{1}_n+\boldsymbol{\gamma})^{-1}\operatorname{diag}(\boldsymbol{\gamma})\bar{\boldsymbol{A}}(G))^{-1}\operatorname{diag}(\beta\mathbf{1}_n+\boldsymbol{\gamma})^{-1}$ is equal to $(\beta\boldsymbol{I}_n-\operatorname{diag}(\boldsymbol{\gamma})(\bar{\boldsymbol{A}}(G)-\boldsymbol{I}_n))^{-1}$. Indeed, if $\beta>0$ and $\gamma_{\min}\geq 0$, then $\operatorname{diag}(\beta\mathbf{1}_n+\boldsymbol{\gamma})^{-1}$ exists and $\rho(\operatorname{diag}(\beta\mathbf{1}_n+\boldsymbol{\gamma})^{-1}\operatorname{diag}(\boldsymbol{\gamma})\bar{\boldsymbol{A}}(G))\leq\|\operatorname{diag}(\beta\mathbf{1}_n+\boldsymbol{\gamma})^{-1}\operatorname{diag}(\boldsymbol{\gamma})\|_\infty\|\bar{\boldsymbol{A}}(G)\|_\infty=\max\{\gamma_i/(\beta+\gamma_i)\mid i\in\mathcal{I}\}<1$ (Lemmata B.7 and B.8), which implies that the matrix $\boldsymbol{I}_n-\operatorname{diag}(\beta\mathbf{1}_n+\boldsymbol{\gamma})^{-1}\operatorname{diag}(\boldsymbol{\gamma})\bar{\boldsymbol{A}}(G)$ is nonsingular (Lemma B.3).

92. Two games are called strategically equivalent if they have the same sets of NEs.

In his anthropological study of urban social organization in the Central African town Ndola, Epstein (1961) distinguishes between a person's effective and extended network, where the effective network consists of those with whom one "interacts most intensely and most regularly, and who are therefore also likely to come to know one another" (p. 57) and the remainder of one's network of social ties constitutes the extended network. This nomenclature of social ties bears some resemblance to Granovetter's (1973) distinction between a person's weak and strong ties—to be interpreted as the result of dichotomizing the strength of a tie defined as a "combination of the amount of time, the emotional intensity, the intimacy (mutual confiding), and the reciprocal services which characterize the tie" (p. 1361).[93] Both nomenclatures presuppose that a person's behavior is affected and shaped by two different types of social ties. In the context of NALA games, this translates into the assumption that a player can have two types of out-neighbors, the actions of whom define two social norms, deviations from which bring about different social costs or benefits. To put it another way, social pressure in social interactions may come from two different sources, for example friends and acquaintances, and in varying degree, for example, high and low.

In order to formalize the extension outlined above, consider the standard setup of a generic NALA game $(\mathcal{I}, G, \mathcal{Y}, \{(\alpha_i, \beta, \gamma)\}_{i\in\mathcal{I}}, f)$, where the players are connected by a digraph G on $\mathcal{I}$. Each player's out-neighborhood in G is divided into two disjoint sets, following the terminology of Granovetter (1973), a set of *weak* and a set of *strong* out-neighbors.[94] It follows that the arc set of G is divided into two disjoint sets. The corresponding digraphs on $\mathcal{I}$ encode the identities of the players' weak and strong out-neighbors in G and are denoted by G_w and G_s, respectively. Alternatively, G_w and G_s may result from a partition of $\mathcal{I}$ into two subsets of players, for example, according to gender,[95] academic, economic, cultural, national, or religious affiliation, political allegiance, or ethnical affinity, as illustrated by the following stylized example.

Example 1.68 Let $\mathcal{I}$ be partitioned into blue and green players, and let arcs between players of the same color be encoded by G_s and those between players of different color by G_w. $\diamond$

The players' preferences over $\mathcal{Y}^n$ are assumed to be representable by a family of utility functions $\{u_i\colon \mathcal{Y}^n \to \mathbb{R}\}_{i\in\mathcal{I}}$ that satisfies the following assumption.

Assumption U'' For all $i \in \mathcal{I}$, player i's utility function $u_i\colon \mathcal{Y}^n \to \mathbb{R}$ is given by

$$u_i(y_1,\ldots,y_n) := p\big(f(y_i) \mid \alpha_i, \beta\big) + s_i\big(f(y_1),\ldots,f(y_n) \mid \gamma_w, G_w\big)$$

93. Granovetter's (1973) contribution lies in distinguishing between weak and strong ties and, more important, in recognizing the importance of weak ties for the diffusion of information about job openings and for community organization.

94. It is not of importance which attributive adjectives are used to designate the two different types of out-neighbors, in particular, being a weak or a strong out-neighbor shall not carry a connotation of any kind whatsoever.

95. For example, Cohen-Cole (2006) argues that "a teenage girl (boy) might care differently about what other young women (men) do than about what young men (women) do—boys and girls are thus important to be considered as distinct reference groups. For example, a girl might be more prone to smoke if girls in her school do so, but less (or more) likely to if the boys do so." (p. 158)

$$+ s_i\big(f(y_1), \ldots, f(y_n) \bigm| \gamma_s, G_s\big),$$

where $(\alpha_i, \beta, \gamma_w, \gamma_s) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^2$ is a quadruple of parameters and $f \in \mathcal{F}(\mathcal{Y})$.[96]

Assumption U'' leads to the notion of a (generic) NALA game with weak and strong ties.

Definition G'' A *NALA game with weak and strong ties* is a NALA game where the players are connected by two digraphs G_w and G_s on $\mathcal{I}$ whose arc sets are disjoint and the players' preferences over $\mathcal{Y}^n$ are represented by a family of utility functions $\{u_i \colon \mathcal{Y}^n \to \mathbb{R}\}_{i \in \mathcal{I}}$ that satisfies Assumption U''. A NALA game with weak and strong ties is denoted by the sextuple $(\mathcal{I}, G_w, G_s, \mathcal{Y}, \{(\alpha_i, \beta, \gamma_w, \gamma_s)\}_{i \in \mathcal{I}}, f)$. A NALA game with weak and strong ties for which $\mathcal{Y}$ is left unspecified is referred to as a *generic NALA game with weak and strong ties*.

Let $\Gamma'' := (\mathcal{I}, G_w, G_s, \mathcal{Y}, \{(\alpha_i, \beta, \gamma_w, \gamma_s)\}_{i \in \mathcal{I}}, f)$ be a generic NALA game with weak and strong ties, and let $\alpha_{\min}$, $\alpha_{\max}$, and $\boldsymbol{\alpha}$ be defined as in Section 1.3.7.2. Propositions 1.69, 1.70, and 1.71 give sufficient conditions for the existence of a unique and interior NE of Γ''. Apart from the case where $\mathcal{Y} = \mathbb{R}$, the results are limited to nonnegative γ_w and γ_s.

Proposition 1.69 *Suppose $\mathcal{Y} = \mathbb{R}$. The NALA game with weak and strong ties Γ'' has a unique NE $\boldsymbol{y}^\star \in \mathbb{R}^n$, which is given by*

$$\boldsymbol{f}(\boldsymbol{y}^\star) = \frac{1}{\beta + \gamma_w + \gamma_s} \left(\boldsymbol{I}_n - \frac{1}{\beta + \gamma_w + \gamma_s} \big(\gamma_w \bar{\boldsymbol{A}}(G_w) + \gamma_s \bar{\boldsymbol{A}}(G_s) \big) \right)^{-1} \boldsymbol{\alpha}, \tag{1.62}$$

if three conditions are satisfied: (1.69.1) $\beta > 0$, (1.69.2) $(\gamma_w, \gamma_s) \in \{(a, b) \in \mathbb{R}^2 \mid \min\{a, b, a + b\} > -\beta/2\}$ (see Figure 1.19), and (1.69.3) f is not bounded below and above.

Proposition 1.70 *Suppose $\mathcal{Y} = \mathbb{R}_+$. The NALA game with weak and strong ties Γ'' has a unique and interior NE $\boldsymbol{y}^\star \in \mathbb{R}^n_{++}$, which is given by (1.62), if four conditions are satisfied: (1.70.1) $\beta > 0$, (1.70.2) $\gamma_w \geq 0$ and $\gamma_s \geq 0$, (1.70.3) $\beta f(0) < \alpha_{\min}$, and (1.70.4) f is not bounded above.*

Proposition 1.71 *Suppose $\mathcal{Y} = [0, \bar{v}]$. The NALA game with weak and strong ties Γ'' has a unique and interior NE $\boldsymbol{y}^\star \in (0, \bar{v})^n$, which is given by (1.62), if four conditions are satisfied: (1.71.1) $\beta > 0$, (1.71.2) $\gamma_w \geq 0$ and $\gamma_s \geq 0$, (1.71.3) $\beta f(0) < \alpha_{\min}$, and (1.71.4) $\alpha_{\max} < \beta f(\bar{v})$.*

An alternative representation of $\boldsymbol{f}(\boldsymbol{y}^\star)$, which is more compact than (1.62), is given by

$$\boldsymbol{f}(\boldsymbol{y}^\star) = \Big(\beta \boldsymbol{I}_n - \gamma_w \big(\bar{\boldsymbol{A}}(G_w) - \boldsymbol{I}_n \big) - \gamma_s \big(\bar{\boldsymbol{A}}(G_s) - \boldsymbol{I}_n \big) \Big)^{-1} \boldsymbol{\alpha}. \tag{1.63}$$

With the normalization $\beta = 1$, (1.63) is equivalent to

$$\boldsymbol{f}(\boldsymbol{y}^\star) = \boldsymbol{\alpha} + \gamma_w \big(\bar{\boldsymbol{A}}(G_w) - \boldsymbol{I}_n \big) \boldsymbol{f}(\boldsymbol{y}^\star) + \gamma_s \big(\bar{\boldsymbol{A}}(G_s) - \boldsymbol{I}_n \big) \boldsymbol{f}(\boldsymbol{y}^\star).$$

96. See Assumption U for the definition of the private component function p and player i's social component function s_i.

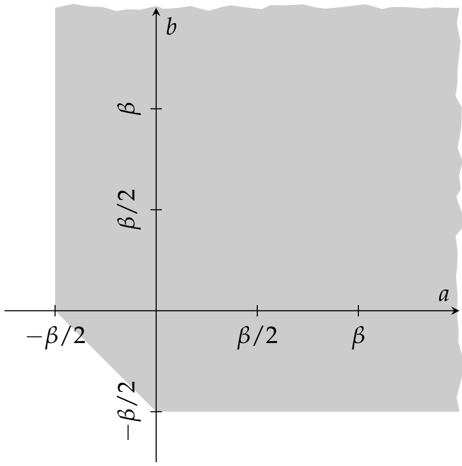

Figure 1.19. The set $\{(a, b) \in \mathbb{R}^2 \mid \min\{a, b, a+b\} > -\beta/2\}$ (Proposition 1.69)

1.3.8 An open problem

In this section, I discuss an open problem about the monotonicity of equilibrium welfare as a function of the players' common social cost or benefit parameter.

Let $\Gamma := (\mathcal{I}, G, \mathcal{Y}, \{(\alpha_i, \beta, \gamma)\}_{i\in\mathcal{I}}, f)$ be a generic NALA game, and let $\boldsymbol{\alpha}$ be defined as in Section 1.3.2. Suppose $\beta > 0$ and $\gamma > -\beta/2$. In addition, suppose Γ has a unique and interior NE $\boldsymbol{y}^\star$, which is given by (1.7). In accordance with the notation introduced in Section 1.3.6.3, equilibrium welfare is written as $w(\boldsymbol{y}^\star)$. The Jacobian (matrix) $\boldsymbol{J}(\beta, \gamma, G)$ is abbreviated to $\boldsymbol{J}$.

The open problem reads as follows.

Conjecture 1.72 If G is symmetric, then $\partial w(\boldsymbol{y}^\star)/\partial\gamma \leq 0$.

If Condition G is not satisfied, then $\partial w(\boldsymbol{y}^\star)/\partial\gamma = 0$ according to (1.54). According to Result 1.55.3, $\partial w(\boldsymbol{y}^\star)/\partial\gamma|_{\gamma=0} \leq 0$. Thus, apart from assuming that G is symmetric, I may also assume that Condition G is satisfied and $\gamma \neq 0$ in what follows.

The subsequent discussion of Conjecture 1.72 is structured as follows. First, I show that $\partial w(\boldsymbol{y}^\star)/\partial\gamma$ is a quadratic form in $\boldsymbol{f}(\boldsymbol{y}^\star) - \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)$. Second, I show that Conjecture 1.72 is true if the following conjecture is true.

Conjecture 1.73 Let $\boldsymbol{H}$ and $\boldsymbol{K}$ be real square matrices of the same order. If $\boldsymbol{H}$ is symmetric and positive definite, $\boldsymbol{K}$ is similar to a symmetric and positive semidefinite but not positive definite matrix, and $\sigma(\boldsymbol{HK}) \subset \mathbb{R}$, then $\min \sigma(\boldsymbol{HK}) \geq 0$.

As regards Conjecture 1.73, a similar statement involving stronger assumptions is true, as the following lemma demonstrates.

Lemma 1.74 *Let $\boldsymbol{H}$ and $\boldsymbol{K}$ be real symmetric matrices of the same order. If $\boldsymbol{H}$ is positive definite and $\boldsymbol{K}$ is positive semidefinite but not positive definite, then $\sigma(\boldsymbol{HK}) \subset \mathbb{R}$ and $\min \sigma(\boldsymbol{HK}) \geq 0$.*

The partial derivative of $w(\boldsymbol{y}^\star)$ with respect to γ has a representation as a quadratic form in $\boldsymbol{f}(\boldsymbol{y}^\star) - \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)$. Indeed, we have

$$\begin{aligned}\frac{\partial w(\boldsymbol{y}^\star)}{\partial\gamma} &= -\frac{1}{2}\|\boldsymbol{f}(\boldsymbol{y}^\star)\|_2^2 + \frac{1}{2}\|\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\|_2^2 + \beta\langle \boldsymbol{f}(\boldsymbol{y}^\star) - \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \boldsymbol{J}\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle \\ &= -\frac{1}{2}\langle \boldsymbol{f}(\boldsymbol{y}^\star) - \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \boldsymbol{f}(\boldsymbol{y}^\star) + \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle \\ &\quad + \beta\langle \boldsymbol{f}(\boldsymbol{y}^\star) - \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \boldsymbol{J}\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle \\ &= -\frac{1}{2}\big\langle \big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)\boldsymbol{f}(\boldsymbol{y}^\star), \big(\boldsymbol{I}_n + \bar{\boldsymbol{A}}(G) - 2\beta\boldsymbol{J}\bar{\boldsymbol{A}}(G)\big)\boldsymbol{f}(\boldsymbol{y}^\star)\big\rangle \\ &= -\frac{1}{2}\big\langle \big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)\boldsymbol{f}(\boldsymbol{y}^\star), \big(2(\beta+\gamma)\boldsymbol{J} - \boldsymbol{I}_n\big)\big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)\boldsymbol{f}(\boldsymbol{y}^\star)\big\rangle \\ &= -\frac{1}{2}\big\langle \big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)\boldsymbol{f}(\boldsymbol{y}^\star), \big((\beta+\gamma)(\boldsymbol{J}+\boldsymbol{J}^\top) - \boldsymbol{I}_n\big)\big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)\boldsymbol{f}(\boldsymbol{y}^\star)\big\rangle,\end{aligned}$$

where the first equality is according to (1.54), the second equality follows from the real polarization identity (see, for example, Kubrusly 2011, Proposition 5.4), and the second to last equality follows from (D.138), in particular,

$$\boldsymbol{I}_n + \bar{\boldsymbol{A}}(G) - 2\beta\boldsymbol{J}\bar{\boldsymbol{A}}(G) = \big(2(\beta+\gamma)\boldsymbol{J} - \boldsymbol{I}_n\big)\big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big).$$

Next, I show that the spectrum of $(\beta+\gamma)(\boldsymbol{J}+\boldsymbol{J}^\top) - \boldsymbol{I}_n$, which is a symmetric matrix of order n, is related to the spectrum of the product of two square matrices $\boldsymbol{H}$ and $\boldsymbol{K}$ of order $2n$, where $\boldsymbol{H}$ is symmetric and positive definite but not diagonal and $\boldsymbol{K}$ is in general not symmetric but similar to a symmetric and positive semidefinite but not positive definite matrix.

First, I show that $\bar{\boldsymbol{A}}(G)$ is similar to a symmetric matrix. Let $\boldsymbol{D} := \boldsymbol{D}^+(\mathrm{sl}(G))$ denote the out-degree matrix for $\mathrm{sl}(G)$, that is, $\boldsymbol{D}$ is a diagonal matrix of order n with

$$\forall i \in \mathcal{I} \quad [\boldsymbol{D}]_{i,i} = \deg^+_{\mathrm{sl}(G)}(i) = \begin{cases} 1 & \text{if } \deg^+_G(i) = 0, \\ \deg^+_G(i) & \text{if } \deg^+_G(i) > 0. \end{cases}$$

Evidently, $\boldsymbol{D}$ is symmetric and positive definite (and therefore nonsingular). It follows that $\boldsymbol{D}$ has a unique positive definite (and therefore nonsingular) square root $\sqrt{\boldsymbol{D}}$, which is diagonal (and therefore symmetric). According to the definitions of $\dot{\boldsymbol{A}}(\mathrm{sl}(G))$ and $\bar{\boldsymbol{A}}(G)$, $\bar{\boldsymbol{A}}(G) = \boldsymbol{D}^{-1}\dot{\boldsymbol{A}}(\mathrm{sl}(G))$. We have

$$\sqrt{\boldsymbol{D}}\bar{\boldsymbol{A}}(G)\sqrt{\boldsymbol{D}}^{-1} = \sqrt{\boldsymbol{D}}\boldsymbol{D}^{-1}\dot{\boldsymbol{A}}\big(\mathrm{sl}(G)\big)\sqrt{\boldsymbol{D}}^{-1} = \sqrt{\boldsymbol{D}}^{-1}\dot{\boldsymbol{A}}\big(\mathrm{sl}(G)\big)\sqrt{\boldsymbol{D}}^{-1}, \tag{1.64}$$

that is, $\bar{\boldsymbol{A}}(G)$ is similar to the symmetric matrix $\sqrt{\boldsymbol{D}}^{-1}\dot{\boldsymbol{A}}(\mathrm{sl}(G))\sqrt{\boldsymbol{D}}^{-1}$. It follows that $\sigma(\bar{\boldsymbol{A}}(G)) \subset \mathbb{R}$. Moreover, $\sigma(\bar{\boldsymbol{A}}(G)) \subset [-1,1]$ because $\rho(\bar{\boldsymbol{A}}(G)) = 1$.

Second, I show that

$$\Sigma_1 := \sigma\left(\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1} - \frac{1}{2}\boldsymbol{I}_n\right) \subset \mathbb{R}_{++}. \tag{1.65}$$

We have $\Sigma_1 \subset \mathbb{R}$ because $(\boldsymbol{I}_n - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G))^{-1} - (1/2)\boldsymbol{I}_n$ is similar to the symmetric matrix $(\boldsymbol{I}_n - \gamma/(\beta+\gamma)\sqrt{\boldsymbol{D}}^{-1}\dot{\boldsymbol{A}}(\mathrm{sl}(G))\sqrt{\boldsymbol{D}}^{-1})^{-1} - (1/2)\boldsymbol{I}_n$ according to (1.64). Note that, for all $\lambda \in \sigma(\bar{\boldsymbol{A}}(G))$, $1 - \gamma\lambda/(\beta+\gamma) \neq 0$ because $\gamma > -\beta/2$ and $\sigma(\bar{\boldsymbol{A}}(G)) \subset [-1,1]$. Let

$$\Sigma_2 := \left\{ \frac{1}{2}\left(1 + \frac{\gamma}{\beta+\gamma}\lambda\right)\left(1 - \frac{\gamma}{\beta+\gamma}\lambda\right)^{-1} \,\middle|\, \lambda \in \sigma\big(\bar{\boldsymbol{A}}(G)\big) \right\}.$$

I show that $\Sigma_1 = \Sigma_2$. Let $(\mu, \boldsymbol{w})$ be an eigenpair of $(\boldsymbol{I}_n - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G))^{-1} - (1/2)\boldsymbol{I}_n$, that is, $(\boldsymbol{I}_n - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G))^{-1}\boldsymbol{w} - (1/2)\boldsymbol{w} = \mu\boldsymbol{w}$ with $\mu \in \Sigma_1$ and $\boldsymbol{w} \in \mathbb{R}^n \setminus \{\boldsymbol{0}_n\}$. Straightforward calculations yield

$$(2\mu+1)\frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\boldsymbol{w} = (2\mu-1)\boldsymbol{w}. \tag{1.66}$$

We have $2\mu+1 \neq 0$ because $\boldsymbol{w} \neq \boldsymbol{0}_n$. Indeed, if $2\mu+1 = 0$, then $\boldsymbol{w}_n = \boldsymbol{0}_n$ according to (1.66), which contradicts $\boldsymbol{w} \neq \boldsymbol{0}_n$. Using (1.66), we find

$$\bar{\boldsymbol{A}}(G)\boldsymbol{w} = \frac{\beta+\gamma}{\gamma}\frac{2\mu-1}{2\mu+1}\boldsymbol{w},$$

which implies that

$$\lambda_\mu := \frac{\beta+\gamma}{\gamma}\frac{2\mu-1}{2\mu+1} \in \sigma\big(\bar{\boldsymbol{A}}(G)\big).$$

We find

$$\mu = \frac{1}{2}\left(1 + \frac{\gamma}{\beta+\gamma}\lambda_\mu\right)\left(1 - \frac{\gamma}{\beta+\gamma}\lambda_\mu\right)^{-1} \in \Sigma_2.$$

This proves $\Sigma_1 \subset \Sigma_2$. Let $(\lambda, \boldsymbol{v})$ be an eigenpair of $\bar{\boldsymbol{A}}(G)$, that is, $\bar{\boldsymbol{A}}(G)\boldsymbol{v} = \lambda\boldsymbol{v}$ with $\lambda \in \sigma(\bar{\boldsymbol{A}}(G))$ and $\boldsymbol{v} \in \mathbb{R}^n \setminus \{\boldsymbol{0}_n\}$. Straightforward calculations yield

$$\left(\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1} - \frac{1}{2}\boldsymbol{I}_n\right)\boldsymbol{v} = \frac{1}{2}\left(1 + \frac{\gamma}{\beta+\gamma}\lambda\right)\left(1 - \frac{\gamma}{\beta+\gamma}\lambda\right)^{-1}\boldsymbol{v}.$$

This proves $\Sigma_2 \subset \Sigma_1$ and concludes the proof of $\Sigma_1 = \Sigma_2$. Next, I prove that $\Sigma_2 \subset \mathbb{R}_{++}$. To this end, I define the auxiliary function $h(\cdot \mid \beta, \gamma)\colon [-1,1] \to \mathbb{R}$ by

$$h(z \mid \beta, \gamma) := \frac{1}{2}\left(1 + \frac{\gamma}{\beta+\gamma}z\right)\left(1 - \frac{\gamma}{\beta+\gamma}z\right)^{-1}.$$

We find

$$\partial h(z \mid \beta, \gamma) = \frac{\gamma}{\beta+\gamma}\left(1 - \frac{\gamma}{\beta+\gamma}z\right)^{-2}.$$

If $\gamma \in (-\beta/2, 0)$, then $h(\cdot \mid \beta, \gamma)$ is strictly decreasing on $[-1,1]$ with a global minimum point at 1, where $h(1 \mid \beta, \gamma) = (\beta+2\gamma)/(2\beta) > 0$. If $\gamma \in (0, +\infty)$, then $h(\cdot \mid \beta, \gamma)$ is strictly increasing on $[-1,1]$ with a global minimum point at -1, where $h(-1 \mid \beta, \gamma) = \beta/(2(\beta+2\gamma)) > 0$. This concludes the proof of $\Sigma_2 \subset \mathbb{R}_{++}$ and also of $\Sigma_1 \subset \mathbb{R}_{++}$ because $\Sigma_1 = \Sigma_2$.

Third, I show that $\sigma((\beta+\gamma)(\boldsymbol{J}+\boldsymbol{J}^\top)-\boldsymbol{I}_n)$ is contained in $\sigma(\boldsymbol{HK})$, where

$$\boldsymbol{H} := \boldsymbol{I}_2 \otimes \left(\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\sqrt{\boldsymbol{D}}^{-1}\dot{\boldsymbol{A}}(\mathrm{sl}(G))\sqrt{\boldsymbol{D}}^{-1}\right)^{-1} - \frac{1}{2}\boldsymbol{I}_n\right)$$

and

$$\boldsymbol{K} := \begin{pmatrix} \boldsymbol{I}_n & \boldsymbol{D} \\ \boldsymbol{D}^{-1} & \boldsymbol{I}_n \end{pmatrix}.$$

Note that $\boldsymbol{H}$ is symmetric because G is symmetric, not diagonal (Condition G), and positive definite because

$$\begin{aligned}
&\sigma\left(\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\sqrt{\boldsymbol{D}}^{-1}\dot{\boldsymbol{A}}(\mathrm{sl}(G))\sqrt{\boldsymbol{D}}^{-1}\right)^{-1} - \frac{1}{2}\boldsymbol{I}_n\right) \\
&\qquad\qquad = \sigma\left(\sqrt{\boldsymbol{D}}\left(\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1} - \frac{1}{2}\boldsymbol{I}_n\right)\sqrt{\boldsymbol{D}}^{-1}\right) \\
&\qquad\qquad\qquad\qquad = \sigma\left(\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1} - \frac{1}{2}\boldsymbol{I}_n\right) \subset \mathbb{R}_{++},
\end{aligned}$$

where the first equality follows from (1.64), the second equality is according to the fact that similar matrices have the same spectrum, and the set inclusion is according to (1.65). Also note that $\boldsymbol{K}$ is in general not symmetric but similar to the symmetric and positive semidefinite but not positive definite matrix

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \otimes \boldsymbol{I}_n$$

because

$$\boldsymbol{K} = \frac{1}{\sqrt{2}}\begin{pmatrix} \boldsymbol{I}_n & -\boldsymbol{I}_n \\ \boldsymbol{D}^{-1} & \boldsymbol{D}^{-1} \end{pmatrix}\left(\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \otimes \boldsymbol{I}_n\right)\left(\frac{1}{\sqrt{2}}\begin{pmatrix} \boldsymbol{I}_n & -\boldsymbol{I}_n \\ \boldsymbol{D}^{-1} & \boldsymbol{D}^{-1} \end{pmatrix}\right)^{-1},$$

where

$$\left(\frac{1}{\sqrt{2}}\begin{pmatrix} \boldsymbol{I}_n & -\boldsymbol{I}_n \\ \boldsymbol{D}^{-1} & \boldsymbol{D}^{-1} \end{pmatrix}\right)^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} \boldsymbol{I}_n & \boldsymbol{D} \\ -\boldsymbol{I}_n & \boldsymbol{D} \end{pmatrix} \quad \text{and} \quad \sigma\left(\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \otimes \boldsymbol{I}_n\right) = \{0, 2\}.$$

We have

$$\begin{aligned}
&\sigma\big((\beta+\gamma)(\boldsymbol{J}+\boldsymbol{J}^\top)-\boldsymbol{I}_n\big) \\
&\quad = \sigma\left(\sqrt{\boldsymbol{D}}((\beta+\gamma)(\boldsymbol{J}+\boldsymbol{J}^\top)-\boldsymbol{I}_n)\sqrt{\boldsymbol{D}}^{-1}\right) \\
&\quad = \sigma\Bigg(\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\sqrt{\boldsymbol{D}}^{-1}\dot{\boldsymbol{A}}(\mathrm{sl}(G))\sqrt{\boldsymbol{D}}^{-1}\right)^{-1} - \frac{1}{2}\boldsymbol{I}_n \\
&\qquad\qquad + \boldsymbol{D}\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\sqrt{\boldsymbol{D}}^{-1}\dot{\boldsymbol{A}}(\mathrm{sl}(G))\sqrt{\boldsymbol{D}}^{-1}\right)^{-1}\boldsymbol{D}^{-1} - \frac{1}{2}\boldsymbol{I}_n\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= \sigma\Bigg(\begin{pmatrix} \boldsymbol{I}_n \\ \boldsymbol{I}_n \end{pmatrix}^{\top} \begin{pmatrix} \boldsymbol{I}_n & \boldsymbol{O}_n \\ \boldsymbol{O}_n & \boldsymbol{D} \end{pmatrix} \\
&\qquad \times \left(\boldsymbol{I}_2 \otimes \left(\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma} \sqrt{\boldsymbol{D}}^{-1} \dot{\boldsymbol{A}}(\mathrm{sl}(G)) \sqrt{\boldsymbol{D}}^{-1} \right)^{-1} - \frac{1}{2} \boldsymbol{I}_n \right) \right) \\
&\qquad \times \begin{pmatrix} \boldsymbol{I}_n & \boldsymbol{O}_n \\ \boldsymbol{O}_n & \boldsymbol{D} \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{I}_n \\ \boldsymbol{I}_n \end{pmatrix} \Bigg) \\
&\subset \sigma\Bigg(\left(\boldsymbol{I}_2 \otimes \left(\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma} \sqrt{\boldsymbol{D}}^{-1} \dot{\boldsymbol{A}}(\mathrm{sl}(G)) \sqrt{\boldsymbol{D}}^{-1} \right)^{-1} - \frac{1}{2} \boldsymbol{I}_n \right) \right) \\
&\qquad \times \begin{pmatrix} \boldsymbol{I}_n & \boldsymbol{O}_n \\ \boldsymbol{O}_n & \boldsymbol{D} \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{I}_n \\ \boldsymbol{I}_n \end{pmatrix} \begin{pmatrix} \boldsymbol{I}_n \\ \boldsymbol{I}_n \end{pmatrix}^{\top} \begin{pmatrix} \boldsymbol{I}_n & \boldsymbol{O}_n \\ \boldsymbol{O}_n & \boldsymbol{D} \end{pmatrix} \Bigg) \\
&= \sigma\Bigg(\left(\boldsymbol{I}_2 \otimes \left(\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma} \sqrt{\boldsymbol{D}}^{-1} \dot{\boldsymbol{A}}(\mathrm{sl}(G)) \sqrt{\boldsymbol{D}}^{-1} \right)^{-1} - \frac{1}{2} \boldsymbol{I}_n \right) \right) \begin{pmatrix} \boldsymbol{I}_n & \boldsymbol{D} \\ \boldsymbol{D}^{-1} & \boldsymbol{I}_n \end{pmatrix} \Bigg) \\
&= \sigma(\boldsymbol{H}\boldsymbol{K}).
\end{aligned}
$$

The first equality is according to the fact that similar matrices have the same spectrum. The second equality follows from

$$
(\beta+\gamma)\sqrt{\boldsymbol{D}}\boldsymbol{J}\sqrt{\boldsymbol{D}}^{-1} = \left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma} \sqrt{\boldsymbol{D}}^{-1} \dot{\boldsymbol{A}}(\mathrm{sl}(G)) \sqrt{\boldsymbol{D}}^{-1} \right)^{-1}
$$

and

$$
(\beta+\gamma)\sqrt{\boldsymbol{D}}\boldsymbol{J}^{\top}\sqrt{\boldsymbol{D}}^{-1} = \boldsymbol{D}\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma} \sqrt{\boldsymbol{D}}^{-1} \dot{\boldsymbol{A}}(\mathrm{sl}(G)) \sqrt{\boldsymbol{D}}^{-1} \right)^{-1} \boldsymbol{D}^{-1}.
$$

The remaining equalities are obvious. For the set inclusion see, for example, Horn and Johnson (2012, Theorem 1.3.22).

To sum up, $\sigma((\beta+\gamma)(\boldsymbol{J}+\boldsymbol{J}^{\top}) - \boldsymbol{I}_n)$ is a subset of $\sigma(\boldsymbol{H}\boldsymbol{K})$, which is equal to $\sigma((\beta+\gamma)(\boldsymbol{J}+\boldsymbol{J}^{\top}) - \boldsymbol{I}_n) \cup \{0\} \subset \mathbb{R}$ (see, for example, Theorem 1.3.22). I conclude that if Conjecture 1.73 is true, that is, $\min \sigma(\boldsymbol{H}\boldsymbol{K}) \geq 0$, then $(\beta+\gamma)(\boldsymbol{J}+\boldsymbol{J}^{\top}) - \boldsymbol{I}_n$ is positive semidefinite, which is sufficient for Conjecture 1.72 to be true.

1.4 Statistical models

This section is concerned with statistical models that are derived from the system of best reply functions at the Nash equilibrium of a NALA game. It is motivated by the desire to take the economic model of Section 1.3 to data and to test hypotheses about the players' common social cost or benefit parameter, in particular, whether a phenomenon is driven, at least partially, by conformist or anti-conformist behavior. The main focus of the section lies on the identification problem, which logically precedes all problems of statistical inference like parameter estimation and hypothesis testing. A discussion of suitable estimators is left for future research.

There are several important findings in this section. First, the economic model translates into a statistical model with correlated error terms, specifically, the error terms follow a moving average of order one. This dependence is due to the assumption of idiosyncrasies with local externalities. Second, the endogenous effects matrix, that is, the matrix whose components determine endogenous effects (see Definition A in Section 1.3.1), and the exogenous effects matrix, that is, the matrix whose components determine exogenous effects (see Definition B in Section 1.4.1), are structurally different in the presence of players without out-neighbors. The endogenous effects matrix is row-normalized with a zero on the main diagonal for a player with at least one out-neighbor and a one for a player without out-neighbors, whereas the exogenous effects matrix is not row-normalized in the presence of players without out-neighbors and has always zeros on its main diagonal, irrespective of the players' out-neighborhoods. Third, a nonidentifying condition in the form of a kernel condition arises if identification is based on the conditional mean of the distribution of the response variable. Fourth, there are two types of identifying conditions for endogenous and exogenous effects (to be defined in Section 1.4.3): rank conditions and linear independence conditions. A rank condition involves both the model's design matrix and the endogenous and exogenous effects matrices and arises if identification is based on the conditional mean. A linear independence condition involves solely the endogenous or exogenous effects matrices and arises if identification is based on the conditional variance. Identifying conditions for endogenous effects that are based on the conditional mean entail without exception restrictions on the parameter space, whereas this is not always the case for identifying conditions that are based on the conditional variance. Fifth, identifying conditions entail restrictions on the topologies of the digraphs from which the endogenous and exogenous effects matrices are derived. In general, a characterization of these restrictions is difficult, especially in case a condition involves more than three matrices that are functions of the endogenous and exogenous effects matrices. A complete characterization is, however, possible and related to the notion of a normally regular digraph (Jørgensen 2015) for two linear independence conditions involving three matrices that are functions of either the endogenous or the exogenous effects matrix. Sixth, players without out-neighbors are critical for the identification of endogenous and exogenous effects if the endogenous and exogenous effects matrices are derived from a single digraph. Seventh, the nonaffine nature of a NALA game admits of aligning a nonnegative action space with the support of a statistical model. Eighth, an important class of statistical models in the social interactions literature can be reconciled with a NALA game by imposing a parameter restriction that is not without loss of generality.

The remainder of this section is structured as follows. Section 1.4.1 details on the NALA game that is translated into two statistical models in Section 1.4.2. The assumptions invoked in the specification and in the translation of the NALA game are guided by the desire to get a close resemblance of the resulting statistical models with existing models in the social interactions literature. Section 1.4.3 is concerned with the identification problem. The focus of the discussion is on the identification of endogenous and exogenous effects in the absence of unobservable correlated effects (Sections 1.4.3.1, 1.4.3.2, 1.4.3.3, and 1.4.3.4) or in the presence of unobservable correlated effects in the form of digraph component specific fixed effects (Sections

1.4.3.5, 1.4.3.6, and 1.4.3.7). Section 1.4.4 examines the existence of a statistical model and a NALA game such that the supports of the former are in alignment with the action space of the latter. Finally, Section 1.4.5 relates NALA games to an important class of existing linear social interactions models.

1.4.1 Towards two statistical models

Let $\Gamma := (\mathcal{I}, G, \mathcal{Y}, \{(\alpha(\chi, H)(i), \beta, \gamma)\}_{i\in\mathcal{I}}, f)$ be a generic NALA game with idiosyncrasies that admit of local externalities, that is, the players' idiosyncrasies are given by a function $\alpha(\chi, H)\colon \mathcal{I} \to \mathbb{R}$, where $\chi\colon \mathcal{I} \to \mathbb{R}$ is the function representing the players' elementary idiosyncrasies and H is a digraph on $\mathcal{I}$.[97] Let $\boldsymbol{\chi}$ denote the profile of the players' elementary idiosyncrasies, which is to be interpreted as a parameter. The profile of the players' idiosyncrasies is denoted by $\boldsymbol{\alpha}$. The following four assumptions are made with respect to Γ. First, the function $\alpha(\chi, H)\colon \mathcal{I} \to \mathbb{R}$ satisfies

$$\alpha(\chi, H)(i) = \begin{cases} \chi(i) & \text{if } \deg_H^+(i) = 0, \\ \chi(i) + \zeta \dfrac{\sum_{j\in\mathcal{N}_H^+(i)} \chi(j)}{\deg_H^+(i)} & \text{if } \deg_H^+(i) > 0, \end{cases}$$

where $\zeta \in \mathbb{R}$ is a parameter whose magnitude is a measure of the strength of the local externalities of the players' idiosyncrasies (cf. Example 1.61). Second, $\beta > 0$. Third, $\beta + \gamma > 0$. Fourth, Γ has a unique and interior NE $\boldsymbol{y}^\star := (y_1^\star, \ldots, y_n^\star)$, which is given by (1.8). The following definition is instrumental in representing $\boldsymbol{\alpha}$ as a function of ζ, $\boldsymbol{\chi}$, and H.

Definition B The *exogenous effects matrix* of H is the square matrix of order n defined by

$$\bar{\boldsymbol{C}}(H) := \bar{\boldsymbol{A}}(H) - \operatorname{diag}\big(\boldsymbol{\iota}_0^+(H)\big),$$

where $\boldsymbol{\iota}_0^+(H) \in \{0,1\}^n$ denotes the (column) vector with component in row i equal to $\mathbb{1}_{\{0\}}(\deg_H^+(i))$ and $\operatorname{diag}(\boldsymbol{\iota}_0^+(H))$ is the diagonal matrix of order n with main diagonal $\boldsymbol{\iota}_0^+(H)$.[98]

It is important to note that $\bar{\boldsymbol{C}}(H)$ is not necessarily row-normalized: its row sums are zero or one, and it has zeros on its main diagonal, in contrast to $\bar{\boldsymbol{A}}(G)$, which is row-normalized and has zeros or ones on its main diagonal. Example 1.75 illustrates Definition B.

Example 1.75 If $H = (\{1,2,3\}, \{(2,1), (3,1)\})$, then

$$\bar{\boldsymbol{C}}(H) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \qquad \diamond$$

97. See Section 1.3.7.1 for the definition and a discussion of the notion of idiosyncrasies with local externalities.

98. If $\deg_H^+(i) = 0$, then $\mathbb{1}_{\{0\}}(\deg_H^+(i)) = 1$, and if $\deg_H^+(i) > 0$, then $\mathbb{1}_{\{0\}}(\deg_H^+(i)) = 0$.

The assumption about $\alpha(\chi, H)$ and Definition B imply that $\boldsymbol{\alpha} = (\boldsymbol{I}_n + \zeta\bar{\boldsymbol{C}}(H))\boldsymbol{\chi}$. Note that ζ need not satisfy the condition $-1 \notin \sigma(\zeta\bar{\boldsymbol{C}}(H))$, which is both necessary and sufficient for $\boldsymbol{I}_n + \zeta\bar{\boldsymbol{C}}(H)$ to be nonsingular. If $-1 \in \sigma(\zeta\bar{\boldsymbol{C}}(H))$, then different values for $\boldsymbol{\chi}$ may be mapped to the same $\boldsymbol{\alpha}$.

To allow for a more compact notation of the difference $\bar{\boldsymbol{A}}(G) - \boldsymbol{I}_n$, let the polynomial q be defined by $q(x) := x^1 - x^0$. Note that for any square matrix $\boldsymbol{A}$ of order n, $q(\boldsymbol{A}) = \boldsymbol{A} - \boldsymbol{I}_n$, in particular, $q(\bar{\boldsymbol{A}}(G)) = \bar{\boldsymbol{A}}(G) - \boldsymbol{I}_n$.

The assumption of a positive β implies that the two generic NALA games Γ and $(\mathcal{I}, G, \mathcal{Y}, \{(\alpha(\chi, H)(i)/\beta, 1, \gamma/\beta)\}_{i\in\mathcal{I}}, f)$ are strategically equivalent. It follows that the players' preference parameters $\{(\alpha(\chi, H)(i), \beta, \gamma)\}_{i\in\mathcal{I}}$ are not identifiable without the imposition of a parameter restriction. The normalization $\beta = 1$ is a restriction that is without loss of generality. With this normalization, the profile of NE actions satisfies

$$\boldsymbol{f}(\boldsymbol{y}^\star) = \Big(\boldsymbol{I}_n - \gamma q\big(\bar{\boldsymbol{A}}(G)\big)\Big)^{-1}\big(\boldsymbol{I}_n + \zeta\bar{\boldsymbol{C}}(H)\big)\boldsymbol{\chi}.$$

This is a system of n equations with $n+2$ parameters, namely, γ, ζ, and $\boldsymbol{\chi}$, in the n variables $f(y_1^\star), \ldots, f(y_n^\star)$. The specification of a statistical model of $\boldsymbol{y}^\star$ calls therefore for an approximation or a parsimonious representation of $\boldsymbol{\chi}$. To this end, assume that all players are attributed with covariates, that is, characteristics, that span in their entirety a linear subspace L of $\mathbb{R}^n$ with dimension K over $\mathbb{R}$, where $0 < K < n-2$. There exists a unique orthogonal decomposition of $\boldsymbol{\chi}$ into the sum of a (column) vector in L and a (column) vector in the orthogonal complement $L^\perp$ to L: $\boldsymbol{\chi} = \boldsymbol{\chi}_L + \boldsymbol{u}$, where $\boldsymbol{\chi}_L \in L$ and $\boldsymbol{u} := \boldsymbol{\chi} - \boldsymbol{\chi}_L \in L^\perp$. Let $\boldsymbol{\phi}$ denote the (column) vector of coordinates of $\boldsymbol{\chi}_L$ with respect to a basis $\{\boldsymbol{b}_k\}_{k\in\{1,\ldots,K\}}$ of L, that is, $\boldsymbol{\chi}_L = \boldsymbol{X\phi}$ with $\boldsymbol{X} := (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_K)$. The decomposition of $\boldsymbol{\chi}$ yields

$$\boldsymbol{f}(\boldsymbol{y}^\star) = \Big(\boldsymbol{I}_n - \gamma q\big(\bar{\boldsymbol{A}}(G)\big)\Big)^{-1}\Big(\boldsymbol{X\phi} + \zeta\bar{\boldsymbol{C}}(H)\boldsymbol{X\phi} + \big(\boldsymbol{I}_n + \zeta\bar{\boldsymbol{C}}(H)\big)\boldsymbol{u}\Big) \tag{1.67}$$

or, equivalently,

$$\boldsymbol{f}(\boldsymbol{y}^\star) = \gamma q\big(\bar{\boldsymbol{A}}(G)\big)\boldsymbol{f}(\boldsymbol{y}^\star) + \boldsymbol{X\phi} + \zeta\bar{\boldsymbol{C}}(H)\boldsymbol{X\phi} + \big(\boldsymbol{I}_n + \zeta\bar{\boldsymbol{C}}(H)\big)\boldsymbol{u}.$$

The system of equations (1.67) forms the basis for the specification of two statistical models of the profile of NE actions in a NALA game.

1.4.2 Specification of two statistical models

This section is concerned with the specification of two statistical models, a model without unobservable correlated effects (Section 1.4.2.1) and a model with unobservable correlated effects in the form of component specific fixed effects (Section 1.4.2.2), of the profile of NE actions in a NALA game with $\mathcal{Y} = \mathbb{R}$.[99] The cases $\mathcal{Y} = \mathbb{R}_+$ and $\mathcal{Y} = [0, \bar{v}]$ are briefly discussed in Section 1.4.4. Each of the two models to

99. A statistical model is a family of probability distributions. For a definition of a statistical model in algebraic terms, using category theory, see McCullagh (2002).

be specified hereinafter is a parameterized conditional statistical model, that is, a family of parameterized regular conditional probability distributions, of the profile of NE actions in a NALA game with $\mathcal{Y} = \mathbb{R}$.[100] The specification of the two models involves some technicalities, some of which are introduced only to provide a consistent probabilistic framework. A few symbols that have been introduced in Section 1.4.1 and in previous sections will be redefined. Even though their mathematical nature changes, they retain to a large extent their original meaning and interpretation.

Let $f \in \mathcal{F}(\mathbb{R})$ be bounded neither below nor above (cf. Condition 1.13.4). Let $(\Omega, \mathfrak{A}, \mathbb{P})$ be a complete probability space. Let G and H be two nonempty random digraphs on $\mathcal{I}$. Both G and H are observable. Let $K > 1$ be an integer. For all $i \in \mathcal{I}$, let $\boldsymbol{x}_i$ be a random vector in $\mathbb{R}^K$. The K components of $\boldsymbol{x}_i$ are random variables in $\mathbb{R}$ and represent player i's covariates, that is, his characteristics. Let $x_{i,k}$ denote the kth component of $\boldsymbol{x}_i$, and let $\boldsymbol{X}$ denote the random $n \times K$ matrix with component $x_{i,k}$ in row i and column k. The matrix $\boldsymbol{X}$ is observable and referred to as the *design matrix*. Let $\mathfrak{F} \subset \mathfrak{A}$ denote the σ-field generated by G, H, and $\boldsymbol{X}$. Let $\boldsymbol{u}$ be a random vector in $\mathbb{R}^n$. The vector $\boldsymbol{u}$ is unobservable. For all $i \in \mathcal{I}$, let u_i denote the ith component of $\boldsymbol{u}$, and assume that $u_i \in \mathcal{L}^2(\Omega, \mathfrak{A}, \mathbb{P})$, that is, u_i is square-integrable. Two assumptions are imposed on $\boldsymbol{X}$ and the first and second conditional moments of $\boldsymbol{u}$ given $\mathfrak{F}$. (All statements that involve or that are based on conditional moments are understood to be true $\mathbb{P}$-a.s.)

Assumption $\mathcal{P}$-$\boldsymbol{X}$ The design matrix is equal to $(\mathbf{1}_n : \boldsymbol{X}_2)$, where $\mathbf{1}_n \notin \text{c-sp}(\boldsymbol{X}_2)$, that is, the first column of $\boldsymbol{X}$ is the constant vector $\mathbf{1}_n$, with $\boldsymbol{X}_2$ denoting all but the first column of $\boldsymbol{X}$.

Assumption $\mathcal{P}$-$\boldsymbol{u}$ The first and second conditional moments of $\boldsymbol{u}$ given $\mathfrak{F}$ satisfy $\mathbb{E}(\boldsymbol{u} \mid \mathfrak{F}) = \mathbf{0}_n$ and $\mathbb{E}(\boldsymbol{u}\boldsymbol{u}^\top \mid \mathfrak{F}) = \boldsymbol{I}_n$.

Assumption $\mathcal{P}$-$\boldsymbol{u}$ implies that the design matrix and the endogenous and exogenous effects matrices are strictly exogenous, that is, $\mathbb{E}(\boldsymbol{u} \mid \boldsymbol{X}) = \mathbb{E}(\boldsymbol{u} \mid \bar{\boldsymbol{A}}(G)) = \mathbb{E}(\boldsymbol{u} \mid \bar{\boldsymbol{C}}(H)) = \mathbf{0}_n$.

The models' common *parameter space* is a subset of $\mathbb{R}^{2K+3}$ that is denoted by Θ. A typical element of Θ is denoted by a subscripted $\boldsymbol{\theta}$, where the subscript is a nonnegative integer, for example, $\boldsymbol{\theta}_0$, $\boldsymbol{\theta}_1$, or $\boldsymbol{\theta}_2$, and is called a *parameter point in* Θ. A parameter point $\boldsymbol{\theta}_0$ in Θ is a quintuple $(\gamma_0, \boldsymbol{\phi}_0, \boldsymbol{\psi}_0, \zeta_0, \varsigma_0)$, where γ_0, ζ_0, and ς_0 are singles and both $\boldsymbol{\phi}_0$ and $\boldsymbol{\psi}_0$ are K-tuples, and it may be considered a (column) vector, in which case it is written as $(\gamma_0, \boldsymbol{\phi}_0^\top, \boldsymbol{\psi}_0^\top, \zeta_0, \varsigma_0)^\top$, where $\boldsymbol{\phi}_0$ and $\boldsymbol{\psi}_0$ are considered (column) vectors with K components.[101] The K-tuple or (column) vector $\boldsymbol{\phi}_0$ without its first component, which is denoted by ϕ_0, is denoted by $\boldsymbol{\phi}_{0,-1}$. An analogous notation applies to $\boldsymbol{\psi}_0$.

It is convenient to introduce the notion of a subparameter to state assumptions about the parameter space Θ and results on identification (Section 1.4.3). A mapping g with domain Θ is called a *subparameter* (cf. Hájek 1967, p. 140). An element of

100. For an exposition of regular conditional distributions see, for example, Klenke (2014, Section 8.3).
101. A single is a tuple of length one.

$g(\Theta)$, the image of Θ under g, is called a *subparameter point in* $g(\Theta)$. The eponymous example of a subparameter is the projection mapping $[\cdot]_{\mathcal{R}} \colon \Theta \to \mathbb{R}^{|\mathcal{R}|}$ that maps a parameter point $\boldsymbol{\theta}_0$ in Θ to the subvector $[\boldsymbol{\theta}_0]_{\mathcal{R}}$ that lies in the rows of $\boldsymbol{\theta}_0$ indexed by $\mathcal{R}$, where $\mathcal{R}$ is a nonempty subset of $\{1,\dots,2K+3\}$. Some special cases deserve separate notation. For $\mathcal{R} = \{1\}$, let $\gamma := [\cdot]_{\mathcal{R}}$; for $\mathcal{R} = \{2,\dots,K+1\}$, let $\boldsymbol{\phi} := [\cdot]_{\mathcal{R}}$; for $\mathcal{R} = \{K+2,\dots,2K+1\}$, let $\boldsymbol{\psi} := [\cdot]_{\mathcal{R}}$; for $\mathcal{R} = \{2K+2\}$, let $\zeta := [\cdot]_{\mathcal{R}}$; and for $\mathcal{R} = \{2K+3\}$, let $\varsigma := [\cdot]_{\mathcal{R}}$, which is referred to as the *dispersion parameter*. In addition, for $\mathcal{R} = \{2\}$, let $\phi := [\cdot]_{\mathcal{R}}$; for $\mathcal{R} = \{3,\dots,K+1\}$, let $\boldsymbol{\phi}_{-1} := [\cdot]_{\mathcal{R}}$; for $\mathcal{R} = \{K+2\}$, let $\psi := [\cdot]_{\mathcal{R}}$; and for $\mathcal{R} = \{K+3,\dots,2K+1\}$, let $\boldsymbol{\psi}_{-1} := [\cdot]_{\mathcal{R}}$. It follows that $\boldsymbol{\phi} = (\phi, \boldsymbol{\phi}_{-1})$ and $\boldsymbol{\psi} = (\psi, \boldsymbol{\psi}_{-1})$. Finally, for $\mathcal{R} = \{1,\dots,2K+3\}$, let $\boldsymbol{\theta} := [\cdot]_{\mathcal{R}}$, that is, $\boldsymbol{\theta}$ is the identity on Θ. The foregoing definitions imply that for a parameter point $\boldsymbol{\theta}_0 = (\gamma_0, \boldsymbol{\phi}_0, \boldsymbol{\psi}_0, \zeta_0, \varsigma_0) = (\gamma_0, \phi_0, \boldsymbol{\phi}_{0,-1}, \psi_0, \boldsymbol{\psi}_{0,-1}, \zeta_0, \varsigma_0)$ in Θ, $\gamma(\boldsymbol{\theta}_0) = \gamma_0$, $\boldsymbol{\phi}(\boldsymbol{\theta}_0) = \boldsymbol{\phi}_0$, $\phi(\boldsymbol{\theta}_0) = \phi_0$, $\boldsymbol{\phi}_{-1}(\boldsymbol{\theta}_0) = \boldsymbol{\phi}_{0,-1}$, $\boldsymbol{\psi}(\boldsymbol{\theta}_0) = \boldsymbol{\psi}_0$, $\psi(\boldsymbol{\theta}_0) = \psi_0$, $\boldsymbol{\psi}_{-1}(\boldsymbol{\theta}_0) = \boldsymbol{\psi}_{0,-1}$, $\zeta(\boldsymbol{\theta}_0) = \zeta_0$, and $\varsigma(\boldsymbol{\theta}_0) = \varsigma_0$.

The parameter space satisfies the following assumptions: $\zeta(\Theta)$ contains a neighborhood of 0, both $\boldsymbol{\phi}(\Theta)$ and $\boldsymbol{\psi}(\Theta)$ contain a neighborhood of $\mathbf{0}_K$, and the dispersion parameter is positive, that is, $\varsigma(\Theta) = \mathbb{R}_{++}$. The assumption about γ is stated separately for easy reference.

Assumption $\mathcal{P}$-γ The set $\gamma(\Theta)$ contains a neighborhood of 0 and all subparameter points γ_0 in $\gamma(\Theta)$ satisfy $1 + \gamma_0 > 0$ and $(1+\gamma_0) \notin \sigma(\gamma_0 \bar{\boldsymbol{A}}(G))$.

Assumption $\mathcal{P}$-γ (cf. Conditions 1.13.2 and 1.13.3) is sufficient for $\boldsymbol{I}_n - \gamma_0 q(\bar{\boldsymbol{A}}(G))$ to be nonsingular.

1.4.2.1 A statistical model without unobservable correlated effects

For a given parameter point $\boldsymbol{\theta}_0 = (\gamma_0, \boldsymbol{\phi}_0^\top, \boldsymbol{\psi}_0^\top, \zeta_0, \varsigma_0)^\top$ in Θ, let $\boldsymbol{y}(\boldsymbol{\theta}_0)$ denote the random vector in $\mathbb{R}^n$ that is defined by the system of equations

$$\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_0)) = \Big(\boldsymbol{I}_n - \gamma_0 q\big(\bar{\boldsymbol{A}}(G)\big)\Big)^{-1} \Big(\boldsymbol{X}\boldsymbol{\phi}_0 + \bar{\boldsymbol{C}}(H)\boldsymbol{X}\boldsymbol{\psi}_0 + \varsigma_0\big(\boldsymbol{I}_n + \zeta_0 \bar{\boldsymbol{C}}(H)\big)\boldsymbol{u}\Big). \tag{1.68}$$

The vector $\boldsymbol{y}(\boldsymbol{\theta}_0)$ is referred to as the *response variable at* $\boldsymbol{\theta}_0$ and represents the profile of NE actions in a NALA game with $\mathcal{Y} = \mathbb{R}$. The *error term at* $\boldsymbol{\theta}_0$ is the random vector in $\mathbb{R}^n$ defined by $\boldsymbol{\varepsilon}(\boldsymbol{\theta}_0) := \varsigma_0(\boldsymbol{I}_n + \zeta_0 \bar{\boldsymbol{C}}(H))\boldsymbol{u}$. It is a first-order moving average with mean $\mathbb{E}(\boldsymbol{\varepsilon}(\boldsymbol{\theta}_0)) = \mathbf{0}_n$ and variance $\operatorname{var}(\boldsymbol{\varepsilon}(\boldsymbol{\theta}_0)) = \varsigma_0^2(\boldsymbol{I}_n + \zeta_0 \bar{\boldsymbol{C}}(H))(\boldsymbol{I}_n + \zeta_0 \bar{\boldsymbol{C}}(H)^\top)$. If $\zeta_0 = 0$, then

$$\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_0)) = \Big(\boldsymbol{I}_n - \gamma_0 q\big(\bar{\boldsymbol{A}}(G)\big)\Big)^{-1} \big(\boldsymbol{X}\boldsymbol{\phi}_0 + \bar{\boldsymbol{C}}(H)\boldsymbol{X}\boldsymbol{\psi}_0 + \varsigma_0 \boldsymbol{u}\big)$$

and $\operatorname{var}(\boldsymbol{\varepsilon}(\boldsymbol{\theta}_0)) = \varsigma_0^2 \boldsymbol{I}_n$, that is, the components of $\boldsymbol{\varepsilon}(\boldsymbol{\theta}_0)$ are uncorrelated.

There exists a regular conditional probability distribution of $\boldsymbol{y}(\boldsymbol{\theta}_0)$ given $\mathfrak{F}$ (see, for example, Klenke 2014, Theorem 8.37), which is denoted by $\mathbb{P}_{\boldsymbol{\theta}_0,\mathfrak{F}}$. The family $\mathcal{P}(\Theta) := \{\mathbb{P}_{\boldsymbol{\theta}_1,\mathfrak{F}}\}_{\boldsymbol{\theta}_1 \in \Theta}$ is a parameterized conditional statistical model (without unobservable correlated effects) of the profile of NE actions in a NALA game with

$\mathcal{Y} = \mathbb{R}$. It is of interest to define a submodel of $\mathcal{P}(\Theta)$ in which the components of the error term are uncorrelated. To this end, let Θ_0 be the largest subset of Θ such that $\zeta\restriction_{\Theta_0} = 0$. The family $\mathcal{P}(\Theta_0) = \{\mathbb{P}_{\boldsymbol{\theta}_1,\mathfrak{F}}\}_{\boldsymbol{\theta}_1 \in \Theta_0}$ is a submodel of $\mathcal{P}(\Theta)$ with the desired property.

A comparison of (1.67) to (1.68) suggests that $\boldsymbol{\phi}$, $\boldsymbol{\psi}$, and ζ are related by the equality $\boldsymbol{\psi} = \zeta\boldsymbol{\phi}$. A corresponding hypothesis test can cast some light on the validity of the theory of generic NALA games and the assumptions involved in the translation to the statistical models.

1.4.2.2 A statistical model with unobservable correlated effects

Suppose G consists of $R > 1$ weakly connected components of orders at least 2. For all $r \in \{1,\ldots,R\}$, let G_r denote the rth component of G, and let $\mathcal{I}_r$ denote its vertex set and n_r its order, that is, $n_r = |\mathcal{I}_r|$. The assumption about the components' orders, that is, $\min\{n_r \mid r \in \{1,\ldots,R\}\} > 1$, is equivalent to the assumption that no player is isolated in G.[102] This assumption does not preclude the existence of players without out-neighbors in G. A player without out-neighbors in G must, however, have at least one in-neighbor in G. Assume without loss of generality that for all $r \in \{1,\ldots,R\}$, $\mathcal{I}_r = \{1 + \sum_{j=1}^{r-1} n_j, \ldots, \sum_{j=1}^{r} n_j\}$. It follows that $\bar{\boldsymbol{A}}(G)$ is block diagonal. For all $r \in \{1,\ldots,R\}$, let η_r be a random variable in $\mathbb{R}$. The family $\{\eta_r\}_{r\in\{1,\ldots,R\}}$ represents the unobservable component specific fixed effects. Let

$$\boldsymbol{\iota} := \begin{pmatrix} \mathbf{1}_{n_1} & \mathbf{0}_{n_1} & \cdots & \mathbf{0}_{n_1} \\ \mathbf{0}_{n_2} & \mathbf{1}_{n_2} & \cdots & \mathbf{0}_{n_2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{n_R} & \mathbf{0}_{n_R} & \cdots & \mathbf{1}_{n_R} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\eta} := \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_R \end{pmatrix}. \tag{1.69}$$

For a given parameter point $\boldsymbol{\theta}_0 = (\gamma_0, \boldsymbol{\phi}_0^\top, \boldsymbol{\psi}_0^\top, \zeta_0, \varsigma_0)^\top$ in Θ, let $\boldsymbol{y}(\boldsymbol{\theta}_0)$ be the random vector in $\mathbb{R}^n$ that is defined by the system of equations

$$\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_0)\big) = \Big(\boldsymbol{I}_n - \gamma_0 q\big(\bar{\boldsymbol{A}}(G)\big)\Big)^{-1}\Big(\boldsymbol{X}\boldsymbol{\phi}_0 + \bar{\boldsymbol{C}}(H)\boldsymbol{X}\boldsymbol{\psi}_0 + \boldsymbol{\iota}\boldsymbol{\eta} + \varsigma_0\big(\boldsymbol{I}_n + \zeta_0\bar{\boldsymbol{C}}(H)\big)\boldsymbol{u}\Big), \tag{1.70}$$

where $\bar{\boldsymbol{C}}(H)\boldsymbol{X}\boldsymbol{\psi}_0 = -\psi_0\boldsymbol{\iota}_0^+(H) + \bar{\boldsymbol{C}}(H)\boldsymbol{X}_2\boldsymbol{\psi}_{0,-1}$.

There exists a regular conditional probability distribution of $\boldsymbol{y}(\boldsymbol{\theta}_0)$ given $\mathfrak{F}$, which is denoted by $\mathbb{Q}_{\boldsymbol{\theta}_0,\mathfrak{F}}$. The family $\mathcal{Q}(\Theta) := \{\mathbb{Q}_{\boldsymbol{\theta}_1,\mathfrak{F}}\}_{\boldsymbol{\theta}_1\in\Theta}$ is a parameterized conditional statistical model (with unobservable correlated effects in the form of component specific fixed effects) of the profile of NE actions in a NALA game with $\mathcal{Y} = \mathbb{R}$. Analogous to Section 1.4.2.1, $\mathcal{Q}(\Theta_0) = \{\mathbb{Q}_{\boldsymbol{\theta}_1,\mathfrak{F}}\}_{\boldsymbol{\theta}_1\in\Theta_0}$ is a submodel of $\mathcal{Q}(\Theta)$ in which the components of the error term are uncorrelated.

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102. A statistical model with unobservable correlated effects in the form of fixed effects can in principle be defined without any restriction on the orders of the components of G. Any suitable method to eliminate the fixed effects from the system of equations upon which the model is defined, like, for example, differencing, leads, however, to a system of equations where the equation corresponding to an isolated player is the identity $0 = 0$.

A comparison of (1.70) to (1.68) shows that $\mathcal{P}(\Theta)$, the model without unobservable correlated effects, accommodates the case of a model with unobservable correlated effects in the form of component specific fixed effects where $R = 1$ because the constant $\mathbf{1}_n$ lies in the column space of $\boldsymbol{X}$ (Assumption $\mathcal{P}$-$\boldsymbol{X}$).

The unobservable fixed effects can be treated as unknown parameters (as in the panel econometrics literature). But they may give rise to the incidental parameter problem (Neyman and Scott 1948) and manifest in inconsistent estimators. Although a discussion of estimators is outside the scope of this paper, the fixed effects are eliminated from (1.70) for the purpose of discussing the identification problem (see Section 1.4.3). This enables in particular to relate the results on identification in this paper to the results in Bramoullé, Djebbari, and Fortin (2009).

The method described hereinafter to eliminate the fixed effects term $\boldsymbol{\iota\eta}$ from (1.70) is based on differencing by means of what will be called a differencing matrix. An $m \times n$ matrix $\boldsymbol{Q}$ with $m \leq n$ is called *differencing matrix for* $\boldsymbol{\iota\eta}$ if $\boldsymbol{Q\iota\eta} = \mathbf{0}_m$. In case $m = n$, $\boldsymbol{Q}$ is of the form $\boldsymbol{I}_n - \boldsymbol{P}$, where $\boldsymbol{P}$ is a block diagonal matrix of order n with the same block structure as $\bar{\boldsymbol{A}}(G)$. The social interactions literature (see, for example, Bramoullé, Djebbari, and Fortin 2009; Lee, Liu, and Lin 2010) distinguishes between two types of differencing (matrices): *local differencing*, LD for short, and *global differencing*. In the current context, where G is a digraph, LD is based on a player's out-neighbors in G, whereas global differencing is based on all players of a given component of G. An example of LD is the matrix $\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)$. An example of global differencing is the matrix $\boldsymbol{I}_n - \boldsymbol{P}$, where $\boldsymbol{P}$ is a block diagonal matrix of order n with the same block structure as $\bar{\boldsymbol{A}}(G)$ and for all $r \in \{1, \ldots, R\}$, the rth block of $\boldsymbol{P}$ is equal to $(1/n_r)\mathbf{1}_{n_r}\mathbf{1}_{n_r}^\top$.

The remainder of this section discusses three examples of differencing in more detail: two local and one global. The related common mathematical problem can be described as follows. Let $\boldsymbol{Q}$ be a differencing matrix for $\boldsymbol{\iota\eta}$. Premultiplying both sides of (1.70) by $\boldsymbol{Q}$ gives

$$\begin{aligned}\boldsymbol{Q}\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_0)\big) = \boldsymbol{Q}\Big(\boldsymbol{I}_n - \gamma_0 q\big(\bar{\boldsymbol{A}}(G)\big)\Big)^{-1} \\ \times \Big(\boldsymbol{X\phi}_0 + \bar{\boldsymbol{C}}(H)\boldsymbol{X\psi}_0 + \boldsymbol{\iota\eta} + \varsigma_0\big(\boldsymbol{I}_n + \zeta_0\bar{\boldsymbol{C}}(H)\big)\boldsymbol{u}\Big).\end{aligned} \tag{1.71}$$

The problem consists—figuratively speaking—in establishing that $\boldsymbol{Q}$ moves from the left to the right side of $(\boldsymbol{I}_n - \gamma_0 q(\bar{\boldsymbol{A}}(G)))^{-1}$, whereby $\boldsymbol{\iota\eta}$ is eliminated from (1.71).[103] A solution to this problem may, subject to the properties of $\boldsymbol{Q}$, call for a modification or a refinement of $\boldsymbol{Q}$.

Local differencing The first example of LD is the matrix $\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)$. For notational convenience, the differencing is stated in terms of the matrix $q(\bar{\boldsymbol{A}}(G))$, that is,

103. Premultiplying both sides of the structural form of (1.70) by $\boldsymbol{Q}$ gives

$$\boldsymbol{Q}\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_0)\big) = \gamma_0\boldsymbol{Q}q\big(\bar{\boldsymbol{A}}(G)\big)\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_0)\big) + \boldsymbol{Q}\Big(\boldsymbol{X\phi}_0 + \bar{\boldsymbol{C}}(H)\boldsymbol{X\psi}_0 + \varsigma_0\big(\boldsymbol{I}_n + \zeta_0\bar{\boldsymbol{C}}(H)\big)\boldsymbol{u}\Big),$$

which is not suitable for the discussion of the identification problem in Section 1.4.3; for that matter, a reduced form of $\boldsymbol{Q}\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_0)\big)$ is indispensable.

$\bar{A}(G) - \boldsymbol{I}_n$ instead of $-q(\bar{A}(G))$, that is, $\boldsymbol{I}_n - \bar{A}(G)$. It is based on the following result.

Lemma 1.76 *The matrix $q(\bar{\boldsymbol{A}}(G))$ has the following properties: (1.76.1) it has rank at most $n-R$; (1.76.2) $q(\bar{\boldsymbol{A}}(G))\mathbf{1}_n = \mathbf{0}_n$; (1.76.3) $q(\bar{\boldsymbol{A}}(G))\boldsymbol{\iota\eta} = \mathbf{0}_n$; and (1.76.4) $q(\bar{\boldsymbol{A}}(G))$ and $(\boldsymbol{I}_n - \gamma_0 q(\bar{\boldsymbol{A}}(G)))^{-1}$ commute.*

It follows from Lemma 1.76 that

$$q(\bar{\boldsymbol{A}}(G))\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_0)) = \Big(\boldsymbol{I}_n - \gamma_0 q(\bar{\boldsymbol{A}}(G))\Big)^{-1} q(\bar{\boldsymbol{A}}(G)) \\ \times \Big(\boldsymbol{X}_2\boldsymbol{\phi}_{0,-1} - \psi_0\boldsymbol{\iota}_0^+(H) + \bar{\boldsymbol{C}}(H)\boldsymbol{X}_2\boldsymbol{\psi}_{0,-1} + \varsigma_0\big(\boldsymbol{I}_n + \zeta_0\bar{\boldsymbol{C}}(H)\big)\boldsymbol{u}\Big). \quad (1.72)$$

Note that the random vector $q(\bar{\boldsymbol{A}}(G))\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_0))$ lies in some subspace of $\mathbb{R}^n$ of dimension at most $n-R$ with probability one because $q(\bar{\boldsymbol{A}}(G))$ is rank deficient with rank at most $n-R$ (Result 1.76.1). It follows that $q(\bar{\boldsymbol{A}}(G))\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_0))$ cannot have a density (with respect to the Lebesgue measure on $\mathbb{R}^n$) if $\boldsymbol{u}$ has a density, even if $-1 \notin \sigma(\zeta_0\bar{\boldsymbol{C}}(H))$.[104] Assuming that f is known, $-1 \notin \sigma(\zeta_0\bar{\boldsymbol{C}}(H))$, and $\boldsymbol{u}$ has a density $d_{\boldsymbol{u}}$ (with respect to the Lebesgue measure on $\mathbb{R}^n$), this result renders estimation of the model parameters based on $d_{\boldsymbol{u}}$, for example, maximum likelihood, impossible. The LD by $q(\bar{\boldsymbol{A}}(G))$ can, however, be refined to address this problem. The refinement is based on a spectral decomposition of $q(\bar{\boldsymbol{A}}(G))q(\bar{\boldsymbol{A}}(G))^\top$ and constitutes the second example of LD and is the subject of the following result.[105]

Lemma 1.77 *There exists an $(n-S)\times n$ matrix $\boldsymbol{Q}_\ell$, where $n-S$ is equal to the rank of $q(\bar{\boldsymbol{A}}(G))$ and $n-S \le n-R$, with the following seven properties: (1.77.1) $\boldsymbol{Q}_\ell$ has full row rank; (1.77.2) $\boldsymbol{Q}_\ell\boldsymbol{Q}_\ell^\top = \boldsymbol{I}_{n-S}$; (1.77.3) $\boldsymbol{Q}_\ell^\top\boldsymbol{Q}_\ell$ is block diagonal with the same block structure as $\bar{\boldsymbol{A}}(G)$ and satisfies $q(\bar{\boldsymbol{A}}(G))\boldsymbol{Q}_\ell^\top\boldsymbol{Q}_\ell q(\bar{\boldsymbol{A}}(G))^\top = q(\bar{\boldsymbol{A}}(G))q(\bar{\boldsymbol{A}}(G))^\top$; (1.77.4) $\boldsymbol{Q}_\ell\mathbf{1}_n = \mathbf{0}_{n-S}$; (1.77.5) $\boldsymbol{Q}_\ell\boldsymbol{\iota\eta} = \mathbf{0}_{n-S}$; (1.77.6) $\boldsymbol{Q}_\ell\bar{\boldsymbol{A}}(G) = \boldsymbol{Q}_\ell\bar{\boldsymbol{A}}(G)\boldsymbol{Q}_\ell^\top\boldsymbol{Q}_\ell$; and (1.77.7) $\boldsymbol{Q}_\ell(\boldsymbol{I}_n - \gamma_0 q(\bar{\boldsymbol{A}}(G)))^{-1} = (\boldsymbol{I}_{n-S} - \gamma_0\boldsymbol{Q}_\ell q(\bar{\boldsymbol{A}}(G))\boldsymbol{Q}_\ell^\top)^{-1}\boldsymbol{Q}_\ell$.*

It follows from Lemma 1.77 that

$$\boldsymbol{Q}_\ell\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_0)) = \Big(\boldsymbol{I}_{n-S} - \gamma_0 q\big(\boldsymbol{Q}_\ell\bar{\boldsymbol{A}}(G)\boldsymbol{Q}_\ell^\top\big)\Big)^{-1}\boldsymbol{Q}_\ell \\ \times \Big(\boldsymbol{X}_2\boldsymbol{\phi}_{0,-1} - \psi_0\boldsymbol{\iota}_0^+(H) + \bar{\boldsymbol{C}}(H)\boldsymbol{X}_2\boldsymbol{\psi}_{0,-1} + \varsigma_0\big(\boldsymbol{I}_n + \zeta_0\bar{\boldsymbol{C}}(H)\big)\boldsymbol{u}\Big). \quad (1.73)$$

If $\bar{\boldsymbol{A}}(G) = \bar{\boldsymbol{C}}(H)$, then Result 1.77.6 implies that

$$\boldsymbol{Q}_\ell\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_0)) = \Big(\boldsymbol{I}_{n-S} - \gamma_0 q\big(\bar{\boldsymbol{A}}_\ell(G)\big)\Big)^{-1} \\ \times \Big(\boldsymbol{X}_\ell\boldsymbol{\phi}_{0,-1} + \bar{\boldsymbol{A}}_\ell(G)\boldsymbol{X}_\ell\boldsymbol{\psi}_{0,-1} + \varsigma_0\big(\boldsymbol{I}_{n-S} + \zeta_0\bar{\boldsymbol{A}}_\ell(G)\boldsymbol{u}_\ell\big)\Big),$$

where $\bar{\boldsymbol{A}}_\ell(G) := \boldsymbol{Q}_\ell\bar{\boldsymbol{A}}(G)\boldsymbol{Q}_\ell^\top$, $\boldsymbol{X}_\ell := \boldsymbol{Q}_\ell\boldsymbol{X}_2$, and $\boldsymbol{u}_\ell := \boldsymbol{Q}_\ell\boldsymbol{u}$ with variance $\operatorname{var}(\boldsymbol{u}_\ell) = \boldsymbol{I}_{n-S}$.[106]

104. The matrix $\boldsymbol{I}_n + \zeta_0\bar{\boldsymbol{C}}(H)$ is nonsingular if and only if $-1 \notin \sigma(\zeta_0\bar{\boldsymbol{C}}(H))$.
105. Lemma 1.77 is inspired by Lee, Liu, and Lin (2010, Appendix F).
106. If $\bar{A}(G) = \bar{\boldsymbol{C}}(H)$, then $\boldsymbol{\iota}_0^+(H) = \mathbf{0}_n$ (see Lemma 1.83, in particular, Result 1.83.3, on p. 91.)

Global differencing The global differencing is based on a spectral decomposition of a block diagonal matrix of order n that has the same block structure as $\bar{\boldsymbol{A}}(G)$ and where for all $r \in \{1,\ldots,R\}$, the rth block of the matrix is equal to $\boldsymbol{I}_{n_r} - (1/n_r)\mathbf{1}_{n_r}\mathbf{1}_{n_r}^\top$. A global differencing can in principle be based on any nonzero scalar multiple of $\boldsymbol{I}_{n_r} - (1/n_r)\mathbf{1}_{n_r}\mathbf{1}_{n_r}^\top$. A natural choice is

$$\boldsymbol{I}_{n_r} - \frac{1}{n_r - 1}\left(\mathbf{1}_{n_r}\mathbf{1}_{n_r}^\top - \boldsymbol{I}_{n_r}\right) = \frac{n_r}{n_r - 1}\left(\boldsymbol{I}_{n_r} - \frac{1}{n_r}\mathbf{1}_{n_r}\mathbf{1}_{n_r}^\top\right)$$

because $(1/(n_r-1))(\mathbf{1}_{n_r}\mathbf{1}_{n_r}^\top - \boldsymbol{I}_{n_r})$ is equal to the row-normalized adjacency matrix of the complete digraph of order n_r. The global differencing is, however, based on $\boldsymbol{I}_{n_r} - (1/n_r)\mathbf{1}_{n_r}\mathbf{1}_{n_r}^\top$ and not on $\boldsymbol{I}_{n_r} - (1/(n_r-1))(\mathbf{1}_{n_r}\mathbf{1}_{n_r}^\top) - \boldsymbol{I}_{n_r}$ because the former matrix entails nicer mathematical properties of the corresponding differencing matrix than does the latter, as the following result demonstrates.[107]

Lemma 1.78 *Let $c \neq 0$. There exists an $(n-R)\times n$ matrix $\boldsymbol{Q}_g$ with the following seven properties: (1.78.1) $\boldsymbol{Q}_g$ has full row rank; (1.78.2) $\boldsymbol{Q}_g\boldsymbol{Q}_g^\top = c\boldsymbol{I}_{n-R}$; (1.78.3) $\boldsymbol{Q}_g^\top\boldsymbol{Q}_g$ is block diagonal with the same block structure as $\bar{\boldsymbol{A}}(G)$, where for all $r \in \{1,\ldots,R\}$, the rth block is equal to $c(\boldsymbol{I}_{n_r} - (1/n_r)\mathbf{1}_{n_r}\mathbf{1}_{n_r}^\top)$; (1.78.4) $\boldsymbol{Q}_g\mathbf{1}_n = \mathbf{0}_{n-R}$; (1.78.5) $\boldsymbol{Q}_g\boldsymbol{\iota}\boldsymbol{\eta} = \mathbf{0}_{n-R}$; (1.78.6) $c\boldsymbol{Q}_g\bar{\boldsymbol{A}}(G) = \boldsymbol{Q}_g\bar{\boldsymbol{A}}(G)\boldsymbol{Q}_g^\top\boldsymbol{Q}_g$; and (1.78.7) $\boldsymbol{Q}_g(\boldsymbol{I}_n - \gamma_0 q(\bar{\boldsymbol{A}}(G)))^{-1}$ is equal to $(\boldsymbol{I}_{n-R} - (\gamma_0/c)\boldsymbol{Q}_g q(\bar{\boldsymbol{A}}(G))\boldsymbol{Q}_g^\top)^{-1}\boldsymbol{Q}_g$.*

It follows from Lemma 1.78 (with $c = 1$) that

$$\begin{aligned}\boldsymbol{Q}_g\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_0)) &= \Big(\boldsymbol{I}_{n-R} - \gamma_0 q\big(\boldsymbol{Q}_g\bar{\boldsymbol{A}}(G)\boldsymbol{Q}_g^\top\big)\Big)^{-1}\boldsymbol{Q}_g \\ &\quad\times\Big(\boldsymbol{X}_2\boldsymbol{\phi}_{0,-1} - \psi_0\boldsymbol{\iota}_0^+(H) + \bar{\boldsymbol{C}}(H)\boldsymbol{X}_2\boldsymbol{\psi}_{0,-1} + \varsigma_0\big(\boldsymbol{I}_n + \zeta_0\bar{\boldsymbol{C}}(H)\big)\boldsymbol{u}\Big). \qquad (1.74)\end{aligned}$$

If G and H have the same component structure (that is, $\bar{\boldsymbol{A}}(G)$ and $\bar{\boldsymbol{C}}(H)$ have the same block structure) and $\mathcal{I}_0^+(H) = \emptyset$ (that is, there are no players without out-neighbors in H) or, equivalently, $\boldsymbol{\iota}_0^+(H) = \mathbf{0}_n$, then Result 1.78.6 implies that

$$\begin{aligned}\boldsymbol{Q}_g\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_0)) &= \Big(\boldsymbol{I}_{n-R} - \gamma_0 q\big(\bar{\boldsymbol{A}}_g(G)\big)\Big)^{-1} \\ &\quad\times\Big(\boldsymbol{X}_g\boldsymbol{\phi}_{0,-1} + \bar{\boldsymbol{C}}_g(H)\boldsymbol{X}_g\boldsymbol{\psi}_{0,-1} + \varsigma_0\big(\boldsymbol{I}_{n-R} + \zeta_0\bar{\boldsymbol{C}}_g(H)\boldsymbol{u}_g\big)\Big),\end{aligned}$$

where $\bar{\boldsymbol{A}}_g(G) := \boldsymbol{Q}_g\bar{\boldsymbol{A}}(G)\boldsymbol{Q}_g^\top$, $\bar{\boldsymbol{C}}_g(H) := \boldsymbol{Q}_g\bar{\boldsymbol{C}}(H)\boldsymbol{Q}_g^\top$, $\boldsymbol{X}_g := \boldsymbol{Q}_g\boldsymbol{X}_2$, and $\boldsymbol{u}_g := \boldsymbol{Q}_g\boldsymbol{u}$ with variance $\mathrm{var}(\boldsymbol{u}_g) = \boldsymbol{I}_{n-R}$.[108]

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107. Lee, Liu, and Lin (2010) prove results analogous to those of Lemma 1.78 for the case where $c = 1$, $q(x) = x$ (see Result 1.78.7), and $\bar{\boldsymbol{A}}(G)$ is a nonnegative and row-normalized block diagonal matrix with zeros on its main diagonal (see, in particular, Section 3 and Lemma C.1).

108. If G and H have the same component structure and $\mathcal{I}_0^+(H) = \emptyset$, then $c\boldsymbol{Q}_g\bar{\boldsymbol{C}}(H) = \boldsymbol{Q}_g\bar{\boldsymbol{C}}(H)\boldsymbol{Q}_g^\top\boldsymbol{Q}_g$. The proof is similar to the proof of Result 1.78.6.

1.4.3 Identification

This section is concerned with the identification problem, that is, the problem of drawing inferences from the conditional probability distribution of the observable response variable to the underlying unobservable parameter point. By construction, a given parameter point is associated with a unique conditional probability distribution of the response variable. A given distribution may, however, be generated by more than one parameter point; it is generated by exactly one parameter point if the mapping that associates with each parameter point a probability distribution is injective. The identification problem and the focus of this section is on finding sufficient conditions—commonly called *identifying conditions* or *identifying restrictions* (see, for example, Hurwicz 1950, p. 248)—for the injectivity of the aforementioned mapping. Identifying conditions may involve relations between different subparameters or relations between subparameters and other structural characteristics of the model, like, for example, the design matrix, the endogenous effects matrix, and the exogenous effects matrix.

The study of the identification problem is based on the same set of assumptions that have been introduced to specify the two statistical models $\mathcal{P}(\Theta)$ and $\mathcal{Q}(\Theta)$ in Section 1.4.2. In addition, as a refinement and clarification of these assumption, it is assumed that the following objects are observable and known a priori: the two nonempty digraphs G and H; the function f; and the realized values of the design matrix and the response variable.

The assumption about G and H posits that survey data are available that include information on all players' out-neighborhoods in both G and H. The *Add Health* data set is an example of survey data with information on G in the form of friendship relationships.[109] At the present time, it appears that no survey data exist that distinguish between different digraphs G and H.[110] In the absence of information on H, one may assume that G and H are related, for example, H may be equal to G or some suitably defined symmetric part of G like $(\mathcal{I}, \{(i,j) \in \mathcal{I}^2 \mid (i,j) \in \mathcal{A}(G) \text{ or } (j,i) \in \mathcal{A}(G)\})$ or $(\mathcal{I}, \{(i,j) \in \mathcal{I}^2 \mid (i,j) \in \mathcal{A}(G) \text{ and } (j,i) \in \mathcal{A}(G)\})$. The assumption about f being known a priori is rather strong and made for technical convenience. The relaxation of this assumption is left for future research.

Apart from the assumptions mentioned above, the study of the identification problem is based on a hypothetical exact knowledge of the conditional probability distribution of the response variable rather than a sample of observations. This is standard practice in the statistical sciences and is reflected in the various notions of identification by the model $\mathcal{P}(\Theta)$ given in Definition I (cf. Gouriéroux and Monfort 1995, Definitions 3.9, 3.10, and 3.11). Analogous definitions apply to the models $\mathcal{P}(\Theta_0)$, $\mathcal{Q}(\Theta)$, and $\mathcal{Q}(\Theta_0)$.

Definition I (I.1) A parameter point $\boldsymbol{\theta}_1$ in Θ is said to be *identified* by $\mathcal{P}(\Theta)$ if for all

109. The *National Longitudinal Study of Adolescent to Adult Health* (*Add Health*) is a longitudinal study of a nationally representative sample of adolescents in grades 7 to 12 in the United States during the 1994/1995 school year. See Harris et al. (2009) for a description of the study design.
110. Blume et al. (2015, p. 471) share this view: "No survey we know of distinguishes between peer- and contextual-effects networks."

parameter points $\boldsymbol{\theta}_2$ in $\Theta \setminus \{\boldsymbol{\theta}_1\}$, $\mathbb{P}_{\boldsymbol{\theta}_1,\mathfrak{F}} \neq \mathbb{P}_{\boldsymbol{\theta}_2,\mathfrak{F}}$. (I.2) The model $\mathcal{P}(\Theta)$ is said to be *identified* if all parameter points in Θ are identified, that is, if the mapping $\boldsymbol{\theta}_0 \mapsto \mathbb{P}_{\boldsymbol{\theta}_0,\mathfrak{F}}$ is injective on Θ. (I.3) A subparameter g is said to be *identified* by $\mathcal{P}(\Theta)$ if for all pairs of parameter points $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ in Θ^2, $\mathbb{P}_{\boldsymbol{\theta}_1,\mathfrak{F}} = \mathbb{P}_{\boldsymbol{\theta}_2,\mathfrak{F}}$ implies that $g(\boldsymbol{\theta}_1) = g(\boldsymbol{\theta}_2)$.

The model $\mathcal{P}(\Theta)$ is identified if and only if $\boldsymbol{\theta}$ is identified by $\mathcal{P}(\Theta)$. The submodel $\mathcal{P}(\Theta_0)$ is identified if the model $\mathcal{P}(\Theta)$ is identified. The submodel $\mathcal{P}(\Theta_0)$ may be identified without $\mathcal{P}(\Theta)$ being identified.

Among all subparameters γ, $\boldsymbol{\phi}$, $\boldsymbol{\psi}$, ζ, and ς, including the components ϕ and $\boldsymbol{\phi}_{-1}$ of $\boldsymbol{\phi}$ and the components ψ and $\boldsymbol{\psi}_{-1}$ of $\boldsymbol{\psi}$, the identification of γ and $\boldsymbol{\psi}_{-1}$ is of special interest, which justifies the introduction of extra terminology: endogenous effects are said to be identified by $\mathcal{P}(\Theta)$ if γ is identified by $\mathcal{P}(\Theta)$, and exogenous effects, also called contextual effects, are said to be identified by $\mathcal{P}(\Theta)$ if $\boldsymbol{\psi}_{-1}$ is identified by $\mathcal{P}(\Theta)$. Similar definitions apply to the models $\mathcal{P}(\Theta_0)$, $\mathcal{Q}(\Theta_0)$, and $\mathcal{Q}(\Theta)$. Both definitions reflect common parlance in the social interactions literature (see, for example, Manski 1993; Bramoullé, Djebbari, and Fortin 2009; Blume et al. 2015).

The rest of this section is structured as follows. Section 1.4.3.1 considers identification of γ, ϕ, $\boldsymbol{\phi}_{-1}$, ψ and $\boldsymbol{\psi}_{-1}$ through the mean by $\mathcal{P}(\Theta)$. Section 1.4.3.2 revisits the results of Section 1.4.3.1 under the assumption of a priori exclusion restrictions with respect to the covariates that appear in the term $\bar{\boldsymbol{C}}(H)\boldsymbol{X}\boldsymbol{\psi}_0$ (see (1.68)). Sections 1.4.3.3 and 1.4.3.4 consider identification through the variance, where identification of γ and ς by $\mathcal{P}(\Theta_0)$ is discussed in Section 1.4.3.3 and identification of γ, ζ, and ς by $\mathcal{P}(\Theta)$ is discussed in Section 1.4.3.4. Sections 1.4.3.5, 1.4.3.6, and 1.4.3.7 consider identification via LD by $q(\bar{\boldsymbol{A}}(G))$ by $\mathcal{Q}(\Theta_0)$ and $\mathcal{Q}(\Theta)$, where Section 1.4.3.5 parallels Section 1.4.3.1, Section 1.4.3.6 parallels Section 1.4.3.3, and Section 1.4.3.7 parallels Section 1.4.3.4. The discussion of the identification of γ, $\boldsymbol{\phi}_{-1}$, ψ, $\boldsymbol{\psi}_{-1}$, and ς by $\mathcal{Q}(\Theta_0)$ and γ, $\boldsymbol{\phi}_{-1}$, ψ, $\boldsymbol{\psi}_{-1}$, ζ, and ς by $\mathcal{Q}(\Theta)$ is confined to the case of LD by $q(\bar{\boldsymbol{A}}(G))$. A comparison of the reduced forms (1.72), (1.73), and (1.74) suggests that identifying conditions similar in nature to those stated in Sections 1.4.3.5, 1.4.3.6, and 1.4.3.7 hold true for LD by $\boldsymbol{Q}_\ell$ and global differencing by $\boldsymbol{Q}_g$.

It is important to note that all results of Sections 1.4.3.1, 1.4.3.2, and 1.4.3.5 are true under assumptions less restrictive than those put forward in Section 1.4.2 as regards the latent variable $\boldsymbol{u}$. It suffices in particular to assume that for all $i \in \mathcal{I}$, $u_i \in \mathcal{L}^1(\Omega, \mathcal{A}, \mathbb{P})$, that is, u_i is integrable, and $\mathbb{E}(\boldsymbol{u} \mid \mathfrak{F}) = \boldsymbol{0}_n$ (cf. Assumption $\mathcal{P}$-$\boldsymbol{u}$).

1.4.3.1 Identification through the mean by $\mathcal{P}(\Theta)$

A subparameter g is called identified through the mean by $\mathcal{P}(\Theta)$ if for all pairs of parameter points $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ in Θ^2, $g(\boldsymbol{\theta}_1) = g(\boldsymbol{\theta}_2)$ is necessary for $\mathbb{E}(\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_1)) \mid \mathfrak{F}) = \mathbb{E}(\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_2)) \mid \mathfrak{F})$. If g is identified through the mean by $\mathcal{P}(\Theta)$, then it is identified by $\mathcal{P}(\Theta)$ because, for all pairs of parameter points $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ in Θ^2, $\mathbb{E}(\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_1)) \mid \mathfrak{F}) = \mathbb{E}(\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_2)) \mid \mathfrak{F})$ is necessary for $\mathbb{P}_{\boldsymbol{\theta}_1,\mathfrak{F}} = \mathbb{P}_{\boldsymbol{\theta}_2,\mathfrak{F}}$. For all parameter points $\boldsymbol{\theta}_0$ in Θ,

$$\mathbb{E}\big(\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_0)) \mid \mathfrak{F}\big) = \Big(\boldsymbol{I}_n - \gamma(\boldsymbol{\theta}_0) q\big(\bar{\boldsymbol{A}}(G)\big)\Big)^{-1} \big(\boldsymbol{X}\boldsymbol{\phi}(\boldsymbol{\theta}_0) + \bar{\boldsymbol{C}}(H)\boldsymbol{X}\boldsymbol{\psi}(\boldsymbol{\theta}_0)\big),$$

from which it follows that the present discussion is confined to the identification of γ, $\boldsymbol{\phi} = (\phi, \boldsymbol{\phi}_{-1})$, and $\boldsymbol{\psi} = (\psi, \boldsymbol{\psi}_{-1})$.

The first result to discuss is a non-identification result (Proposition 1.79). It originates from the conditional mean $\mathbb{E}(\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_0)) \mid \mathfrak{F})$ being functionally independent of $\gamma(\boldsymbol{\theta}_0)$ if $\boldsymbol{X}\boldsymbol{\phi}(\boldsymbol{\theta}_0) + \bar{\boldsymbol{C}}(H)\boldsymbol{X}\boldsymbol{\psi}(\boldsymbol{\theta}_0) = \bar{\boldsymbol{A}}(G)(\boldsymbol{X}\boldsymbol{\phi}(\boldsymbol{\theta}_0) + \bar{\boldsymbol{C}}(H)\boldsymbol{X}\boldsymbol{\psi}(\boldsymbol{\theta}_0))$, which is the statistical model analogue of the system of equations $\boldsymbol{\alpha} = \bar{\boldsymbol{A}}(G)\boldsymbol{\alpha}$ defining the notion of weakly ex ante homogeneous players (Definition H). It is therefore not surprising that γ is not identified through the mean by $\mathcal{P}(\Theta)$ if there exists a parameter point $\boldsymbol{\theta}_0$ in Θ such that $\boldsymbol{X}\boldsymbol{\phi}(\boldsymbol{\theta}_0) + \bar{\boldsymbol{C}}(H)\boldsymbol{X}\boldsymbol{\psi}(\boldsymbol{\theta}_0) \in \ker(q(\bar{\boldsymbol{A}}(G)))$ because $\boldsymbol{f}(\boldsymbol{y}^\star) = \boldsymbol{\alpha}$ if $\beta = 1$ and $\boldsymbol{\alpha} = \bar{\boldsymbol{A}}(G)\boldsymbol{\alpha}$ in the economic model (Result 1.35.1 and Remark 1.36).

Proposition 1.79 *If*

$$\exists \boldsymbol{\theta}_0 \in \Theta \quad \boldsymbol{X}\boldsymbol{\phi}(\boldsymbol{\theta}_0) + \bar{\boldsymbol{C}}(H)\boldsymbol{X}\boldsymbol{\psi}(\boldsymbol{\theta}_0) \in \ker\big(q\big(\bar{\boldsymbol{A}}(G)\big)\big), \tag{1.75}$$

where the dimension of $\ker(q(\bar{\boldsymbol{A}}(G)))$ *over* $\mathbb{R}$ *is at least one because* $q(\bar{\boldsymbol{A}}(G))$ *is singular, then* γ *is not identified through the mean by* $\mathcal{P}(\Theta)$.[111]

Proposition 1.79 entails a necessary condition for γ to be identified through the mean by $\mathcal{P}(\Theta)$: the response variable cannot be functionally independent of all covariates. A precise statement is given in Corollary 1.80. Corollaries 1.81 and 1.82 cover special cases of Proposition 1.79.

Corollary 1.80 *The kernel condition* (1.75) *cannot be satisfied if* γ *is identified through the mean by* $\mathcal{P}(\Theta)$. *If* $\mathcal{I}_0^+(H) \neq \emptyset$ *(respectively,* $\mathcal{I}_0^+(H) = \emptyset$*) and the kernel condition* (1.75) *is not satisfied, then* $\boldsymbol{\phi}_{-1} \neq \boldsymbol{0}_{K-1}$ *or* $\boldsymbol{\psi} \neq \boldsymbol{0}_K$ *(respectively,* $\boldsymbol{\phi}_{-1} \neq \boldsymbol{0}_{K-1}$ *or* $\boldsymbol{\psi}_{-1} \neq \boldsymbol{0}_{K-1}$*) must be true.*[112]

Corollary 1.81 *Suppose* $\boldsymbol{\psi} = \zeta\boldsymbol{\phi}$. *If there exists a parameter point* $\boldsymbol{\theta}_0$ *in* Θ *such that* $\boldsymbol{\phi}(\boldsymbol{\theta}_0) \in \ker(q(\bar{\boldsymbol{A}}(G))(\boldsymbol{I}_n + \zeta(\boldsymbol{\theta}_0)\bar{\boldsymbol{C}}(H))\boldsymbol{X})$, *then* γ *is not identified through the mean by* $\mathcal{P}(\Theta)$.

Corollary 1.82 *Suppose* $\bar{\boldsymbol{A}}(G) = \bar{\boldsymbol{C}}(H)$, $\boldsymbol{\psi} = \zeta\boldsymbol{\phi}$, *and for all subparameter points* ζ_0 *in* $\zeta(\Theta)$, $-1 \notin \sigma(\zeta_0\bar{\boldsymbol{A}}(G))$.[113] *If there exists a parameter point* $\boldsymbol{\theta}_0$ *in* Θ *such that* $\boldsymbol{\phi}(\boldsymbol{\theta}_0) \in \ker(q(\bar{\boldsymbol{A}}(G))\boldsymbol{X})$, *then* γ *is not identified through the mean by* $\mathcal{P}(\Theta)$.[114]

Corollaries 1.81 and 1.82 hold true under weaker conditions; namely, γ is not identified through the mean by $\mathcal{P}(\Theta)$ if there exists a parameter point $\boldsymbol{\theta}_0$ in Θ such that $\boldsymbol{\psi}(\boldsymbol{\theta}_0) = \zeta(\boldsymbol{\theta}_0)\boldsymbol{\phi}(\boldsymbol{\theta}_0)$ and $\boldsymbol{\phi}(\boldsymbol{\theta}_0) \in \ker(q(\bar{\boldsymbol{A}}(G))(\boldsymbol{I}_n + \zeta(\boldsymbol{\theta}_0)\bar{\boldsymbol{C}}(H))\boldsymbol{X})$, and γ is not identified through the mean by $\mathcal{P}(\Theta)$ if $\bar{\boldsymbol{A}}(G) = \bar{\boldsymbol{C}}(H)$ and there exists a parameter point $\boldsymbol{\theta}_0$ in Θ such that $\boldsymbol{\psi}(\boldsymbol{\theta}_0) = \zeta(\boldsymbol{\theta}_0)\boldsymbol{\phi}(\boldsymbol{\theta}_0)$, $-1 \notin \sigma(\zeta(\boldsymbol{\theta}_0)\bar{\boldsymbol{A}}(G))$, and $\boldsymbol{\phi}(\boldsymbol{\theta}_0) \in \ker(q(\bar{\boldsymbol{A}}(G))\boldsymbol{X})$.

111. Note that $q(\bar{\boldsymbol{A}}(G)) \neq \boldsymbol{O}_n$ because G is not empty.
112. Recall that $\mathcal{I}_0^+(H)$ is the set of all players without out-neighbors in H. If $\mathcal{I}_0^+(H) \neq \emptyset$, then there is at least one player without out-neighbors in H.
113. See Footnote 104.
114. The condition $\boldsymbol{\phi}(\boldsymbol{\theta}_0) \in \ker(q(\bar{\boldsymbol{A}}(G))\boldsymbol{X})$ is equivalent to $\boldsymbol{X}\boldsymbol{\phi}(\boldsymbol{\theta}_0) = \bar{\boldsymbol{A}}(G)\boldsymbol{X}\boldsymbol{\phi}(\boldsymbol{\theta}_0)$.

The remaining results state identifying conditions for γ, $\boldsymbol{\phi} = (\phi, \boldsymbol{\phi}_{-1})$, and $\boldsymbol{\psi} = (\psi, \boldsymbol{\psi}_{-1})$. The conditions depend critically on the inequality of $\bar{\boldsymbol{A}}(G)$ and $\bar{\boldsymbol{C}}(H)$. It is therefore important to know when the two matrices are different and when they are equal, especially in case $G = H$.

Lemma 1.83 *(1.83.1) If $G = H$ and $\mathcal{I}_0^+(H) = \emptyset$, then $\bar{\boldsymbol{A}}(G) = \bar{\boldsymbol{C}}(H)$.*

(1.83.2) If $G = H$ and $\mathcal{I}_0^+(H) \neq \emptyset$, then $\bar{\boldsymbol{A}}(G) \neq \bar{\boldsymbol{C}}(H)$.

(1.83.3) If $\bar{\boldsymbol{A}}(G) = \bar{\boldsymbol{C}}(H)$, then $\mathcal{I}_0^+(G) = \mathcal{I}_0^+(H) = \emptyset$.

In case $G = H$, $\bar{\boldsymbol{A}}(G)$ and $\bar{\boldsymbol{C}}(H)$ are different if and only if $\mathcal{I}_0^+(H) \neq \emptyset$, that is, there are players without out-neighbors (Results 1.83.1 and 1.83.2).

The presentation of the remaining results is organized as follows. Proposition 1.84 gives identifying conditions for the case $\bar{\boldsymbol{A}}(G) \neq \bar{\boldsymbol{C}}(H)$ and Proposition 1.86 for the case $\bar{\boldsymbol{A}}(G) = \bar{\boldsymbol{C}}(H)$. Proposition 1.86 stresses the importance of players without out-neighbors for identification in case $G = H$. Proposition 1.89 covers the case of players without out-neighbors (in H) that share a common covariate, that is, characteristic. Both Propositions 1.86 and 1.89 may be considered non-identification results because their identifying conditions entail a strong restriction on the parameter space via γ, $\boldsymbol{\phi}_{-1}$, and $\boldsymbol{\psi}_{-1}$. Proposition 1.90 considers the case where γ is identified through the variance by $\mathcal{P}(\Theta)$. All results but Proposition 1.90 entail restrictions on the parameter space (because of the kernel condition (1.75)) and have in common that their identifying conditions involve a rank condition.

Proposition 1.84 *Suppose $\bar{\boldsymbol{A}}(G) \neq \bar{\boldsymbol{C}}(H)$.*

(1.84.1) Suppose $\mathcal{I}_0^+(H) \neq \emptyset$. If the kernel condition (1.75) is not satisfied and the $n \times (4K - 1)$ matrix

$$\left(\mathbf{1}_n : \boldsymbol{\iota}_0^+(H) : \bar{\boldsymbol{A}}(G)\boldsymbol{\iota}_0^+(H) : \boldsymbol{X}_2 : \bar{\boldsymbol{A}}(G)\boldsymbol{X}_2 : \bar{\boldsymbol{C}}(H)\boldsymbol{X}_2 : \bar{\boldsymbol{A}}(G)\bar{\boldsymbol{C}}(H)\boldsymbol{X}_2\right) \quad (1.76)$$

has full column rank, then γ, $\boldsymbol{\phi}$, and $\boldsymbol{\psi}$ are identified by $\mathcal{P}(\Theta)$.[115] *If γ, $\boldsymbol{\phi}$, and $\boldsymbol{\psi}$ are identified through the mean by $\mathcal{P}(\Theta)$, then $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G)$, $\bar{\boldsymbol{C}}(H)$, $\bar{\boldsymbol{A}}(G)\bar{\boldsymbol{C}}(H)$ are linearly independent.*

(1.84.2) Suppose $\mathcal{I}_0^+(H) = \emptyset$. If the kernel condition (1.75) is not satisfied and the $n \times (4K - 3)$ matrix

$$\left(\mathbf{1}_n : \boldsymbol{X}_2 : \bar{\boldsymbol{A}}(G)\boldsymbol{X}_2 : \bar{\boldsymbol{C}}(H)\boldsymbol{X}_2 : \bar{\boldsymbol{A}}(G)\bar{\boldsymbol{C}}(H)\boldsymbol{X}_2\right) \quad (1.77)$$

has full column rank, then γ, $\boldsymbol{\phi}_{-1}$, and $\boldsymbol{\psi}_{-1}$ are identified by $\mathcal{P}(\Theta)$. If γ, $\boldsymbol{\phi}_{-1}$, and $\boldsymbol{\psi}_{-1}$ are identified through the mean by $\mathcal{P}(\Theta)$, then $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G)$, $\bar{\boldsymbol{C}}(H)$, $\bar{\boldsymbol{A}}(G)\bar{\boldsymbol{C}}(H)$ are linearly independent.

115. Note that, for all $i \in \mathcal{I}$, the component of $\boldsymbol{\iota}_0^+(H)$ in row i is equal to 1 if $\deg_H^+(i) = 0$ and equal to 0 if $\deg_H^+(i) > 0$, and the component of $\bar{\boldsymbol{A}}(G)\boldsymbol{\iota}_0^+(H)$ in row i is equal to 0 if $\deg_G^+(i) = 0$ and $\deg_H^+(i) > 0$, equal to 1 if $\deg_G^+(i) = \deg_H^+(i) = 0$, and equal to $|\mathcal{N}_G^+(i) \cap \mathcal{I}_0^+(H)|/\deg_G^+(i) \in [0,1]$ if $\deg_G^+(i) > 0$.

The rank conditions of Results 1.84.1 and 1.84.2 entail three necessary conditions. First, $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G)$, $\bar{\boldsymbol{C}}(H)$, $\bar{\boldsymbol{A}}(G)\bar{\boldsymbol{C}}(H)$ are linearly independent. Their linear independence is, however, in general not sufficient for identification through the mean, as illustrated by Example 1.85. Second, all four matrices $\boldsymbol{X}_2$, $\bar{\boldsymbol{A}}(G)\boldsymbol{X}_2$, $\bar{\boldsymbol{C}}(H)\boldsymbol{X}_2$, and $\bar{\boldsymbol{A}}(G)\bar{\boldsymbol{C}}(H)\boldsymbol{X}_2$ have full column rank. Third, the order condition $n \geq 4K-1$ (Result 1.84.1) or $n \geq 4K-3$ (Result 1.84.2).

Example 1.85 Suppose $\mathcal{I} = \{1,2,3,4\}$, $\mathcal{A}(G) = \bigcup_{i\in\mathcal{I}\setminus\{1\}}\{(i,1)\}$ and $\mathcal{A}(H) = \bigcup_{i\in\mathcal{I}\setminus\{1\}}\{(1,i),(i,1)\}$, that is, both G and H are star-shaped, $\boldsymbol{X} = (\boldsymbol{1}_4 : \boldsymbol{e}_1)$, and $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ are two parameter points in Θ with $(\gamma_1, \boldsymbol{\phi}_1, \boldsymbol{\psi}_1) = (1/3, 3/4, 4/5, 0, 1)$ and $(\gamma_2, \boldsymbol{\phi}_2, \boldsymbol{\psi}_2) = (2/3, 13/10, 1/4, 0, 1/2)$. Note that $\ker(q(\bar{\boldsymbol{A}}(G))) = \{c\boldsymbol{1}_4 \mid c \in \mathbb{R}\}$ and $\boldsymbol{X}\boldsymbol{\phi}_1 + \bar{\boldsymbol{C}}(H)\boldsymbol{X}\boldsymbol{\psi}_1 \notin \ker(q(\bar{\boldsymbol{A}}(G)))$ and $\boldsymbol{X}\boldsymbol{\phi}_2 + \bar{\boldsymbol{C}}(H)\boldsymbol{X}\boldsymbol{\psi}_2 \notin \ker(q(\bar{\boldsymbol{A}}(G)))$ (cf. the kernel condition (1.75)). The matrices $\boldsymbol{I}_4$, $\bar{\boldsymbol{A}}(G)$, $\bar{\boldsymbol{C}}(H)$, $\bar{\boldsymbol{A}}(G)\bar{\boldsymbol{C}}(H)$ are linearly independent, but

$$\mathbb{E}\big(\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_1)\big) \mid \mathfrak{F}\big) = \frac{1}{20}\begin{pmatrix}31\\34\\34\\34\end{pmatrix} = \mathbb{E}\big(\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_2)\big) \mid \mathfrak{F}\big). \qquad \diamond$$

A comparison of Results 1.84.1 and 1.84.2 suggests that ϕ and ψ are not identified by $\mathcal{P}(\Theta)$ if $\mathcal{I}_0^+(H) = \emptyset$. This is straightforward to see. If $\mathcal{I}_0^+(H) = \emptyset$, then $\bar{\boldsymbol{C}}(H)\boldsymbol{1}_n = \boldsymbol{1}_n$, so that there are two constant covariates in (1.68), namely, $\boldsymbol{1}_n$ and $\bar{\boldsymbol{C}}(H)\boldsymbol{1}_n$, from which it follows that ϕ and ψ are not identified (through the mean) by $\mathcal{P}(\Theta)$. In case $\mathcal{I}_0^+(H) = \emptyset$, one could simply posit that $\psi = 0$ to overcome non-identification of ϕ. The resulting submodel is derived from the following system of equations:

$$\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_0)\big) = \Big(\boldsymbol{I}_n - \gamma_0 q\big(\bar{\boldsymbol{A}}(G)\big)\Big)^{-1}\Big(\boldsymbol{X}\boldsymbol{\phi}_0 + \bar{\boldsymbol{C}}(H)\boldsymbol{X}_2\boldsymbol{\psi}_{0,-1} + \varsigma_0\big(\boldsymbol{I}_n + \zeta_0\bar{\boldsymbol{C}}(H)\big)\boldsymbol{u}\Big).$$

Proposition 1.86 *Suppose $\bar{\boldsymbol{A}}(G) = \bar{\boldsymbol{C}}(H)$. If the kernel condition (1.75) is not satisfied, $\gamma\boldsymbol{\phi}_{-1} + (1+\gamma)\boldsymbol{\psi}_{-1} \neq \boldsymbol{0}_{K-1}$, and the $n \times (3K-2)$ matrix*

$$\big(\boldsymbol{1}_n : \boldsymbol{X}_2 : \bar{\boldsymbol{A}}(G)\boldsymbol{X}_2 : \bar{\boldsymbol{A}}(G)^2\boldsymbol{X}_2\big) \tag{1.78}$$

has full column rank, then γ, $\boldsymbol{\phi}_{-1}$, and $\boldsymbol{\psi}_{-1}$ are identified by $\mathcal{P}(\Theta)$. If γ, $\boldsymbol{\phi}_{-1}$, and $\boldsymbol{\psi}_{-1}$ are identified through the mean by $\mathcal{P}(\Theta)$, then $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G)$, $\bar{\boldsymbol{A}}(G)^2$ are linearly independent.

Proposition 1.86 may be considered a non-identification result because of the inequality $\gamma\boldsymbol{\phi}_{-1} + (1+\gamma)\boldsymbol{\psi}_{-1} \neq \boldsymbol{0}_{K-1}$ and the restriction it imposes on the parameter space Θ. The inequality is not true if, for example, there exists a parameter point $\boldsymbol{\theta}_1$ in Θ with $\gamma(\boldsymbol{\theta}_1) = 0$ and $\boldsymbol{\psi}_{-1}(\boldsymbol{\theta}_1) = \boldsymbol{0}_{K-1}$. A hypothesis test with the null hypothesis $(\gamma(\boldsymbol{\theta}_0), \boldsymbol{\psi}_{-1}(\boldsymbol{\theta}_0)) = \boldsymbol{0}_K$ and the alternative hypothesis $(\gamma(\boldsymbol{\theta}_0), \boldsymbol{\psi}_{-1}(\boldsymbol{\theta}_0)) \neq \boldsymbol{0}_K$ about the true data generating parameter point, denoted by $\boldsymbol{\theta}_0$, is, however, the most interesting one to carry out in empirical work. In other words, the case where both endogenous and exogenous effects are not present in the data is not identified through the mean by $\mathcal{P}(\Theta)$. In the light of these considerations, Proposition 1.86 underscores the importance of players without out-neighbors for identification in case $G = H$ because $\bar{\boldsymbol{A}}(G) = \bar{\boldsymbol{C}}(H)$ if $G = H$ and $\mathcal{I}_0^+(H) = \emptyset$ (Result 1.83.1) but $\bar{\boldsymbol{A}}(G) \neq \bar{\boldsymbol{C}}(H)$ if $G = H$ and $\mathcal{I}_0^+(H) \neq \emptyset$ (Result 1.83.2).

Bramoullé, Djebbari, and Fortin (2009) give a sufficient condition for the linear independence of $\boldsymbol{I}_n$, $\bar{\boldsymbol{C}}(G)$, $\bar{\boldsymbol{C}}(G)^2$ in terms of the topology of G (see Section 2.4.2).[116] The result is also true for $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G)$, $\bar{\boldsymbol{A}}(G)^2$ and is stated in Lemma 1.87. It is based on the following notion: a triple (x, y, z) of pairwise distinct vertices of a digraph D is called *intransitive triple in D* if both arcs (x, y) and (y, z) are in D but the arc (x, z) is not in D.

Lemma 1.87 (Bramoullé, Djebbari, and Fortin 2009) *If there exists an intransitive triple in G, then $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G)$, $\bar{\boldsymbol{A}}(G)^2$ are linearly independent.*

Similar to Proposition 1.84, the linear independence of $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G)$, $\bar{\boldsymbol{A}}(G)^2$ is in general not sufficient for the identification of γ, $\boldsymbol{\phi}_{-1}$, and $\boldsymbol{\psi}_{-1}$ through the mean by $\mathcal{P}(\Theta)$, as illustrated by Example 1.88.

Example 1.88 Suppose $\mathcal{I} = \{1, 2, 3, 4\}$, $\mathcal{A}(G) = \bigcup_{i \in \mathcal{I} \setminus \{1\}} \{(1, i), (i, 1)\}$, that is, G is symmetric and star-shaped, $G = H$, $\boldsymbol{X} = (\boldsymbol{1}_4 : \boldsymbol{e}_1)$, and $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ are two parameter points in Θ with $(\gamma_1, \boldsymbol{\phi}_1, \boldsymbol{\psi}_1) = (1/3, 3/4, 4/5, 0, 1)$ and $(\gamma_2, \boldsymbol{\phi}_2, \boldsymbol{\psi}_2) = (2/3, 19/25, 3/4, 0, 103/100)$. Note that $\ker(q(\bar{\boldsymbol{A}}(G))) = \{c\boldsymbol{1}_4 \mid c \in \mathbb{R}\}$ and $\boldsymbol{X}\boldsymbol{\phi}_1 + \bar{\boldsymbol{C}}(H)\boldsymbol{X}\boldsymbol{\psi}_1 \notin \ker(q(\bar{\boldsymbol{A}}(G)))$ and $\boldsymbol{X}\boldsymbol{\phi}_2 + \bar{\boldsymbol{C}}(H)\boldsymbol{X}\boldsymbol{\psi}_2 \notin \ker(q(\bar{\boldsymbol{A}}(G)))$ (cf. the kernel condition (1.75)). The matrices $\boldsymbol{I}_4$, $\bar{\boldsymbol{A}}(G)$, $\bar{\boldsymbol{A}}(G)^2$ are linearly independent because $(2, 1, 3)$ is an intransitive triple in G (Lemma 1.87), but

$$\mathbb{E}\big(\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_1)\big) \mid \mathfrak{F}\big) = \frac{1}{100}\begin{pmatrix}159\\171\\171\\171\end{pmatrix} = \mathbb{E}\big(\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_2)\big) \mid \mathfrak{F}\big). \qquad \diamond$$

Even though players without out-neighbors are critical for identification in case $G = H$, they may also pose a threat to identification if they share a common covariate, that is, characteristic. The players without out-neighbors (in H) have a common covariate if $\operatorname{diag}(\boldsymbol{\iota}_0^+(H))\boldsymbol{x}$ is a scalar multiple of $\boldsymbol{\iota}_0^+(H)$ for some column $\boldsymbol{x}$ of $\boldsymbol{X}_2$. In general, players without out-neighbors are a threat to identification if $G = H$ and some linear combination of the covariates of the players without out-neighbors is a scalar multiple of $\boldsymbol{\iota}_0^+(G)$, formally, $\boldsymbol{0}_n \neq \boldsymbol{\iota}_0^+(G) \in \text{c-sp}(\operatorname{diag}(\boldsymbol{\iota}_0^+(G))\boldsymbol{X}_2)$.[117] The corresponding result is stated in Proposition 1.89. The identifying condition involves the subparameter restriction of Proposition 1.86. Proposition 1.89 may for this reason also be considered a non-identification result. The problem of non-identification does not arise if the players without out-neighbors are sufficiently different in terms of their characteristics.

116. Note that Bramoullé, Djebbari, and Fortin (2009) use a different notation.
117. Note that $\boldsymbol{\iota}_0^+(G) \neq \boldsymbol{0}_n$ if and only if $\mathcal{I}_0^+(G) \neq \emptyset$.

Proposition 1.89 *Suppose $G = H$ and $\mathbf{0}_n \neq \boldsymbol{\iota}_0^+(G) \in$ c-sp$(\mathrm{diag}(\boldsymbol{\iota}_0^+(G))\boldsymbol{X}_2)$. If the kernel condition (1.75) is not satisfied, $\gamma\boldsymbol{\phi}_{-1} + (1+\gamma)\boldsymbol{\psi}_{-1} \neq \mathbf{0}_{K-1}$, and the $n \times (5K-4)$ matrix*

$$\left(\mathbf{1}_n : \boldsymbol{X}_2 : \bar{\boldsymbol{A}}(G)\boldsymbol{X}_2 : \bar{\boldsymbol{C}}(G)\boldsymbol{X}_2 : \bar{\boldsymbol{A}}(G)^2\boldsymbol{X}_2 : \bar{\boldsymbol{A}}(G)\bar{\boldsymbol{C}}(G)\boldsymbol{X}_2\right) \tag{1.79}$$

has full column rank, then γ, $\boldsymbol{\phi}$, and $\boldsymbol{\psi}$ are identified by $\mathcal{P}(\Theta)$. If γ, $\boldsymbol{\phi}$, and $\boldsymbol{\psi}$ are identified through the mean by $\mathcal{P}(\Theta)$, then $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G)$, $\bar{\boldsymbol{C}}(G)$, $\bar{\boldsymbol{A}}(G)^2$, $\bar{\boldsymbol{A}}(G)\bar{\boldsymbol{C}}(G)$ are linearly independent.

The last result states identifying conditions for the case where γ is identified through the variance by $\mathcal{P}(\Theta)$.

Proposition 1.90 *Suppose γ is identified by $\mathcal{P}(\Theta)$.*

(1.90.1) Suppose $\mathcal{I}_0^+(H) \neq \emptyset$. If the $n \times 2K$ matrix $(\mathbf{1}_n : \boldsymbol{\iota}_0^+(H) : \boldsymbol{X}_2 : \bar{\boldsymbol{C}}(H)\boldsymbol{X}_2)$ has full column rank, then $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are identified by $\mathcal{P}(\Theta)$.

(1.90.2) Suppose $\mathcal{I}_0^+(H) = \emptyset$. If the $n \times (2K-1)$ matrix $(\mathbf{1}_n : \boldsymbol{X}_2 : \bar{\boldsymbol{C}}(H)\boldsymbol{X}_2)$ has full column rank, then $\boldsymbol{\phi}_{-1}$ and $\boldsymbol{\psi}_{-1}$ are identified by $\mathcal{P}(\Theta)$.

Note that Proposition 1.90 does not involve the kernel condition (1.75). Note also that $\boldsymbol{I}_n$ and $\bar{\boldsymbol{C}}(H)$ are linearly independent because $\bar{\boldsymbol{C}}(H) \neq \boldsymbol{I}_n$ (see Definition B) and, by assumption, H is not empty, which implies that $\bar{\boldsymbol{C}}(H) \neq \boldsymbol{O}_n$.

1.4.3.2 Identification through the mean by a submodel of $\mathcal{P}(\Theta)$ involving a priori exclusion restrictions

The rank conditions of Propositions 1.84 and 1.89 are not satisfied if $\boldsymbol{X}_2$, $\bar{\boldsymbol{A}}(G)\boldsymbol{X}_2$, or $\bar{\boldsymbol{C}}(H)\boldsymbol{X}_2$ are rank deficient. In empirical work, one may choose covariates that give rise to a matrix $\boldsymbol{X}_2$ such that $\mathbf{1}_n \notin$ c-sp$(\boldsymbol{X}_2)$ and both $\boldsymbol{X}_2$ and $\bar{\boldsymbol{A}}(G)\boldsymbol{X}_2$ have full column rank. Specifically, given a family of nonconstant and linearly independent covariates $\{\boldsymbol{x}_k\}_{k\in\mathcal{K}}$ with $\mathcal{K} \subset \mathbb{Z}_+$, one may choose a subset $\{k_2, \ldots, k_K\} \subset \mathcal{K}$ such that $\boldsymbol{X}_2$ is equal to $(\boldsymbol{x}_{k_2} : \ldots : \boldsymbol{x}_{k_K})$ and $\bar{\boldsymbol{A}}(G)\boldsymbol{X}_2$ has full column rank. Even if $\boldsymbol{X}_2$ and $\bar{\boldsymbol{A}}(G)\boldsymbol{X}_2$ have full column rank, $\bar{\boldsymbol{C}}(H)\boldsymbol{X}_2$ need not necessarily have full column rank. Choosing a subset $\{k_2, \ldots, k_K\} \subset \mathcal{K}$ such that $\boldsymbol{X}_2$ is equal to $(\boldsymbol{x}_{k_2} : \ldots : \boldsymbol{x}_{k_K})$ and both $\bar{\boldsymbol{A}}(G)\boldsymbol{X}_2$ and $\bar{\boldsymbol{C}}(H)\boldsymbol{X}_2$ have full column rank may result in a small value of K (small in comparison with n) and represent a poor approximation of the profile of the players' idiosyncracies. The case of a rank deficient matrix $\bar{\boldsymbol{C}}(H)\boldsymbol{X}$ will for this reason be addressed below. The exposition rests on the assumptions that $K > 2$ and the $n \times K$ matrix $\bar{\boldsymbol{C}}(H)\boldsymbol{X}$ has rank r with $r < K$. Note that $r \geq 1$ because H is not empty and $\mathbf{1}_n \in$ c-sp$(\boldsymbol{X})$ (Assumption $\mathcal{P}$-$\boldsymbol{X}$).

There are at least two solutions to the problem that a rank deficient matrix $\bar{\boldsymbol{C}}(H)\boldsymbol{X}$ causes for identification. Both of them consist in replacing $\bar{\boldsymbol{C}}(H)\boldsymbol{X}$ in the system of equations (1.68) with a substitute of full column rank.

The first solution involves a rank decomposition of $\bar{\boldsymbol{C}}(H)\boldsymbol{X}$. A rank decomposition of $\bar{\boldsymbol{C}}(H)\boldsymbol{X}$ is a factorization of $\bar{\boldsymbol{C}}(H)\boldsymbol{X}$ into a product of two matrices $\boldsymbol{C}$ and $\boldsymbol{D}$, where $\boldsymbol{C}$ is an $n \times r$ matrix of full column rank that has the same column space as $\bar{\boldsymbol{C}}(H)\boldsymbol{X}$ and $\boldsymbol{D}$ is an $r \times K$ matrix of full row rank.[118] Given a rank decomposition $\boldsymbol{CD}$ of $\bar{\boldsymbol{C}}(H)\boldsymbol{X}$, the term $\bar{\boldsymbol{C}}(H)\boldsymbol{X}\boldsymbol{\psi}_0$ in (1.68) can be written as $\boldsymbol{C}\boldsymbol{\psi}_{\boldsymbol{D},0}$, where $\boldsymbol{\psi}_{\boldsymbol{D},0} := \boldsymbol{D}\boldsymbol{\psi}_0$ is a vector in $\mathbb{R}^r$ that acts as a new parameter point. Substituting $\boldsymbol{C}\boldsymbol{\psi}_{\boldsymbol{D},0}$ for $\bar{\boldsymbol{C}}(H)\boldsymbol{X}\boldsymbol{\psi}_0$ in (1.68) is almost without loss of generality because $\bar{\boldsymbol{C}}(H)\boldsymbol{X}$ and $\boldsymbol{C}$ have identical column spaces, except that $\boldsymbol{\psi}_{\boldsymbol{D},0}$ may not be as straightforward to interpret as $\boldsymbol{\psi}_0$.

The second solution involves a priori exclusion restrictions with respect to the covariates that appear in the term $\bar{\boldsymbol{C}}(H)\boldsymbol{X}\boldsymbol{\psi}_0$, which can be written as a sum $\sum_{k=1}^{K} \bar{\boldsymbol{C}}(H)\boldsymbol{x}_k\psi_{0,k}$ with $\boldsymbol{x}_k$ denoting the kth column of $\boldsymbol{X}$ and $\psi_{0,k}$ the kth component of $\boldsymbol{\psi}_0$. There exists a rank-maximizing $n \times L$ submatrix $\boldsymbol{Z}$ of $\boldsymbol{X}$ with $1 \leq L \leq r$ such that $\bar{\boldsymbol{C}}(H)\boldsymbol{Z}$ has full column rank, specifically, $\boldsymbol{Z} = (\boldsymbol{x}_{k_1} : \ldots : \boldsymbol{x}_{k_L})$ for some maximal subset $\mathcal{R} := \{k_1, \ldots, k_L\} \subset \{1, \ldots, K\}$. Let $\bar{\boldsymbol{\psi}}_0$ denote the subparameter point that lies in the rows of $\boldsymbol{\psi}_0$ indexed by $\mathcal{R}$, that is, $\bar{\boldsymbol{\psi}}_0 := [\boldsymbol{\psi}_0]_{\mathcal{R}}$. Substituting $\bar{\boldsymbol{C}}(H)\boldsymbol{Z}\bar{\boldsymbol{\psi}}_0$ for $\bar{\boldsymbol{C}}(H)\boldsymbol{X}\boldsymbol{\psi}_0$ in (1.68) yields a new system of equations, which may be considered the result of imposing a priori exclusion restrictions on the summands in $\sum_{k=1}^{K} \bar{\boldsymbol{C}}(H)\boldsymbol{x}_k\psi_{0,k}$, namely, for all $k \in \{1, \ldots, K\} \setminus \mathcal{R}$, $\psi_{0,k} = 0$. The substitution is not without loss of generality unless $r = L$ or $r < L$ and the a priori exclusion restrictions are true in the original model.

Both solutions give rise to statistical models for which results similar to those of Section 1.4.3.1 can be obtained. This shall be exemplified by the model of the second solution, denoted by $\mathcal{P}(\Theta_1)$, which is based on the following system of equations:

$$\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_0)\big) = \Big(\boldsymbol{I}_n - \gamma_0 q\big(\bar{\boldsymbol{A}}(G)\big)\Big)^{-1}\Big(\boldsymbol{X}\boldsymbol{\phi}_0 + \bar{\boldsymbol{C}}(H)\boldsymbol{Z}\bar{\boldsymbol{\psi}}_0 + \varsigma_0\big(\boldsymbol{I}_n + \zeta_0\bar{\boldsymbol{C}}(H)\big)\boldsymbol{u}\Big). \tag{1.80}$$

Let the subparameter $\bar{\boldsymbol{\phi}}$ be defined by $\bar{\boldsymbol{\phi}} := [\boldsymbol{\phi}]_{\mathcal{R}}$, where $\mathcal{R}$ is the index set defined above of the covariates that constitute the columns of $\boldsymbol{Z}$. Let $\bar{\boldsymbol{\phi}}_{-1}$ denote the subparameter $\bar{\boldsymbol{\phi}}$ without its first component. The two subparameters $\bar{\boldsymbol{\psi}}$ and $\bar{\boldsymbol{\psi}}_{-1}$ are defined similarly. Let Θ_1 be the largest subset of Θ such that $[\boldsymbol{\psi}]_{\{1,\ldots,K\}\setminus\mathcal{R}}\restriction_{\Theta_1} = \boldsymbol{0}_{K-L}$, that is, all subparameter points $(\psi_{0,1}, \ldots, \psi_{0,K})$ in $\boldsymbol{\psi}(\Theta_1)$ satisfy the exclusion restriction, for all $k \in \{1, \ldots, K\} \setminus \mathcal{R}$, $\psi_{0,k} = 0$. The family of regular conditional probability distributions $\mathcal{P}(\Theta_1) = \{\mathbb{P}_{\boldsymbol{\theta}_1,\mathfrak{F}}\}_{\boldsymbol{\theta}_1\in\Theta_1}$ is a submodel of $\mathcal{P}(\Theta)$.

To ensure comparability, the main results of Section 1.4.3.1 are restated for the model $\mathcal{P}(\Theta_1)$ on the basis of the following assumption.

Assumption $\boldsymbol{\mathcal{P}}$-$\boldsymbol{Z}$ The $n \times L$ matrix $\boldsymbol{Z}$ is equal to $(\boldsymbol{1}_n : \boldsymbol{Z}_2)$, that is, the first column of $\boldsymbol{Z}$ is the constant vector $\boldsymbol{1}_n$, with $\boldsymbol{Z}_2$ denoting all but the first column of $\boldsymbol{Z}$.

The definition of $\boldsymbol{Z}$ and Assumptions $\mathcal{P}$-$\boldsymbol{X}$ and $\mathcal{P}$-$\boldsymbol{Z}$ imply that $\boldsymbol{1}_n \notin \text{c-sp}(\boldsymbol{Z}_2)$.

Proposition 1.79' *If*

$$\exists \boldsymbol{\theta}_0 \in \Theta_1 \quad \boldsymbol{X}\boldsymbol{\phi}(\boldsymbol{\theta}_0) + \bar{\boldsymbol{C}}(H)\boldsymbol{Z}\bar{\boldsymbol{\psi}}(\boldsymbol{\theta}_0) \in \ker\big(q\big(\bar{\boldsymbol{A}}(G)\big)\big), \tag{1.81}$$

then γ is not identified through the mean by $\mathcal{P}(\Theta_1)$.

118. A rank decomposition of $\bar{\boldsymbol{C}}(H)\boldsymbol{X}$ always exists.

Corollary 1.80' *The kernel condition (1.81) cannot be satisfied if γ is identified through the mean by $\mathcal{P}(\Theta_1)$. If $\mathcal{I}_0^+(H) \neq \emptyset$ (respectively, $\mathcal{I}_0^+(H) = \emptyset$) and the kernel condition (1.81) is not satisfied, then $\boldsymbol{\phi}_{-1} \neq \mathbf{0}_{K-1}$ or $\bar{\boldsymbol{\psi}} \neq \mathbf{0}_L$ (respectively, $\boldsymbol{\phi}_{-1} \neq \mathbf{0}_{K-1}$ or $\bar{\boldsymbol{\psi}}_{-1} \neq \mathbf{0}_{L-1}$) must be true.*

Proposition 1.84' *Suppose $\bar{\boldsymbol{A}}(G) \neq \bar{\boldsymbol{C}}(H)$.*

(1.84'.1) Suppose $\mathcal{I}_0^+(H) \neq \emptyset$. If the kernel condition (1.81) is not satisfied and the $n \times (2K+2L-1)$ matrix

$$\left(\mathbf{1}_n : \boldsymbol{\iota}_0^+(H) : \bar{\boldsymbol{A}}(G)\boldsymbol{\iota}_0^+(H) : \boldsymbol{X}_2 : \bar{\boldsymbol{A}}(G)\boldsymbol{X}_2 : \bar{\boldsymbol{C}}(H)\boldsymbol{Z}_2 : \bar{\boldsymbol{A}}(G)\bar{\boldsymbol{C}}(H)\boldsymbol{Z}_2\right)$$

has full column rank, then γ, $\boldsymbol{\phi}$, and $\bar{\boldsymbol{\psi}}$ are identified by $\mathcal{P}(\Theta_1)$. If γ, $\boldsymbol{\phi}$, and $\bar{\boldsymbol{\psi}}$ are identified through the mean by $\mathcal{P}(\Theta_1)$, then $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G)$, $\bar{\boldsymbol{C}}(H)$, $\bar{\boldsymbol{A}}(G)\bar{\boldsymbol{C}}(H)$ are linearly independent.

(1.84'.2) Suppose $\mathcal{I}_0^+(H) = \emptyset$. If the kernel condition (1.81) is not satisfied and the $n \times (2K+2L-3)$ matrix

$$\left(\mathbf{1}_n : \boldsymbol{X}_2 : \bar{\boldsymbol{A}}(G)\boldsymbol{X}_2 : \bar{\boldsymbol{C}}(H)\boldsymbol{Z}_2 : \bar{\boldsymbol{A}}(G)\bar{\boldsymbol{C}}(H)\boldsymbol{Z}_2\right)$$

has full column rank, then γ, $\boldsymbol{\phi}_{-1}$, and $\bar{\boldsymbol{\psi}}_{-1}$ are identified by $\mathcal{P}(\Theta_1)$. If γ, $\boldsymbol{\phi}_{-1}$, and $\bar{\boldsymbol{\psi}}_{-1}$ are identified through the mean by $\mathcal{P}(\Theta_1)$, then $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G)$, $\bar{\boldsymbol{C}}(H)$, $\bar{\boldsymbol{A}}(G)\bar{\boldsymbol{C}}(H)$ are linearly independent.

Proposition 1.86' *Suppose $\bar{\boldsymbol{A}}(G) = \bar{\boldsymbol{C}}(H)$. If the kernel condition (1.81) is not satisfied, $\gamma\bar{\boldsymbol{\phi}}_{-1} + (1+\gamma)\bar{\boldsymbol{\psi}}_{-1} \neq \mathbf{0}_{L-1}$, and the $n \times (2K+L-2)$ matrix*

$$\left(\mathbf{1}_n : \boldsymbol{X}_2 : \bar{\boldsymbol{A}}(G)\boldsymbol{X}_2 : \bar{\boldsymbol{A}}(G)^2\boldsymbol{Z}_2\right) \tag{1.82}$$

has full column rank, then γ, $\boldsymbol{\phi}_{-1}$, and $\bar{\boldsymbol{\psi}}_{-1}$ are identified by $\mathcal{P}(\Theta_1)$. If γ, $\boldsymbol{\phi}_{-1}$, and $\bar{\boldsymbol{\psi}}_{-1}$ are identified through the mean by $\mathcal{P}(\Theta_1)$, then $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G)$, $\bar{\boldsymbol{A}}(G)^2$ are linearly independent.

Proposition 1.89' *Suppose $G = H$ and $\mathbf{0}_n \neq \boldsymbol{\iota}_0^+(G) \in \text{c-sp}(\text{diag}(\boldsymbol{\iota}_0^+(G))\boldsymbol{Z}_2)$. If the kernel condition (1.81) is not satisfied, $\gamma\bar{\boldsymbol{\phi}}_{-1} + (1+\gamma)\bar{\boldsymbol{\psi}}_{-1} \neq \mathbf{0}_{L-1}$, and the $n \times (3K+2L-4)$ matrix*

$$\left(\mathbf{1}_n : \boldsymbol{X}_2 : \bar{\boldsymbol{A}}(G)\boldsymbol{X}_2 : \bar{\boldsymbol{C}}(G)\boldsymbol{Z}_2 : \bar{\boldsymbol{A}}(G)^2\boldsymbol{X}_2 : \bar{\boldsymbol{A}}(G)\bar{\boldsymbol{C}}(G)\boldsymbol{Z}_2\right) \tag{1.83}$$

has full column rank, then γ, $\boldsymbol{\phi}$, and $\bar{\boldsymbol{\psi}}$ are identified by $\mathcal{P}(\Theta_1)$. If γ, $\boldsymbol{\phi}$, and $\bar{\boldsymbol{\psi}}$ are identified through the mean by $\mathcal{P}(\Theta_1)$, then $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G)$, $\bar{\boldsymbol{C}}(G)$, $\bar{\boldsymbol{A}}(G)^2$, $\bar{\boldsymbol{A}}(G)\bar{\boldsymbol{C}}(G)$ are linearly independent.

Proposition 1.90' *Suppose γ is identified by $\mathcal{P}(\Theta_1)$.*

(1.90'.1) Suppose $\mathcal{I}_0^+(H) \neq \emptyset$. If the $n \times (K+L)$ matrix $(\mathbf{1}_n : \boldsymbol{\iota}_0^+(H) : \boldsymbol{X}_2 : \bar{\boldsymbol{C}}(H)\boldsymbol{Z}_2)$ has full column rank, then $\boldsymbol{\phi}$ and $\bar{\boldsymbol{\psi}}$ are identified by $\mathcal{P}(\Theta_1)$.

(1.90'.2) Suppose $\mathcal{I}_0^+(H) = \emptyset$. If the $n \times (K+L-1)$ matrix $(\mathbf{1}_n : \boldsymbol{X}_2 : \bar{\boldsymbol{C}}(H)\boldsymbol{Z}_2)$ has full column rank, then $\boldsymbol{\phi}_{-1}$ and $\bar{\boldsymbol{\psi}}_{-1}$ are identified by $\mathcal{P}(\Theta_1)$.

1.4.3.3 Identification through the variance by $\mathcal{P}(\Theta_0)$

A subparameter g is called identified through the variance by $\mathcal{P}(\Theta_0)$ if for all pairs of parameter points $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ in Θ_0^2, $g(\boldsymbol{\theta}_1) = g(\boldsymbol{\theta}_2)$ is necessary for $\operatorname{var}(\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_1)) \mid \mathfrak{F}) = \operatorname{var}(\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_2)) \mid \mathfrak{F})$. If g is identified through the variance by $\mathcal{P}(\Theta_0)$, then it is identified by $\mathcal{P}(\Theta_0)$ because, for all pairs of parameter points $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ in Θ_0^2, $\operatorname{var}(\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_1)) \mid \mathfrak{F}) = \operatorname{var}(\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_2)) \mid \mathfrak{F})$ is necessary for $\mathbb{P}_{\boldsymbol{\theta}_1,\mathfrak{F}} = \mathbb{P}_{\boldsymbol{\theta}_2,\mathfrak{F}}$. For all parameter points $\boldsymbol{\theta}_0$ in Θ_0,

$$\operatorname{var}\big(\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_0)) \mid \mathfrak{F}\big) = \varsigma(\boldsymbol{\theta}_0)^2\Big(\boldsymbol{I}_n - \gamma(\boldsymbol{\theta}_0)q\big(\bar{\boldsymbol{A}}(G)\big)\Big)^{-1}\Big(\boldsymbol{I}_n - \gamma(\boldsymbol{\theta}_0)q\big(\bar{\boldsymbol{A}}(G)\big)^{\top}\Big)^{-1},$$

from which it follows that the present discussion is confined to the identification of γ and ς.

An identifying condition for γ and ς is stated in Proposition 1.91. An equivalent condition follows from Lemma 1.92.

Proposition 1.91 *If $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G) + \bar{\boldsymbol{A}}(G)^{\top}$, $\bar{\boldsymbol{A}}(G)\bar{\boldsymbol{A}}(G)^{\top}$ are linearly independent, then γ and ς are identified by $\mathcal{P}(\Theta_0)$. If γ and ς are identified through the variance by $\mathcal{P}(\Theta_0)$, then $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G) + \bar{\boldsymbol{A}}(G)^{\top}$, $\bar{\boldsymbol{A}}(G)\bar{\boldsymbol{A}}(G)^{\top}$ are linearly independent.*

Lemma 1.92 *The matrices $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G) + \bar{\boldsymbol{A}}(G)^{\top}$, $\bar{\boldsymbol{A}}(G)\bar{\boldsymbol{A}}(G)^{\top}$ are linearly independent if and only if $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G) + \bar{\boldsymbol{A}}(G)^{\top}$, $\bar{\boldsymbol{A}}(G)^{\top}\bar{\boldsymbol{A}}(G)$ are linearly independent.*

Proposition 1.91 implies that γ and ς are not identified through the variance by $\mathcal{P}(\Theta_0)$ if G is complete. Indeed, $\bar{\boldsymbol{A}}(G) = (1/(n-1))(\mathbf{1}_n\mathbf{1}_n^{\top} - \boldsymbol{I}_n)$ if G is complete, from which it follows that

$$\bar{\boldsymbol{A}}(G)\bar{\boldsymbol{A}}(G)^{\top} = \frac{1}{n-1}\boldsymbol{I}_n + \frac{n-2}{2(n-1)}\big(\bar{\boldsymbol{A}}(G) + \bar{\boldsymbol{A}}(G)^{\top}\big),$$

that is, $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G) + \bar{\boldsymbol{A}}(G)^{\top}$, $\bar{\boldsymbol{A}}(G)\bar{\boldsymbol{A}}(G)^{\top}$ are linearly dependent.

The identifying condition of Proposition 1.91 is satisfied if and only if the $n^2 \times 3$ matrix

$$\Big(\operatorname{vec}_{n,n}(\boldsymbol{I}_n) : \operatorname{vec}_{n,n}\big(\bar{\boldsymbol{A}}(G) + \bar{\boldsymbol{A}}(G)^{\top}\big) : \operatorname{vec}_{n,n}\big(\bar{\boldsymbol{A}}(G)\bar{\boldsymbol{A}}(G)^{\top}\big)\Big) \tag{1.84}$$

has full column rank.[119] Example 1.93 demonstrates the usefulness of this equivalence.

Example 1.93 Suppose $\mathcal{A}(G) = \bigcup_{i \in \mathcal{I}\setminus\{1\}}\{(i,1)\}$, that is, G is star-shaped with arcs from every peripheral player $i \in \mathcal{I} \setminus \{1\}$ to the central player 1. The matrices $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G) + \bar{\boldsymbol{A}}(G)^{\top} = \mathbf{1}_n\boldsymbol{e}_1^{\top} + \boldsymbol{e}_1\mathbf{1}_n^{\top}$, $\bar{\boldsymbol{A}}(G)\bar{\boldsymbol{A}}(G)^{\top} = \mathbf{1}_n\mathbf{1}_n^{\top}$ are linearly independent. Indeed, the matrix (1.84) has full column rank because it is equal to

$$\begin{pmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_1 + \mathbf{1}_n & \mathbf{1}_n \\ \boldsymbol{e}_2 & \boldsymbol{e}_1 & \mathbf{1}_n \\ \vdots & \vdots & \vdots \\ \boldsymbol{e}_n & \boldsymbol{e}_1 & \mathbf{1}_n \end{pmatrix}$$

119. For any pair (r,s) of positive integers, $\operatorname{vec}_{r,s} \colon \mathcal{M}(r,s,\mathbb{R}) \to \mathcal{M}(rs,1,\mathbb{R})$ denotes the vectorization operator that maps an $r \times s$ matrix $\boldsymbol{A}$ to an $rs \times 1$ matrix $\operatorname{vec}_{r,s}(\boldsymbol{A})$, that is, a column vector with rs components, by stacking its columns on top of one another.

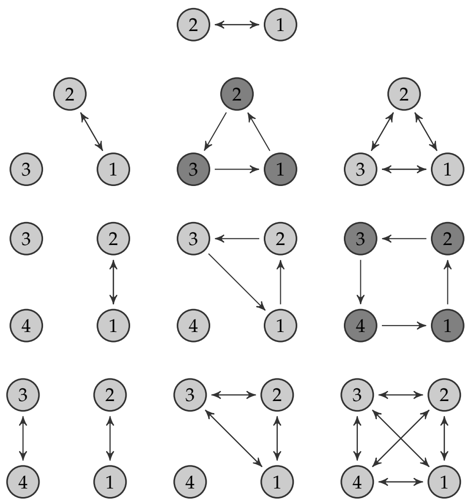

Figure 1.20. Nonempty digraphs of orders at least 2 and at most 4 that do not satisfy the identifying condition of Proposition 1.91 (digraphs in light or dark gray)

and its Gram determinant is equal to $2(n-1)^2n^2$ and therefore different from zero or, alternatively, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\mathbf{1}_n$ are linearly independent.[120] ◇

It is instructive to give an overview of all nonempty digraphs of orders at least 2 and at most 4 that satisfy the identifying condition of Proposition 1.91. (All figures reported hereinafter refer to nonempty digraphs.) In a negative sense, there are 10 (out of 234) isomorphism classes of digraphs of orders at least 2 and at most 4 that do not satisfy the condition. A representative of each class is depicted in Figure 1.20. In a positive sense, any digraph of order at most 4 that is not isomorphic to one of the digraphs depicted in Figure 1.20 satisfies the condition. Some count statistics may help complete the picture. Among the 3 digraphs of order 2, which are partitioned into 2 isomorphism classes, only 2 isomorphic graphs, namely, $(\{1,2\},\{(1,2)\})$ and $(\{1,2\},\{(2,1)\})$, satisfy the condition. There are 63 digraphs of order 3, which are partitioned into 15 isomorphism classes. The representatives of 12 classes (57 digraphs altogether, which corresponds to a share of approximately 90 per cent of 63) satisfy the condition. There are 4,095 digraphs of order 4, which are partitioned into 217 isomorphism classes. The representatives of 211 classes (4,067 digraphs altogether, which corresponds to a share of approximately 99 per cent of 4,095) satisfy the condition.

The rest of this section is about a characterization of the identifying condition of Proposition 1.91 in terms of the topology of G and results that follow therefrom. The set of all nonempty digraphs of order n can be partitioned into two subsets: a set for which the condition is satisfied and a set for which it is not satisfied. Proposition 1.95 gives a characterization of the latter set in terms of normally regular

120. Note that an $r \times s$ matrix $\boldsymbol{A}$ over $\mathbb{R}$ with $r \geq s$ has full column rank if and only if its Gram determinant is nonzero, that is, $\det(\boldsymbol{A}^\top \boldsymbol{A}) \neq 0$.

digraphs. The notion of a normally regular digraph was introduced in Jørgensen (1994) and slightly generalized in Jørgensen (2015). In the current context, only the definition in latter paper is of interest. Jørgensen's (2015) theory of normally regular digraphs requires that in the following definition v be a positive integer, k be a nonnegative integer, and λ and μ be integers.

Definition NRD (Jørgensen 2015, Definition 1) A *normally regular digraph with parameters* (v, k, λ, μ), also denoted by NRD(v, k, λ, μ), is a digraph D of order v for which the following conditions are satisfied: (NRD.1) all vertices have out-degree k; (NRD.2) all adjacent vertices x and y for which exactly one of the arcs (x, y) or (y, x) is in D have λ common out-neighbors; (NRD.3) all adjacent vertices x and y for which both arcs (x, y) and (y, x) are in D have $2\lambda - \mu$ common out-neighbors; and (NRD.4) all nonadjacent vertices x and y have μ common out-neighbors.

An NRD(v, k, λ, μ) is normal, that is, two distinct vertices have the same number of common in-neighbors and common out-neighbors, and k-regular, that is, all vertices have in-degree k and out-degree k (see Jørgensen 2015, Corollary 11). A normal and regular digraph is, however, not necessarily a normally regular digraph (see Section 1, p. 3). An NRD$(v, k, \lambda, 0)$ need not be weakly connected, but all of its weakly connected components are normally regular digraphs with the same values of (k, λ) (see Section 4.1, p. 10). Moreover, all weakly connected components of an NRD$(v, k, \lambda, 0)$ are strongly connected (see Section 4.1, p. 10).[121] Examples of normally regular digraphs of order v include the cycle digraph, for which $(v, k, \lambda, \mu) = (v, 1, 0, 0)$, and the complete digraph, for which $(v, k, \lambda, \mu) = (v, v-1, l, m)$ for a pair of integers (l, m) such that $2l - m = v - 2$, for example, $(v, k, \lambda, \mu) = (v, v-1, (v-2)/2, 0)$. Another example of a normally regular digraph with $\mu = 0$ is given in Example 1.94, which is taken from Jørgensen (2015).

Example 1.94 (Jørgensen 2015, Example 2) Let D be the digraph of order 8 with adjacency matrix $\dot{A}(D)$ given by

$$\dot{A}(D) = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

See Figure 1.21 for an illustration of D. The digraph D is an NRD$(8, 3, 1, 0)$. It is the digraph of smallest order among all normally regular digraphs with $\mu = 0$ that are different from the empty digraph, the cycle digraph, and the complete digraph. ⋄

121. This can be seen as follows. If $k = 0$, then all weakly connected components of an NRD$(v, k, \lambda, 0)$ are of order 1 and therefore strongly connected. If $k > 0$, then all weakly connected components of an NRD$(v, k, \lambda, 0)$ are at least 1-regular, from which it follows that they are strongly connected.

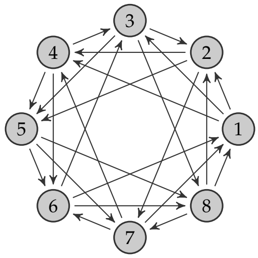

Figure 1.21. The NRD$(8,3,1,0)$ (Example 1.94)

A digraph D is an NRD(v,k,λ,μ) if and only if its adjacency matrix $\dot{\boldsymbol{A}}(D)$ satisfies (see Jørgensen 2015, Proposition 1)

$$\dot{\boldsymbol{A}}(D)\dot{\boldsymbol{A}}(D)^\top = k\boldsymbol{I}_v + \lambda\big(\dot{\boldsymbol{A}}(D) + \dot{\boldsymbol{A}}(D)^\top\big) + \mu\big(\mathbf{1}_v\mathbf{1}_v^\top - \boldsymbol{I}_v - \dot{\boldsymbol{A}}(D) - \dot{\boldsymbol{A}}(D)^\top\big). \tag{1.85}$$

In case $\mu = 0$, (1.85) is equivalent to $k\boldsymbol{I}_v + \lambda(\dot{\boldsymbol{A}}(D) + \dot{\boldsymbol{A}}(D)^\top) - \dot{\boldsymbol{A}}(D)\dot{\boldsymbol{A}}(D)^\top = \boldsymbol{O}_v$, that is, $\boldsymbol{I}_v$, $\dot{\boldsymbol{A}}(D) + \dot{\boldsymbol{A}}(D)^\top$, $\dot{\boldsymbol{A}}(D)\dot{\boldsymbol{A}}(D)^\top$ are linearly dependent. This result gives rise to the conjecture that G is a normally regular digraph if $\mathcal{I}_0(G) = \varnothing$ (a nonempty normally regular digraph cannot have isolated vertices) and $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G) + \bar{\boldsymbol{A}}(G)^\top$, $\bar{\boldsymbol{A}}(G)\bar{\boldsymbol{A}}(G)^\top$ are linearly dependent. Proposition 1.95 demonstrates that an even more general result is true. In order to state the result, let $G\langle\mathcal{I}^\star(G)\rangle$ denote the subdigraph of G induced by the vertex subset $\mathcal{I}^\star(G) := \mathcal{I} \setminus \mathcal{I}_0(G)$ of $\mathcal{I}$, that is, $G\langle\mathcal{I}^\star(G)\rangle$ is the largest subdigraph of G that has no isolated vertices. Note that $|\mathcal{I}^\star(G)| > 1$ and $G\langle\mathcal{I}^\star(G)\rangle$ is not empty because G is not empty.

Proposition 1.95 *The matrices $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G) + \bar{\boldsymbol{A}}(G)^\top$, $\bar{\boldsymbol{A}}(G)\bar{\boldsymbol{A}}(G)^\top$ are linearly dependent if and only if $G\langle\mathcal{I}^\star(G)\rangle$ is an* NRD$(|\mathcal{I}^\star(G)|, d, (d-1)/2, 0)$ *for some positive integer d.*

It is worth mentioning that one conditional statement of Proposition 1.95, namely, if $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G) + \bar{\boldsymbol{A}}(G)^\top$, $\bar{\boldsymbol{A}}(G)\bar{\boldsymbol{A}}(G)^\top$ are linearly dependent, then $G\langle\mathcal{I}^\star(G)\rangle$ is an NRD$(|\mathcal{I}^\star(G)|, d, (d-1)/2, 0)$ for some positive integer d, does not follow from the characterization of a normally regular digraph given in (1.85). The reason is that the adjacency matrix of $G\langle\mathcal{I}^\star(G)\rangle$ and the endogenous effects matrix of $G\langle\mathcal{I}^\star(G)\rangle$ are—although related—distinct mathematical objects unless $G\langle\mathcal{I}^\star(G)\rangle$ is 1-regular.

Proposition 1.95 states that the identifying condition of Proposition 1.91, namely, $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G) + \bar{\boldsymbol{A}}(G)^\top$, $\bar{\boldsymbol{A}}(G)\bar{\boldsymbol{A}}(G)^\top$ are linearly independent, is not satisfied if and only if $G\langle\mathcal{I}^\star(G)\rangle$ is an NRD$(|\mathcal{I}^\star(G)|, d, (d-1)/2, 0)$ for some positive integer d. Proposition 1.95 therefore constitutes a negative characterization of the topology of digraphs of order n that satisfy the identifying condition of Proposition 1.91.

Proposition 1.95 entails a necessary and sufficient condition for the linear independence of $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G) + \bar{\boldsymbol{A}}(G)^\top$, $\bar{\boldsymbol{A}}(G)\bar{\boldsymbol{A}}(G)^\top$ in case G consists solely of complete components.

Corollary 1.96 *Suppose G has at least two complete components of orders at least* 2.

(1.96.1) *If not all complete components are of the same order, then $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G) + \bar{\boldsymbol{A}}(G)^\top$, $\bar{\boldsymbol{A}}(G)\bar{\boldsymbol{A}}(G)^\top$ are linearly independent.*

(1.96.2) *Suppose G consists solely of complete components. Not all components are of the same order if and only if $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G) + \bar{\boldsymbol{A}}(G)^{\intercal}$, $\bar{\boldsymbol{A}}(G)\bar{\boldsymbol{A}}(G)^{\intercal}$ are linearly independent.*

In case G has at least two complete components of orders at least 2, γ and ς are identified by $\mathcal{P}(\Theta_0)$ if not all complete components are of the same order (Proposition 1.91 and Result 1.96.1). In case G consists solely of complete components, γ and ς are identified by $\mathcal{P}(\Theta_0)$ if not all components are of the same order (Proposition 1.91 and Result 1.96.2), and not all components are of the same order if γ and ς are identified through the variance by $\mathcal{P}(\Theta_0)$ (Proposition 1.91 and Result 1.96.2).[122]

The necessary condition for identification of Proposition 1.86 is in general not stronger than the identifying condition of Proposition 1.91. In other words, there exist digraphs D with $\mathcal{I}_0^+(D) = \emptyset$ for which $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(D)$, $\bar{\boldsymbol{A}}(D)^2$ are linearly independent and $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(D) + \bar{\boldsymbol{A}}(D)^{\intercal}$, $\bar{\boldsymbol{A}}(D)\bar{\boldsymbol{A}}(D)^{\intercal}$ are linearly dependent (see the digraphs depicted in dark gray in Figure 1.20).[123,124] By virtue of Proposition 1.95, the two conditions are equivalent in case G is symmetric (without imposing the restriction that $\mathcal{I}_0^+(G) = \emptyset$). For example, friendship may be considered a symmetric relation, which gives rise to a symmetric digraph G.

Proposition 1.97 *Suppose G is symmetric. The matrices $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G)$, $\bar{\boldsymbol{A}}(G)^2$ are linearly independent if and only if $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G) + \bar{\boldsymbol{A}}(G)^{\intercal}$, $\bar{\boldsymbol{A}}(G)\bar{\boldsymbol{A}}(G)^{\intercal}$ are linearly independent.*[125]

122. Lee (2007) and Bramoullé, Djebbari, and Fortin (2009) discuss similar results. Lee (2007) discusses identification by and estimation of a statistical model of social interactions where the individuals interact in groups, that is, the individuals are connected by a digraph that consists solely of complete components. He shows that endogenous and exogenous effects are identified in the presence of group specific fixed effects if there is sufficient variation in group sizes. Bramoullé, Djebbari, and Fortin (2009) discuss identification of endogenous and exogenous effects by two statistical models that may be considered variations of Manski's (1993) linear-in-means model and Moffitt's (2001) model. For the case where individuals interact in groups, they show that in the absence of group specific fixed effects endogenous and exogenous effects are identified if not all groups are of the same size and a parameter restriction is satisfied (see Bramoullé, Djebbari, and Fortin 2009, Proposition 2).

123. Within the setup of Proposition 1.86, $\bar{\boldsymbol{A}}(G) = \bar{\boldsymbol{C}}(H)$, which implies that $\mathcal{I}_0(G) \subset \mathcal{I}_0^+(G) = \emptyset$ (Result 1.83.3), which in turn implies that $\mathcal{I}^{\star}(G) = \mathcal{I}$.

124. It is an open problem whether there exist digraphs D of order n (at least 5) with $\mathcal{I}_0^+(D) = \emptyset$ for which $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(D)$, $\bar{\boldsymbol{A}}(D)^2$ are linearly dependent and $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(D) + \bar{\boldsymbol{A}}(D)^{\intercal}$, $\bar{\boldsymbol{A}}(D)\bar{\boldsymbol{A}}(D)^{\intercal}$ are linearly independent. To this end, note that for a digraph D of order n with $\mathcal{I}_0^+(D) = \emptyset$ and whose weakly connected components are strongly connected, if $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(D) + \bar{\boldsymbol{A}}(D)^{\intercal}$, $\bar{\boldsymbol{A}}(D)\bar{\boldsymbol{A}}(D)^{\intercal}$ are linearly independent, then $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(D)$, $\bar{\boldsymbol{A}}(D)^2$ are linearly independent. The proof is as follows. Suppose D is a digraph of order n with $\mathcal{I}_0^+(D) = \emptyset$ (and therefore $\mathcal{I}_0(D) = \emptyset$) and all its weakly connected components are strongly connected. Suppose $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(D)$, $\bar{\boldsymbol{A}}(D)^2$ are linearly dependent. There do not exist intransitive triples in D because $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(D)$, $\bar{\boldsymbol{A}}(D)^2$ are linearly dependent (Lemma 1.87). It follows that D is transitive, which implies that all of its strongly connected components are transitive. Note that a strongly connected digraph is transitive if and only if it is complete (see Bang-Jensen and Gutin 2009, p. 37). It follows that all strongly connected components of D are complete. All complete components of D must be of the same order, denoted by m, because $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(D)$, $\bar{\boldsymbol{A}}(D)^2$ are linearly dependent (see Bramoullé, Djebbari, and Fortin 2009, Section 2.4.1.3). It follows that D is an NRD$(n, m-1, (m-2)/2, 0)$, from which it follows that $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(D) + \bar{\boldsymbol{A}}(D)^{\intercal}$, $\bar{\boldsymbol{A}}(D)\bar{\boldsymbol{A}}(D)^{\intercal}$ are linearly dependent (Proposition 1.95).

125. Note that $\bar{\boldsymbol{A}}(G)$ is not symmetric if G is symmetric, unless all weakly connected components of G of order at least 2 are 1-regular.

Propositions 1.95 and 1.97 allow of a negative characterization of the topology of symmetric digraphs D of order n that satisfy the necessary condition for identification of Proposition 1.86: they must not be normally regular digraphs with parameters $(n, d, (d-1)/2, 0)$ for some positive integer d (Corollary 1.98).[126]

Corollary 1.98 *Suppose G is symmetric. The matrices $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G)$, $\bar{\boldsymbol{A}}(G)^2$ are linearly dependent if and only if $G\langle \mathcal{I}^\star(G)\rangle$ is an* $\mathrm{NRD}(|\mathcal{I}^\star(G)|, d, (d-1)/2, 0)$ *for some positive integer d.*

Propositions 1.86, 1.91, and 1.97 give rise to the following result, which establishes a nexus between identification through the mean by $\mathcal{P}(\Theta_0)$ and identification through the variance by $\mathcal{P}(\Theta_0)$.

Corollary 1.99 *Suppose G is symmetric and $\bar{\boldsymbol{A}}(G) = \bar{\boldsymbol{C}}(H)$. If γ, $\boldsymbol{\phi}_{-1}$, and $\boldsymbol{\psi}_{-1}$ are identified through the mean by $\mathcal{P}(\Theta_0)$, then ς is identified by $\mathcal{P}(\Theta_0)$.*

1.4.3.4 Identification through the variance by $\mathcal{P}(\Theta)$

A subparameter g is identified by $\mathcal{P}(\Theta)$ if it is identified through the variance by $\mathcal{P}(\Theta)$. For all parameter points $\boldsymbol{\theta}_0$ in Θ,

$$\begin{aligned}\operatorname{var}\big(\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_0)) \mid \mathfrak{F}\big) = \varsigma(\boldsymbol{\theta}_0)^2 \Big(\boldsymbol{I}_n - \gamma(\boldsymbol{\theta}_0) q\big(\bar{\boldsymbol{A}}(G)\big)\Big)^{-1} \big(\boldsymbol{I}_n + \zeta(\boldsymbol{\theta}_0)\bar{\boldsymbol{C}}(H)\big) \\ \times \big(\boldsymbol{I}_n + \zeta(\boldsymbol{\theta}_0)\bar{\boldsymbol{C}}(H)^\top\big) \Big(\boldsymbol{I}_n - \gamma(\boldsymbol{\theta}_0) q\big(\bar{\boldsymbol{A}}(G)\big)^\top\Big)^{-1},\end{aligned}$$

from which it follows that the present discussion is confined to the identification of γ, ζ, and ς.[127]

The presentation of the results is organized as follows. Proposition 1.100 gives an identifying condition for ζ and ς under the premise that γ is identified through the mean by $\mathcal{P}(\Theta)$. An equivalent condition follows from Lemma 1.101. Proposition 1.102 gives a sufficient condition for the identifying condition of Proposition 1.100, namely, $\mathcal{I}_0^+(H) \neq \emptyset$, that is, there are players of Γ without out-neighbors in H. A characterization of the identifying condition of Proposition 1.100 in terms of the topology of H is given in Proposition 1.103. In order to put into context these results, note that Proposition 1.100 parallels Proposition 1.91, Lemma 1.101 parallels Lemma 1.92, and Proposition 1.103 parallels Proposition 1.95. Propositions 1.104 and 1.105 state identifying conditions for γ, ζ, and ς, where the former result covers the case $\bar{\boldsymbol{A}}(G) \neq \bar{\boldsymbol{C}}(H)$ and the latter the case $\bar{\boldsymbol{A}}(G) = \bar{\boldsymbol{C}}(H)$.

Proposition 1.100 *Suppose γ is identified by $\mathcal{P}(\Theta)$. If $\boldsymbol{I}_n$, $\bar{\boldsymbol{C}}(H) + \bar{\boldsymbol{C}}(H)^\top$, $\bar{\boldsymbol{C}}(H)\bar{\boldsymbol{C}}(H)^\top$ are linearly independent, then ζ and ς are identified by $\mathcal{P}(\Theta)$. If ζ and ς are identified through the variance by $\mathcal{P}(\Theta)$, then $\boldsymbol{I}_n$, $\bar{\boldsymbol{C}}(H) + \bar{\boldsymbol{C}}(H)^\top$, $\bar{\boldsymbol{C}}(H)\bar{\boldsymbol{C}}(H)^\top$ are linearly independent.*

Lemma 1.101 *The matrices $\boldsymbol{I}_n$, $\bar{\boldsymbol{C}}(H) + \bar{\boldsymbol{C}}(H)^\top$, $\bar{\boldsymbol{C}}(H)\bar{\boldsymbol{C}}(H)^\top$ are linearly independent if and only if $\boldsymbol{I}_n$, $\bar{\boldsymbol{C}}(H) + \bar{\boldsymbol{C}}(H)^\top$, $\bar{\boldsymbol{C}}(H)^\top\bar{\boldsymbol{C}}(H)$ are linearly independent.*

126. See Footnote 123.
127. Note that $\operatorname{var}(\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_0)) \mid \mathfrak{F})$ is positive definite if and only if $-1 \notin \sigma(\zeta(\boldsymbol{\theta}_0)\bar{\boldsymbol{C}}(H))$.

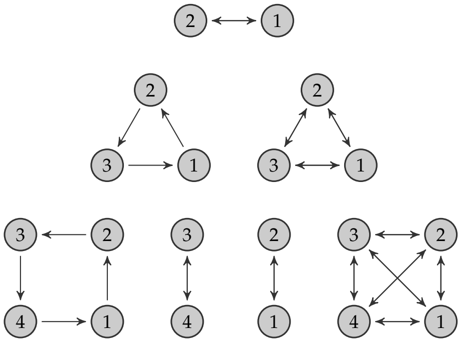

Figure 1.22. Nonempty digraphs of orders at least 2 and at most 4 that do not satisfy the identifying condition of Proposition 1.100

Proposition 1.102 *If $\mathcal{I}_0^+(H) \neq \emptyset$, then $\boldsymbol{I}_n$, $\bar{\boldsymbol{C}}(H) + \bar{\boldsymbol{C}}(H)^\top$, $\bar{\boldsymbol{C}}(H)\bar{\boldsymbol{C}}(H)^\top$ are linearly independent.*

Note that the condition $\mathcal{I}_0^+(H) \neq \emptyset$ is not necessary for the linear independence of $\boldsymbol{I}_n$, $\bar{\boldsymbol{C}}(H) + \bar{\boldsymbol{C}}(H)^\top$, $\bar{\boldsymbol{C}}(H)\bar{\boldsymbol{C}}(H)^\top$.

Proposition 1.103 *The matrices $\boldsymbol{I}_n$, $\bar{\boldsymbol{C}}(H) + \bar{\boldsymbol{C}}(H)^\top$, $\bar{\boldsymbol{C}}(H)\bar{\boldsymbol{C}}(H)^\top$ are linearly dependent if and only if H is an $\mathrm{NRD}(n, d, (d-1)/2, 0)$ for some positive integer d.*

If H is an $\mathrm{NRD}(n, d, (d-1)/2, 0)$ for some positive integer d, then it is d-regular, from which $\mathcal{I}_0^+(H) = \emptyset$ follows. Proposition 1.102 may therefore be considered a corollary to Proposition 1.103.

Propositions 1.95 and 1.103 imply that the identifying condition of Proposition 1.91 is stronger than the identifying condition of Proposition 1.100 in case $G = H$. The two conditions are equivalent if $G = H$ and $\mathcal{I}_0^+(H) = \emptyset$, in which case $\bar{\boldsymbol{A}}(G) = \bar{\boldsymbol{C}}(H)$ (Result 1.83.1).

There are 6 (out of 234) isomorphism classes of nonempty digraphs of orders at least 2 and at most 4 whose representatives do not satisfy the identifying condition of Proposition 1.100. A representative of each class is depicted in Figure 1.22. They are all normally regular digraphs (Proposition 1.103). The digraphs of Figure 1.22 are a proper subset of those of Figure 1.20 because the identifying condition of Proposition 1.91 is stronger than that of Proposition 1.100 in case $G = H$.

Proposition 1.104 *Suppose $\bar{\boldsymbol{A}}(G) \neq \bar{\boldsymbol{C}}(H)$.*[128] *If the nine matrices*

$$\begin{gathered}\boldsymbol{I}_n, \quad \bar{\boldsymbol{A}}(G) + \bar{\boldsymbol{A}}(G)^\top, \quad \bar{\boldsymbol{A}}(G)\bar{\boldsymbol{A}}(G)^\top, \quad \bar{\boldsymbol{C}}(H) + \bar{\boldsymbol{C}}(H)^\top, \\ \bar{\boldsymbol{C}}(H)\bar{\boldsymbol{C}}(H)^\top, \quad \bar{\boldsymbol{A}}(G)\big(\bar{\boldsymbol{C}}(H) + \bar{\boldsymbol{C}}(H)^\top\big) + \big(\bar{\boldsymbol{C}}(H) + \bar{\boldsymbol{C}}(H)^\top\big)\bar{\boldsymbol{A}}(G)^\top, \\ \bar{\boldsymbol{A}}(G)\bar{\boldsymbol{C}}(H)\bar{\boldsymbol{C}}(H)^\top + \bar{\boldsymbol{C}}(H)\big(\bar{\boldsymbol{A}}(G)\bar{\boldsymbol{C}}(H)\big)^\top, \quad \bar{\boldsymbol{A}}(G)\big(\bar{\boldsymbol{C}}(H) + \bar{\boldsymbol{C}}(H)^\top\big)\bar{\boldsymbol{A}}(G)^\top, \\ \bar{\boldsymbol{A}}(G)\bar{\boldsymbol{C}}(H)\big(\bar{\boldsymbol{A}}(G)\bar{\boldsymbol{C}}(H)\big)^\top\end{gathered}$$

are linearly independent, then γ, ζ, and ς are identified by $\mathcal{P}(\Theta)$.

128. Note that $\bar{\boldsymbol{A}}(G) \neq \bar{\boldsymbol{C}}(H)$ if $G = H$ and $\mathcal{I}_0^+(H) \neq \emptyset$ (Result 1.83.2).

Proposition 1.105 *Suppose $\bar{A}(G) = \bar{C}(H)$.*[129] *If $\gamma\zeta \geq 0$ and the six matrices*

$$\boldsymbol{I}_n, \quad \bar{\boldsymbol{C}}(H) + \bar{\boldsymbol{C}}(H)^\top, \quad \bar{\boldsymbol{C}}(H)\bar{\boldsymbol{C}}(H)^\top, \quad \bar{\boldsymbol{C}}(H)^2 + \bar{\boldsymbol{C}}(H)^{\top 2},$$
$$\bar{\boldsymbol{C}}(H)\big(\bar{\boldsymbol{C}}(H) + \bar{\boldsymbol{C}}(H)^\top\big)\bar{\boldsymbol{C}}(H)^\top, \quad \bar{\boldsymbol{C}}(H)^2\bar{\boldsymbol{C}}(H)^{\top 2}$$

are linearly independent, then γ, ζ, and ς are identified by $\mathcal{P}(\Theta)$. In general, γ, ζ, and ς are not identified through the variance by $\mathcal{P}(\Theta)$, even if the above six matrices are linearly independent.

The inequality $\gamma\zeta \geq 0$ entails a restriction on the parameter space Θ. It is true if and only if all pairs of subparameter points (γ_0, ζ_0) lie in the first or third closed orthant in $\mathbb{R}^2$, in particular, if both γ_0 and ζ_0 are different from zero, then they must have the same sign. The discussion of idiosyncrasies that admit of local externalities (see Section 1.3.7.1) suggests that a positive γ_0 (that is, a preference for conformist behavior) is associated with a positive ζ_0 (that is, idiosyncrasies with positive local externalities) in order that a nonempty digraph G that is identical to H emerges from a network formation game because arcs in G entail a social cost if γ_0 is positive and arcs in H entail a social benefit in terms of positive local externalities if ζ_0 is positive.

There are 32 (out of 131) isomorphism classes of digraphs H of orders 3 or 4 with $\mathcal{I}_0^+(H) = \emptyset$ whose representatives do not satisfy the identifying condition of Proposition 1.105. A representative of each class is depicted in Figure 1.23. The digraphs of Figure 1.22 of orders at least 3 are a proper subset of those of Figure 1.23 because the identifying condition of Proposition 1.105 is stronger than that of Proposition 1.100.

1.4.3.5 Identification through the mean by $\mathcal{Q}(\Theta)$

A subparameter g is called identified via $q(\bar{\boldsymbol{A}}(G))$-LD through the mean by $\mathcal{Q}(\Theta)$ if for all pairs of parameter points $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ in Θ^2, $g(\boldsymbol{\theta}_1) = g(\boldsymbol{\theta}_2)$ is necessary for $\mathbb{E}(q(\bar{\boldsymbol{A}}(G))\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_1)) \mid \mathfrak{F}) = \mathbb{E}(q(\bar{\boldsymbol{A}}(G))\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_2)) \mid \mathfrak{F})$. If g is identified via $q(\bar{\boldsymbol{A}}(G))$-LD through the mean by $\mathcal{Q}(\Theta)$, then it is identified by $\mathcal{Q}(\Theta)$. For all parameter points $\boldsymbol{\theta}_0$ in Θ, if $\mathcal{I}_0^+(H) \neq \emptyset$, then

$$\mathbb{E}\big(q(\bar{\boldsymbol{A}}(G))\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_0)) \mid \mathfrak{F}\big) = \Big(\boldsymbol{I}_n - \gamma(\boldsymbol{\theta}_0)q(\bar{\boldsymbol{A}}(G))\Big)^{-1} q(\bar{\boldsymbol{A}}(G))$$
$$\times \big(\boldsymbol{X}_2\boldsymbol{\phi}_{-1}(\boldsymbol{\theta}_0) - \psi(\boldsymbol{\theta}_0)\boldsymbol{\iota}_0^+(H) + \bar{\boldsymbol{C}}(H)\boldsymbol{X}_2\boldsymbol{\psi}_{-1}(\boldsymbol{\theta}_0)\big),$$

and if $\mathcal{I}_0^+(H) = \emptyset$, then

$$\mathbb{E}\big(q(\bar{\boldsymbol{A}}(G))\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_0)) \mid \mathfrak{F}\big) = \Big(\boldsymbol{I}_n - \gamma(\boldsymbol{\theta}_0)q(\bar{\boldsymbol{A}}(G))\Big)^{-1} q(\bar{\boldsymbol{A}}(G))$$
$$\times \big(\boldsymbol{X}_2\boldsymbol{\phi}_{-1}(\boldsymbol{\theta}_0) + \bar{\boldsymbol{C}}(H)\boldsymbol{X}_2\boldsymbol{\psi}_{-1}(\boldsymbol{\theta}_0)\big),$$

129. Note that $\mathcal{I}_0^+(G) = \mathcal{I}_0^+(H) = \emptyset$ if $\bar{A}(G) = \bar{\boldsymbol{C}}(H)$ (Result 1.83.3).

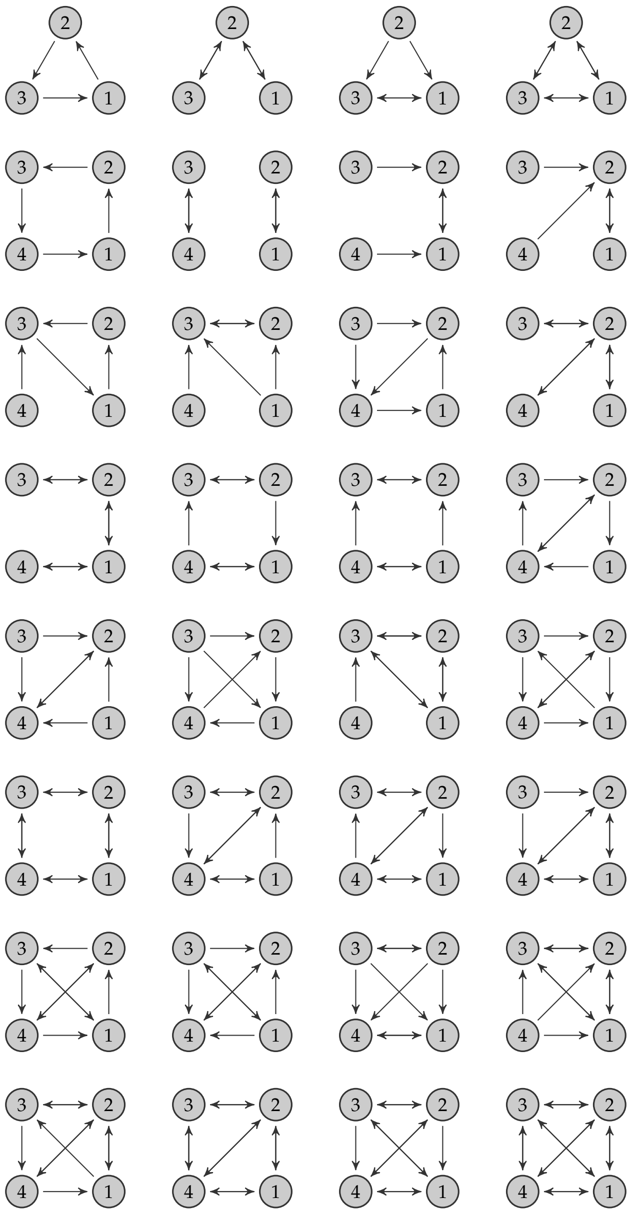

Figure 1.23. Digraphs of orders 3 or 4 in which all vertices have positive out-degrees that do not satisfy the identifying condition of Proposition 1.105

from which it follows that the present discussion is confined to the identification of γ, $\boldsymbol{\phi}_{-1}$, ψ, and $\boldsymbol{\psi}_{-1}$.

Analogous to Proposition 1.79, if

$$\exists \boldsymbol{\theta}_0 \in \Theta \quad \boldsymbol{X}_2\boldsymbol{\phi}_{-1}(\boldsymbol{\theta}_0) - \psi(\boldsymbol{\theta}_0)\boldsymbol{\iota}_0^+(H) + \bar{\boldsymbol{C}}(H)\boldsymbol{X}_2\boldsymbol{\psi}_{-1}(\boldsymbol{\theta}_0) \in \ker\big(q\big(\bar{\boldsymbol{A}}(G)\big)\big), \tag{1.86}$$

then γ is not identified via $q(\bar{\boldsymbol{A}}(G))$-LD through the mean by $\mathcal{Q}(\Theta)$. The kernel condition (1.86) cannot be satisfied if γ is identified via $q(\bar{\boldsymbol{A}}(G))$-LD through the mean by $\mathcal{Q}(\Theta)$. If $\mathcal{I}_0^+(H) \neq \emptyset$ (respectively, $\mathcal{I}_0^+(H) = \emptyset$) and the kernel condition (1.86) is not satisfied, then $\boldsymbol{\phi}_{-1} \neq \boldsymbol{0}_{K-1}$ or $\boldsymbol{\psi} \neq \boldsymbol{0}_K$ (respectively, $\boldsymbol{\phi}_{-1} \neq \boldsymbol{0}_{K-1}$ or $\boldsymbol{\psi}_{-1} \neq \boldsymbol{0}_{K-1}$) must be true.

The identifying conditions given hereinafter are similar to those of Section 1.4.3.1 (Proposition 1.106 parallels Proposition 1.84, Proposition 1.107 parallels Proposition 1.86, Proposition 1.111 parallels Proposition 1.89, and Proposition 1.112 parallels Proposition 1.90) but entail stronger conditions with respect to the topologies of G and H.

Proposition 1.106 *Suppose $\bar{\boldsymbol{A}}(G) \neq \bar{\boldsymbol{C}}(H)$.*

(*1.106.1*) *Suppose $\mathcal{I}_0^+(H) \neq \emptyset$. If the kernel condition* (1.86) *is not satisfied and the $n \times (4K-2)$ matrix resulting from matrix* (1.76) *by eliminating the constant $\boldsymbol{1}_n$ and premultiplying all blocks by $q(\bar{\boldsymbol{A}}(G))$ has full column rank, then γ, $\boldsymbol{\phi}_{-1}$, and $\boldsymbol{\psi}$ are identified by $\mathcal{Q}(\Theta)$. If γ, $\boldsymbol{\phi}_{-1}$, and $\boldsymbol{\psi}$ are identified via $q(\bar{\boldsymbol{A}}(G))$-LD through the mean by $\mathcal{Q}(\Theta)$, then $q(\bar{\boldsymbol{A}}(G))$, $q(\bar{\boldsymbol{A}}(G))\bar{\boldsymbol{A}}(G)$, $q(\bar{\boldsymbol{A}}(G))\bar{\boldsymbol{C}}(H)$, $q(\bar{\boldsymbol{A}}(G))\bar{\boldsymbol{A}}(G)\bar{\boldsymbol{C}}(H)$ are linearly independent.*

(*1.106.2*) *Suppose $\mathcal{I}_0^+(H) = \emptyset$. If the kernel condition* (1.86) *is not satisfied and the $n \times (4K-4)$ matrix resulting from matrix* (1.77) *by eliminating the constant $\boldsymbol{1}_n$ and premultiplying all blocks by $q(\bar{\boldsymbol{A}}(G))$ has full column rank, then γ, $\boldsymbol{\phi}_{-1}$, and $\boldsymbol{\psi}_{-1}$ are identified by $\mathcal{Q}(\Theta)$. If γ, $\boldsymbol{\phi}_{-1}$, and $\boldsymbol{\psi}_{-1}$ are identified via $q(\bar{\boldsymbol{A}}(G))$-LD through the mean by $\mathcal{Q}(\Theta)$, then $q(\bar{\boldsymbol{A}}(G))$, $q(\bar{\boldsymbol{A}}(G))\bar{\boldsymbol{A}}(G)$, $q(\bar{\boldsymbol{A}}(G))\bar{\boldsymbol{C}}(H)$, $q(\bar{\boldsymbol{A}}(G))\bar{\boldsymbol{A}}(G)\bar{\boldsymbol{C}}(H)$ are linearly independent.*

Proposition 1.107 *Suppose $\bar{\boldsymbol{A}}(G) = \bar{\boldsymbol{C}}(H)$. If the kernel condition* (1.86) *is not satisfied, $\gamma\boldsymbol{\phi}_{-1} + (1+\gamma)\boldsymbol{\psi}_{-1} \neq \boldsymbol{0}_{K-1}$, and the $n \times (3K-3)$ matrix resulting from matrix* (1.78) *by eliminating the constant $\boldsymbol{1}_n$ and premultiplying all blocks by $q(\bar{\boldsymbol{A}}(G))$ has full column rank, then γ, $\boldsymbol{\phi}_{-1}$, and $\boldsymbol{\psi}_{-1}$ are identified by $\mathcal{Q}(\Theta)$. If γ, $\boldsymbol{\phi}_{-1}$, and $\boldsymbol{\psi}_{-1}$ are identified via $q(\bar{\boldsymbol{A}}(G))$-LD through the mean by $\mathcal{Q}(\Theta)$, then $q(\bar{\boldsymbol{A}}(G))$, $q(\bar{\boldsymbol{A}}(G))\bar{\boldsymbol{A}}(G)$, $q(\bar{\boldsymbol{A}}(G))\bar{\boldsymbol{A}}(G)^2$ are linearly independent.*

Within the setup of Proposition 1.107, a necessary condition for identification is that the *three* matrices $q(\bar{\boldsymbol{A}}(G))$, $q(\bar{\boldsymbol{A}}(G))\bar{\boldsymbol{A}}(G)$, $q(\bar{\boldsymbol{A}}(G))\bar{\boldsymbol{A}}(G)^2$ are linearly independent, which is true if and only if the *four* matrices $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G)$, $\bar{\boldsymbol{A}}(G)^2$, $\bar{\boldsymbol{A}}(G)^3$ are linearly independent (Lemma 1.108). It follows that Proposition 1.107 entails a stronger condition with respect to the topology of G than does Proposition 1.86, wherein the linear independence of $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G)$, $\bar{\boldsymbol{A}}(G)^2$ is necessary for identification.

Figure 1.24. A digraph of order 4 with diameter 3 (Example 1.110)

Lemma 1.108 *For all integers $p > 1$, the p matrices $q(\bar{\boldsymbol{A}}(G))\bar{\boldsymbol{A}}(G)^0 = q(\bar{\boldsymbol{A}}(G)), \ldots,$ $q(\bar{\boldsymbol{A}}(G))\bar{\boldsymbol{A}}(G)^{p-1}$ are linearly independent if and only if the $p+1$ matrices $\bar{\boldsymbol{A}}(G)^0 = \boldsymbol{I}_n$, $\ldots$, $\bar{\boldsymbol{A}}(G)^p$ are linearly independent.*

Bramoullé, Djebbari, and Fortin (2009) give a sufficient condition for the linear independence of $\boldsymbol{I}_n$, $\bar{\boldsymbol{C}}(G)$, $\bar{\boldsymbol{C}}(G)^2$, $\bar{\boldsymbol{C}}(G)^3$ (see Section 3.2) in terms of the diameter of a subdigraph of G.[130,131] The result is also true for $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G)$, $\bar{\boldsymbol{A}}(G)^2$, $\bar{\boldsymbol{A}}(G)^3$ and is stated in Lemma 1.109.

Lemma 1.109 (Bramoullé, Djebbari, and Fortin 2009) *If G has a subdigraph of diameter (at least) three, then $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G)$, $\bar{\boldsymbol{A}}(G)^2$, $\bar{\boldsymbol{A}}(G)^3$ are linearly independent.*[132]

The linear independence of $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G)$, $\bar{\boldsymbol{A}}(G)^2$, $\bar{\boldsymbol{A}}(G)^3$ is in general not sufficient for the identification of γ, $\boldsymbol{\phi}_{-1}$, and $\boldsymbol{\psi}_{-1}$ via $q(\bar{\boldsymbol{A}}(G))$-LD through the mean by $\mathcal{Q}(\Theta)$, as illustrated by Example 1.110.

Example 1.110 Suppose $\mathcal{I} = \{1,2,3,4\}$, $\mathcal{A}(G) = \{(1,2),(2,1),(2,3),(3,2),(3,4),$ $(4,3)\}$, $G = H$, $\boldsymbol{X} = (\mathbf{1}_4 : \boldsymbol{e}_2 + \boldsymbol{e}_3)$, and $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ are two parameter points in Θ with $(\gamma_1, \boldsymbol{\phi}_1, \boldsymbol{\psi}_1) = (1/3, 3/4, 4/5, 0, 1)$ and $(\gamma_2, \boldsymbol{\phi}_2, \boldsymbol{\psi}_2) = (2/3, 59/60, 3/4, 0, 7/10)$. See Figure 1.24 for an illustration of G. Note that $\ker(q(\bar{\boldsymbol{A}}(G))) = \{c\mathbf{1}_4 \mid c \in \mathbb{R}\}$ and $\boldsymbol{X}\boldsymbol{\phi}_1 + \bar{\boldsymbol{C}}(H)\boldsymbol{X}\boldsymbol{\psi}_1 \notin \ker(q(\bar{\boldsymbol{A}}(G)))$ and $\boldsymbol{X}\boldsymbol{\phi}_2 + \bar{\boldsymbol{C}}(H)\boldsymbol{X}\boldsymbol{\psi}_2 \notin \ker(q(\bar{\boldsymbol{A}}(G)))$ (cf. the kernel condition (1.86)). The matrices $\boldsymbol{I}_4$, $\bar{\boldsymbol{A}}(G)$, $\bar{\boldsymbol{A}}(G)^2$, $\bar{\boldsymbol{A}}(G)^3$ are linearly independent because $\operatorname{diam}(G) = 3$ (Lemma 1.109), but

$$\mathbb{E}\big(q\big(\bar{\boldsymbol{A}}(G)\big)\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_1)\big) \mid \mathfrak{F}\big) = \frac{1}{10}\begin{pmatrix} 2 \\ -1 \\ -1 \\ 2 \end{pmatrix} = \mathbb{E}\big(q\big(\bar{\boldsymbol{A}}(G)\big)\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_2)\big) \mid \mathfrak{F}\big). \qquad \diamond$$

Proposition 1.111 *Suppose $G = H$ and $\mathbf{0}_n \neq \boldsymbol{\iota}_0^+(G) \in \text{c-sp}(\operatorname{diag}(\boldsymbol{\iota}_0^+(G))\boldsymbol{X}_2)$. If the kernel condition (1.86) is not satisfied, $\gamma\boldsymbol{\phi}_{-1} + (1+\gamma)\boldsymbol{\psi}_{-1} \neq \mathbf{0}_{K-1}$, and the $n \times (5K-5)$ matrix resulting from matrix (1.79) by eliminating the constant $\mathbf{1}_n$ and premultiplying all blocks by $q(\bar{\boldsymbol{A}}(G))$ has full column rank, then γ, $\boldsymbol{\phi}_{-1}$, and $\boldsymbol{\psi}$ are identified by $\mathcal{Q}(\Theta)$. If γ, $\boldsymbol{\phi}_{-1}$, and $\boldsymbol{\psi}$ are identified via $q(\bar{\boldsymbol{A}}(G))$-LD through the mean by $\mathcal{Q}(\Theta)$, then $q(\bar{\boldsymbol{A}}(G))$, $q(\bar{\boldsymbol{A}}(G))\bar{\boldsymbol{A}}(G)$, $q(\bar{\boldsymbol{A}}(G))\bar{\boldsymbol{C}}(G)$, $q(\bar{\boldsymbol{A}}(G))\bar{\boldsymbol{A}}(G)^2$, $q(\bar{\boldsymbol{A}}(G))\bar{\boldsymbol{A}}(G)\bar{\boldsymbol{C}}(G)$ are linearly independent.*

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130. The diameter of a digraph D, denoted by $\operatorname{diam}(D)$, is equal to the length of the longest shortest path between any two vertices in D.

131. Note that Bramoullé, Djebbari, and Fortin (2009) use a different notation. They claim that $\boldsymbol{I}_n$, $\bar{\boldsymbol{C}}(G)$, $\bar{\boldsymbol{C}}(G)^2$, $\bar{\boldsymbol{C}}(G)^3$ are linearly independent if G has diameter at least three (see Section 3.2). This claim is, however, not necessarily true if the diameter of G is infinite (which is true if and only if G is not strongly connected).

132. Note that G has a subdigraph of diameter three if G has a subdigraph of diameter at least three.

Proposition 1.112 *Suppose γ is identified by $\mathcal{Q}(\Theta)$.*

(1.112.1) *Suppose $\mathcal{I}_0^+(H) \neq \emptyset$. If the $n \times (2K-1)$ matrix*

$$\left(q\big(\bar{\boldsymbol{A}}(G)\big)\boldsymbol{\iota}_0^+(H) : q\big(\bar{\boldsymbol{A}}(G)\big)\boldsymbol{X}_2 : q\big(\bar{\boldsymbol{A}}(G)\big)\bar{\boldsymbol{C}}(H)\boldsymbol{X}_2\right)$$

has full column rank, then $\boldsymbol{\phi}_{-1}$ and $\boldsymbol{\psi}$ are identified by $\mathcal{Q}(\Theta)$. If $\boldsymbol{\phi}_{-1}$ and $\boldsymbol{\psi}$ are identified via $q(\bar{\boldsymbol{A}}(G))$-LD through the mean by $\mathcal{Q}(\Theta)$, then $q(\bar{\boldsymbol{A}}(G))$ and $q(\bar{\boldsymbol{A}}(G))\bar{\boldsymbol{C}}(H)$ are linearly independent.

(1.112.2) *Suppose $\mathcal{I}_0^+(H) = \emptyset$. If the $n \times (2K-2)$ matrix*

$$\left(q\big(\bar{\boldsymbol{A}}(G)\big)\boldsymbol{X}_2 : q\big(\bar{\boldsymbol{A}}(G)\big)\bar{\boldsymbol{C}}(H)\boldsymbol{X}_2\right)$$

has full column rank, then $\boldsymbol{\phi}_{-1}$ and $\boldsymbol{\psi}_{-1}$ are identified by $\mathcal{Q}(\Theta)$. If $\boldsymbol{\phi}_{-1}$ and $\boldsymbol{\psi}_{-1}$ are identified via $q(\bar{\boldsymbol{A}}(G))$-LD through the mean by $\mathcal{Q}(\Theta)$, then $q(\bar{\boldsymbol{A}}(G))$ and $q(\bar{\boldsymbol{A}}(G))\bar{\boldsymbol{C}}(H)$ are linearly independent.

In case $\bar{\boldsymbol{A}}(G) = \bar{\boldsymbol{C}}(H)$, the *two* matrices $q(\bar{\boldsymbol{A}}(G))$ and $q(\bar{\boldsymbol{A}}(G))\bar{\boldsymbol{C}}(H)$ are linearly independent if and only if the *three* matrices $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}(G)$, $\bar{\boldsymbol{A}}(G)^2$ are linearly independent (Lemma 1.108).

1.4.3.6 Identification through the variance by $\mathcal{Q}(\Theta_0)$

A subparameter g is called identified via $q(\bar{\boldsymbol{A}}(G))$-LD through the variance by $\mathcal{Q}(\Theta_0)$ if for all pairs of parameter points $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ in Θ_0^2, $g(\boldsymbol{\theta}_1) = g(\boldsymbol{\theta}_2)$ is necessary for $\operatorname{var}(q(\bar{\boldsymbol{A}}(G))\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_1)) \mid \mathfrak{F}) = \operatorname{var}(q(\bar{\boldsymbol{A}}(G))\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_2)) \mid \mathfrak{F})$. If g is identified via $q(\bar{\boldsymbol{A}}(G))$-LD through the variance by $\mathcal{Q}(\Theta_0)$, then it is identified by $\mathcal{Q}(\Theta_0)$. For all parameter points $\boldsymbol{\theta}_0$ in Θ_0,

$$\begin{aligned}\operatorname{var}\big(q\big(\bar{\boldsymbol{A}}(G)\big)\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_0)\big) \mid \mathfrak{F}\big) &= \varsigma(\boldsymbol{\theta}_0)^2\Big(\boldsymbol{I}_n - \gamma(\boldsymbol{\theta}_0)q\big(\bar{\boldsymbol{A}}(G)\big)\Big)^{-1} q\big(\bar{\boldsymbol{A}}(G)\big) \\ &\quad \times q\big(\bar{\boldsymbol{A}}(G)\big)^{\top}\Big(\boldsymbol{I}_n - \gamma(\boldsymbol{\theta}_0)q\big(\bar{\boldsymbol{A}}(G)\big)^{\top}\Big)^{-1},\end{aligned}$$

from which it follows that the present discussion is confined to the identification of γ and ς. Note that $\operatorname{var}(q(\bar{\boldsymbol{A}}(G))\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_0)) \mid \mathfrak{F})$ is positive semidefinite but not positive definite because $q(\bar{\boldsymbol{A}}(G))$ is singular.

Proposition 1.113 *If $q(\bar{\boldsymbol{A}}(G))q(\bar{\boldsymbol{A}}(G))^{\top}$,*

$$q\big(\bar{\boldsymbol{A}}(G)\big)\Big(q\big(\bar{\boldsymbol{A}}(G)\big) + q\big(\bar{\boldsymbol{A}}(G)\big)^{\top}\Big)q\big(\bar{\boldsymbol{A}}(G)\big)^{\top}, \quad q^2\big(\bar{\boldsymbol{A}}(G)\big)q^2\big(\bar{\boldsymbol{A}}(G)\big)^{\top}$$

are linearly independent, then γ and ς are identified by $\mathcal{Q}(\Theta_0)$.

Proposition 1.114 *The identifying condition of Proposition 1.113 is stronger than (but not equivalent to) that of Proposition 1.91.*

There are 30 (out of 217) isomorphism classes of digraphs G of orders at least 2 and at most 4 with weakly connected components of orders at least 2 whose representatives do not satisfy the identifying condition of Proposition 1.113. A representative of each class is depicted in light gray in Figure 1.25.

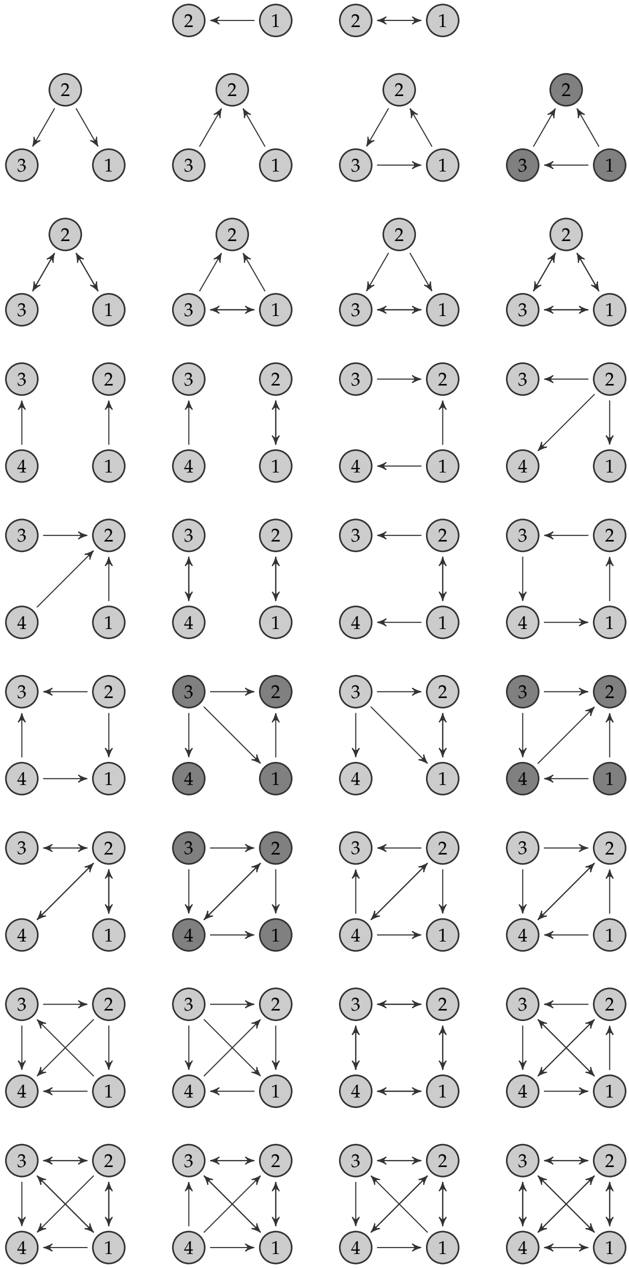


Figure 1.25. Digraphs G of orders at least 2 and at most 4 with weakly connected components of orders at least 2 that do not satisfy the identifying condition of Proposition 1.113 (digraphs in light gray) and that of Proposition 1.115 in case $G = H$ (digraphs in light or dark gray)

1.4.3.7 Identification through the variance by $\mathcal{Q}(\Theta)$

A subparameter g is identified by $\mathcal{Q}(\Theta)$ if it is identified via $q(\bar{A}(G))$-LD through the variance by $\mathcal{Q}(\Theta)$. For all parameter points $\boldsymbol{\theta}_0$ in Θ,

$$\begin{aligned}\operatorname{var}\bigl(q(\bar{A}(G))\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_0)) \mid \mathfrak{F}\bigr) = \varsigma(\boldsymbol{\theta}_0)^2 \Bigl(\boldsymbol{I}_n - \gamma(\boldsymbol{\theta}_0) q(\bar{A}(G))\Bigr)^{-1} q(\bar{A}(G)) \\ \times \bigl(\boldsymbol{I}_n + \zeta(\boldsymbol{\theta}_0)\bar{\boldsymbol{C}}(H)\bigr)\bigl(\boldsymbol{I}_n + \zeta(\boldsymbol{\theta}_0)\bar{\boldsymbol{C}}(H)^{\top}\bigr) \\ \times q(\bar{A}(G))^{\top}\Bigl(\boldsymbol{I}_n - \gamma(\boldsymbol{\theta}_0) q(\bar{A}(G))^{\top}\Bigr)^{-1},\end{aligned}$$

from which it follows that the present discussion is confined to the identification of γ, ζ, and ς.

Proposition 1.115 *Suppose γ is identified by $\mathcal{Q}(\Theta)$. If $q(\bar{A}(G))q(\bar{A}(G))^{\top}$,*

$$q(\bar{A}(G))\bigl(\bar{\boldsymbol{C}}(H) + \bar{\boldsymbol{C}}(H)^{\top}\bigr)q(\bar{A}(G))^{\top}, \quad q(\bar{A}(G))\bar{\boldsymbol{C}}(H)\bar{\boldsymbol{C}}(H)^{\top}q(\bar{A}(G))^{\top}$$

are linearly independent, then ζ and ς are identified by $\mathcal{Q}(\Theta)$.

The identifying condition of Proposition 1.115 is stronger than (but not equivalent to) that of Proposition 1.100.

There are 34 (out of 217) isomorphism classes of digraphs G of orders at least 2 and at most 4 with weakly connected components of orders at least 2 whose representatives do not satisfy the identifying condition of Proposition 1.115 in case $G = H$. A representative of each class is depicted in light or dark gray in Figure 1.25.

Proposition 1.116 *Suppose $\bar{A}(G) \neq \bar{\boldsymbol{C}}(H)$.*[133] *If the nine matrices*

$$\begin{gathered} q(\bar{A}(G))q(\bar{A}(G))^{\top}, \quad q(\bar{A}(G))\bigl(\bar{\boldsymbol{C}}(H) + \bar{\boldsymbol{C}}(H)^{\top}\bigr)q(\bar{A}(G))^{\top}, \\ q(\bar{A}(G))\bar{\boldsymbol{C}}(H)\bar{\boldsymbol{C}}(H)^{\top}q(\bar{A}(G))^{\top}, \quad q(\bar{A}(G))\Bigl(q(\bar{A}(G)) + q(\bar{A}(G))^{\top}\Bigr)q(\bar{A}(G))^{\top}, \\ q(\bar{A}(G))\Bigl(q(\bar{A}(G))\bigl(\bar{\boldsymbol{C}}(H) + \bar{\boldsymbol{C}}(H)^{\top}\bigr) + \bigl(\bar{\boldsymbol{C}}(H) + \bar{\boldsymbol{C}}(H)^{\top}\bigr)q(\bar{A}(G))^{\top}\Bigr)q(\bar{A}(G))^{\top}, \\ q(\bar{A}(G))\Bigl(q(\bar{A}(G))\bar{\boldsymbol{C}}(H)\bar{\boldsymbol{C}}(H)^{\top} + \bar{\boldsymbol{C}}(H)\bar{\boldsymbol{C}}(H)^{\top}q(\bar{A}(G))^{\top}\Bigr)q(\bar{A}(G))^{\top}, \\ q^2(\bar{A}(G))q^2(\bar{A}(G))^{\top}, \quad q^2(\bar{A}(G))\bigl(\bar{\boldsymbol{C}}(H) + \bar{\boldsymbol{C}}(H)^{\top}\bigr)q^2(\bar{A}(G))^{\top}, \\ q^2(\bar{A}(G))\bar{\boldsymbol{C}}(H)\bar{\boldsymbol{C}}(H)^{\top}q^2(\bar{A}(G))^{\top} \end{gathered}$$

are linearly independent, then γ, ζ, and ς are identified by $\mathcal{Q}(\Theta)$.

Proposition 1.117 *The identifying condition of Proposition 1.116 is stronger than (but not equivalent to) that of Proposition 1.104.*

There are no digraphs G of orders 3 or 4 with weakly connected components of orders at least 2 and $\mathcal{I}_0^+(G) \neq \emptyset$ that satisfy the identifying condition of Proposition 1.116 in case $G = H$. A weakly connected digraph G of order 5 with a single vertex without out-neighbors that satisfies the identifying condition of Proposition 1.116 in case $G = H$ is depicted in Figure 1.26.

133. See Footnote 128.

Figure 1.26. A digraph of order 5 that satisfies the identifying condition of Proposition 1.116 in case $G = H$

Proposition 1.118 *Suppose $\bar{\boldsymbol{A}}(G) = \bar{\boldsymbol{C}}(H)$.*[134] *If $\gamma\zeta \geq 0$ and the six matrices*

$$
\begin{gathered}
q\big(\bar{\boldsymbol{C}}(H)\big)q\big(\bar{\boldsymbol{C}}(H)\big)^{\mathsf{T}},\\
q\big(\bar{\boldsymbol{C}}(H)\big)\big(\bar{\boldsymbol{C}}(H)+\bar{\boldsymbol{C}}(H)^{\mathsf{T}}\big)q\big(\bar{\boldsymbol{C}}(H)\big)^{\mathsf{T}}, \quad q\big(\bar{\boldsymbol{C}}(H)\big)\bar{\boldsymbol{C}}(H)\bar{\boldsymbol{C}}(H)^{\mathsf{T}}q\big(\bar{\boldsymbol{C}}(H)\big)^{\mathsf{T}},\\
q\big(\bar{\boldsymbol{C}}(H)\big)\Big(q\big(\bar{\boldsymbol{C}}(H)\big)\big(\bar{\boldsymbol{C}}(H)+\bar{\boldsymbol{C}}(H)^{\mathsf{T}}\big)+\big(\bar{\boldsymbol{C}}(H)+\bar{\boldsymbol{C}}(H)^{\mathsf{T}}\big)q\big(\bar{\boldsymbol{C}}(H)\big)^{\mathsf{T}}\Big)q\big(\bar{\boldsymbol{C}}(H)\big)^{\mathsf{T}},\\
q\big(\bar{\boldsymbol{C}}(H)\big)\Big(q\big(\bar{\boldsymbol{C}}(H)\big)\bar{\boldsymbol{C}}(H)\bar{\boldsymbol{C}}(H)^{\mathsf{T}}+\bar{\boldsymbol{C}}(H)\bar{\boldsymbol{C}}(H)^{\mathsf{T}}q\big(\bar{\boldsymbol{C}}(H)\big)^{\mathsf{T}}\Big)q\big(\bar{\boldsymbol{C}}(H)\big)^{\mathsf{T}},\\
q^2\big(\bar{\boldsymbol{C}}(H)\big)\bar{\boldsymbol{C}}(H)\bar{\boldsymbol{C}}(H)^{\mathsf{T}}q^2\big(\bar{\boldsymbol{C}}(H)\big)^{\mathsf{T}}
\end{gathered}
$$

are linearly independent, then γ, ζ, and ς are identified by $\mathcal{Q}(\Theta)$.

Proposition 1.119 *The identifying condition of Proposition 1.118 is stronger than (but not equivalent to) that of Proposition 1.105.*

1.4.4 Existence of valid statistical models

This section discusses the existence of a statistical model without unobservable correlated effects and a NALA game such that the former is a valid model of the latter for a given action space.[135] Definition V gives a precise meaning to the notion of a valid model.

Definition V A statistical model $\mathcal{M}$ is called a *valid* model of a NALA game with action space $\mathcal{Y}^n$ if for all distributions μ in $\mathcal{M}$, (V.1) $\operatorname{supp}(\mu) \subset \mathcal{Y}^n \cup \partial\,\mathcal{Y}^n$ and (V.2) $\partial\,\mathcal{Y}^n \cap \mathcal{Y}^n$ is a μ-null set.[136]

A statistical model is a valid model of a NALA game if every distribution in the model satisfies two conditions: its support lies in the game's action space (Condition V.1) and the boundary of the action space has measure zero (Condition V.2). The second condition is motivated by the fact that the statistical models of Section 1.4.2 are derived from a system of equations characterizing the interior NE of a NALA game.

Whether and under which conditions a statistical model and a NALA game exist such that the former is a valid model of the latter depends on the game's action space: In case it is $\mathbb{R}^n$, such a pair exists; specifically, the model $\mathcal{P}(\Theta)$ (see

134. See Footnote 129.
135. The case of a statistical model with unobservable correlated effects is similar.
136. Note that $\mathcal{Y}^n \cup \partial\,\mathcal{Y}^n = \mathcal{Y}^n$ and $\partial\,\mathcal{Y}^n \cap \mathcal{Y}^n = \partial\,\mathcal{Y}^n$ because $\mathcal{Y}$ is equal to $\mathbb{R}$, $\mathbb{R}_+$, or $[0, \bar{v}]$. In order that Definition V also applies to the limit of a sequence of NALA games where the action space of the limit is $\operatorname{int}(\mathcal{Y})^n$, Condition V.1 is stated in terms of $\mathcal{Y}^n \cup \partial\,\mathcal{Y}^n$ instead of $\mathcal{Y}^n$ and Condition V.2 in terms of $\partial\,\mathcal{Y}^n \cap \mathcal{Y}^n$ instead of $\partial\,\mathcal{Y}^n$.

Section 1.4.2.1) is a valid model of the NALA game of Section 1.4.1 with $\mathcal{Y} = \mathbb{R}$ if f is bounded neither below nor above and, for example, the conditional distribution of the latent variable $\boldsymbol{u}$ given $\mathfrak{F}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$.[137] In case it is $\mathbb{R}_+^n$ or $[0, \bar{v}]^n$, there exists a statistical model that is a valid model of the limit of a sequence of NALA games.

Let $\Gamma(\mathbb{R}, f)$ denote the NALA game of Section 1.4.1 with $\mathcal{Y} = \mathbb{R}$. Let $\{f_m\}_{m \in \mathbb{Z}_{++}}$ be a sequence of functions in $\mathcal{F}(\mathcal{Y})$. For all $m \in \mathbb{Z}_{++}$, let $\Gamma(\mathcal{Y}, f_m)$ denote the generic NALA game of Section 1.4.1 with $f = f_m$. The sequence $\{\Gamma(\mathcal{Y}, f_m)\}_{m \in \mathbb{Z}_{++}}$ is said to converge if $\{f_m \restriction_{\mathrm{int}(\mathcal{Y})}\}_{m \in \mathbb{Z}_{++}}$ converges to a function f_∞ that satisfies Assumption F. If $\{\Gamma(\mathcal{Y}, f_m)\}_{m \in \mathbb{Z}_{++}}$ converges, then its limit is denoted by $\Gamma(\mathrm{int}(\mathcal{Y}), f_\infty)$. Note that $\Gamma(\mathrm{int}(\mathcal{Y}), f_\infty)$ is not a NALA game unless $\mathcal{Y} = \mathbb{R}$. The notion of a NALA game can, however, be extended to cover the case where the players' action space is $\mathrm{int}(\mathcal{Y})$. If $\{\Gamma(\mathcal{Y}, f_m)\}_{m \in \mathbb{Z}_{++}}$ converges with limit $\Gamma(\mathrm{int}(\mathcal{Y}), f_\infty)$ and for all $m \in \mathbb{Z}_{++}$, $\Gamma(\mathcal{Y}, f_m)$ has a unique and interior NE $\boldsymbol{y}_m^\star$, then the sequence $\{\boldsymbol{y}_m^\star\}_{m \in \mathbb{Z}_{++}}$ converges and its limit is the unique NE of $\Gamma(\mathrm{int}(\mathcal{Y}), f_\infty)$.

Let $\mathcal{P}_1(\Theta)$ denote the model $\mathcal{P}(\Theta)$ with $f = \mathrm{id}_{\mathbb{R}}$. Let B be a nonempty Borel set on $\mathbb{R}$.[138] Let $\mathcal{G}(B)$ denote the set of all real-valued functions with domain B that are bijective and Borel measurable. For any $g \in \mathcal{G}(B)$, let the statistical model $\mathcal{P}_g(\Theta)$ be defined by

$$\mathcal{P}_g(\Theta) := \{\mathbb{P}_1 \restriction_{\mathrm{im}(\boldsymbol{g})} \circ\, \boldsymbol{g} \mid \mathbb{P}_1 \in \mathcal{P}_1(\Theta)\},$$

where $\boldsymbol{g}\colon B^n \to g(B)^n$ is the vector field associated with the function g that is defined by, for all $\boldsymbol{x} := (x_1, \ldots, x_n) \in B^n$, $\boldsymbol{g}(\boldsymbol{x}) := (g(x_1), \ldots, g(x_n))$ (see also Section 1.3.3), $\mathbb{P}_1 \restriction_{\mathrm{im}(\boldsymbol{g})}$ is the restriction of $\mathbb{P}_1$ to the image of $\boldsymbol{g}$, and $\mathbb{P}_1 \restriction_{\mathrm{im}(\boldsymbol{g})} \circ\, \boldsymbol{g}$ is the image distribution of $\mathbb{P}_1 \restriction_{\mathrm{im}(\boldsymbol{g})}$ under the inverse of $\boldsymbol{g}$.[139,140] Note that all distributions $\mathbb{P}_g$ in $\mathcal{P}_g(\Theta)$ satisfy, by definition, $\mathrm{supp}(\mathbb{P}_g) \subset B^n \cup \partial B^n$ (cf. Condition V.1).

In order to shed light on the definition of $\mathcal{P}_g(\Theta)$, let $\mathcal{S} \subset \mathbb{R}^n$ denote the union of the supports of the distributions in $\mathcal{P}_1(\Theta)$, let $\boldsymbol{\theta}_0$ be a parameter point in Θ, and let $\boldsymbol{y}_1(\boldsymbol{\theta}_0)$ denote the response variable $\boldsymbol{y}(\boldsymbol{\theta}_0)$ for the case $f = \mathrm{id}_{\mathbb{R}}$ (see (1.68) for the definition of $\boldsymbol{y}(\boldsymbol{\theta}_0)$). The function $\boldsymbol{y}_g(\boldsymbol{\theta}_0) := \boldsymbol{g}^{-1} \circ \boldsymbol{y}_1(\boldsymbol{\theta}_0)\colon \Omega \to B^n$ is defined $\mathbb{P}$-almost everywhere on Ω if $\mathcal{S} \subset \mathrm{im}(\boldsymbol{g})$ (see Figure 1.27 for an illustration), in which case the conditional distribution of $\boldsymbol{y}_g(\boldsymbol{\theta}_0)$ given $\mathfrak{F}$ is equal to $\mathbb{P}_1 \restriction_{\mathrm{im}(\boldsymbol{g})} \circ\, \boldsymbol{g}$ for some $\mathbb{P}_1$ in $\mathcal{P}_1(\Theta)$.[141] The model $\mathcal{P}_g(\Theta)$ is, therefore, the family of conditional distributions of $\boldsymbol{y}_g(\boldsymbol{\theta}_0)$ given $\mathfrak{F}$ indexed by Θ if $\mathcal{S} \subset \mathrm{im}(\boldsymbol{g})$.

Provided that $\{f_m \restriction_{\mathrm{int}(\mathcal{Y})}\}_{m \in \mathbb{Z}_{++}}$ converges to a function f_∞ that satisfies Assumption F and therefore lies in $\mathcal{G}(\mathrm{int}(\mathcal{Y}))$, Definition V also applies to the model $\mathcal{P}_{f_\infty}(\Theta)$ and the limit $\Gamma(\mathrm{int}(\mathcal{Y}), f_\infty)$ of $\{\Gamma(\mathcal{Y}, f_m)\}_{m \in \mathbb{Z}_{++}}$; specifically, $\mathcal{P}_{f_\infty}(\Theta)$ is a valid

137. The distributions in $\mathcal{P}(\Theta)$ satisfy Condition V.1.

138. In case $\mathcal{Y} = \mathbb{R}$, $B = \mathbb{R}$; in case $\mathcal{Y} = \mathbb{R}_+$, $B = \mathbb{R}_{++}$; and in case $\mathcal{Y} = [0, \bar{v}]$, $B = (0, \bar{v})$.

139. Note that $\mathbb{P}_1 \restriction_{\mathrm{im}(\boldsymbol{g})}$ is defined and therefore a probability measure because B^n is a Borel set on $\mathbb{R}^n$ and $\boldsymbol{g}$ is injective (see, for example, Kechris 1995, Corollary 15.2).

140. In case $\mathcal{Y} = \mathbb{R}$, $g = f$, and in case $\mathcal{Y} = \mathbb{R}_+$ or $\mathcal{Y} = [0, \bar{v}]$, g is the limit f_∞ of $\{f_m \restriction_{\mathrm{int}(\mathcal{Y})}\}_{m \in \mathbb{Z}_{++}}$.

141. More precisely, $\boldsymbol{y}_g(\boldsymbol{\theta}_0)$ is defined on $\boldsymbol{y}_1(\boldsymbol{\theta}_0)^{-1}(\mathcal{S}) \subset \Omega$, the preimage of $\mathcal{S}$ under $\boldsymbol{y}_1(\boldsymbol{\theta}_0)$, and can be extended to Ω by setting $\boldsymbol{y}_g(\boldsymbol{\theta}_0)$ equal to $\mathbf{0}_n$ (or any other vector in $\mathbb{R}^n$) on the $\mathbb{P}$-null set $\Omega \setminus \boldsymbol{y}_1(\boldsymbol{\theta}_0)^{-1}(\mathcal{S})$; the conditional distribution of the resulting random n-vector given $\mathfrak{F}$ is equal to $\mathbb{P}_1 \restriction_{\mathrm{im}(\boldsymbol{g})} \circ\, \boldsymbol{g}$ for some $\mathbb{P}_1$ in $\mathcal{P}_1(\Theta)$.

$$\begin{tikzcd}
\Omega \arrow[r, "\boldsymbol{y}_1(\boldsymbol{\theta}_0)"] \arrow[d, dashed, "\boldsymbol{y}_g(\boldsymbol{\theta}_0)"'] & \mathcal{S} \cup \complement_{\mathbb{R}^n} \mathcal{S} \arrow[d, hook] \\
B^n \arrow[r, shift left, "g"] & \mathrm{im}(g) \arrow[l, shift left, "g^{-1}"]
\end{tikzcd}$$

Figure 1.27. Definition of $\boldsymbol{y}_g(\boldsymbol{\theta}_0)$ under the assumption that $\mathcal{S} \subset \mathrm{im}(g)$

model of $\Gamma(\mathrm{int}(\mathcal{Y}), f_\infty)$ because $\mathrm{int}(\mathcal{Y})^n \cup \partial\,\mathrm{int}(\mathcal{Y})^n = \mathcal{Y}^n$ (cf. Condition V.1) and $\partial\,\mathrm{int}(\mathcal{Y})^n \cap \mathrm{int}(\mathcal{Y})^n = \emptyset$ (cf. Condition V.2).

Proposition 1.120 *(1.120.1) Suppose $\mathcal{Y} = \mathbb{R}$. The model $\mathcal{P}_f(\Theta)$ is a valid model of the NALA game $\Gamma(\mathbb{R}, f)$ if f is bounded neither below nor above and Condition V.2 is satisfied.*

(1.120.2) Suppose $\mathcal{Y} = \mathbb{R}_+$ and $\{f_m\}_{m\in\mathbb{Z}_{++}}$ is defined by, for all $m \in \mathbb{Z}_{++}$,

$$f_m(y) := \log\left(y + \frac{1}{m}\right).$$

The sequence $\{f_m\restriction_{\mathbb{R}_{++}}\}_{m\in\mathbb{Z}_{++}}$ converges with limit log. *The model $\mathcal{P}_{\log}(\Theta)$ is a valid model of the limit of the sequence of NALA games $\{\Gamma(\mathbb{R}_+, f_m)\}_{m\in\mathbb{Z}_{++}}$.*

(1.120.3) Suppose $\mathcal{Y} = [0, \bar{v}]$ and $\{f_m\}_{m\in\mathbb{Z}_{++}}$ is defined by, for all $m \in \mathbb{Z}_{++}$,

$$f_m(y) := \log\left(y + \frac{1}{m}\right) - \log\left(\bar{v} - y + \frac{1}{m}\right).$$

Let f_∞ denote the inverse of the logistic function $f_\infty^{-1}\colon \mathbb{R} \to (0, \bar{v})$ that is defined by $f_\infty^{-1}(y) := \bar{v}/(1 + \exp(-y))$. The sequence $\{f_m\restriction_{(0,\bar{v})}\}_{m\in\mathbb{Z}_{++}}$ converges with limit f_∞. The model $\mathcal{P}_{f_\infty}(\Theta)$ is a valid model of the limit of the sequence of NALA games $\{\Gamma([0, \bar{v}], f_m)\}_{m\in\mathbb{Z}_{++}}$.

Result 1.120.1 follows from the definition of $\mathcal{P}(\Theta)$ because $\mathcal{P}_f(\Theta) = \mathcal{P}(\Theta)$ if f is bounded neither below nor above. Results 1.120.2 and 1.120.3 state that there exists a statistical model that is a valid model of the limit of a sequence of NALA games in case the action space is $\mathbb{R}_+^n$ or $[0, \bar{v}]^n$; specifically, they show that a logarithm and the inverse of a logistic function allow for aligning the action spaces $\mathbb{R}_+^n$ and $[0, \bar{v}]^n$ with the supports of the distributions in the corresponding statistical models.[142] As regards Result 1.120.2, note that the condition of Proposition 1.16 involving $\alpha_{\min}$ and f_m is not binding in the limit as $m \to \infty$ if $\beta > 0$ (Condition 1.16.1) because $\lim_{m\to\infty} f_m(0) = -\infty$. Likewise, the conditions of Proposition 1.17 involving $\alpha_{\min}$ or $\alpha_{\max}$ and f_m are not binding in the limit as $m \to \infty$ if $\beta > 0$ (Condition 1.17.1)

142. Sommer and Sulger (2012) also recognize that a logarithm is instrumental in aligning the nonnegative action space of their network game with the supports of the distributions in the corresponding statistical model (see pp. 82–85).

and $-0.38\beta \approx (\sqrt{5}-3)\beta/2 < \gamma < 0$. As regards Result 1.120.3, note that the conditions of Proposition 1.14 involving $\alpha_{\min}$ or $\alpha_{\max}$ and f_m are not binding in the limit as $m \to \infty$ if $\beta > 0$ (Condition 1.14.1) because $\lim_{m\to\infty} f_m(0) = -\infty$ and $\lim_{m\to\infty} f_m(\bar{v}) = +\infty$. Likewise, the conditions of Proposition 1.15 involving $\alpha_{\min}$ or $\alpha_{\max}$ and f_m are not binding in the limit as $m \to \infty$ if $\beta > 0$ (Condition 1.15.1) and $-\beta/2 < \gamma < 0$ (Condition 1.15.2).

1.4.5 Reconciling existing statistical models with NALA games

The model $\mathcal{P}(\Theta)$ is a translation of the NALA game Γ of Section 1.4.1 with $\mathcal{Y} = \mathbb{R}$ under the assumption that $\beta = 1$. A different model arises from Γ when assuming $\beta + \gamma = 1$ instead of $\beta = 1$, while maintaining the assumption that $\beta > 0$.[143] In this respect, it is important to note that the parameter restriction $\beta + \gamma = 1$ is not without loss of generality, as opposed to the normalization $\beta = 1$.[144] Moreover, $\beta > 0$ and $\beta + \gamma = 1$ imply that $\gamma < 1$. In contrast, the normalization $\beta = 1$ does not impose an upper bound on γ. Using the same assumptions as in the translation of Γ to $\mathcal{P}(\Theta)$, except for imposing the restriction $\beta + \gamma = 1$ instead of the normalization $\beta = 1$, the following system of equations emerges for the response variable at a parameter point $\boldsymbol{\theta}_0$ in Θ:

$$\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_0)\big) = \big(\boldsymbol{I}_n - \gamma_0 \bar{\boldsymbol{A}}(G)\big)^{-1}\Big(\boldsymbol{X}\boldsymbol{\phi}_0 + \bar{\boldsymbol{C}}(H)\boldsymbol{X}\boldsymbol{\psi}_0 + \varsigma_0\big(\boldsymbol{I}_n + \zeta_0\bar{\boldsymbol{C}}(H)\big)\boldsymbol{u}\Big). \quad (1.87)$$

Note that the two systems of equations (1.68) and (1.87) differ only in one respect, the term involving the subparameter point γ_0, which is equal to $\gamma_0(\bar{\boldsymbol{A}}(G) - \boldsymbol{I}_n)$ in (1.68) and equal to $\gamma_0\bar{\boldsymbol{A}}(G)$ in (1.87). Identification results similar to those discussed in Sections 1.4.3.1 and 1.4.3.4 are true for the model derived from (1.87); specifically, the restriction $\beta + \gamma = 1$ does not change the nature of any of the non-identification results stated in Propositions 1.86 and 1.89.[145] Under the assumptions that $G = H$, all players of Γ have at least one out-neighbor in G (that is, $\mathcal{I}_0^+(G) = \emptyset$), $f = \mathrm{id}_{\mathcal{Y}}$, and $\psi = 0$, the structural form of the system of equations (1.87) is given by

$$\boldsymbol{y} = \gamma_0\bar{\boldsymbol{C}}(G)\boldsymbol{y} + \boldsymbol{1}_n\phi_0 + \boldsymbol{X}_2\boldsymbol{\phi}_{0,-1} + \bar{\boldsymbol{C}}(G)\boldsymbol{X}_2\boldsymbol{\psi}_{0,-1} + \boldsymbol{\epsilon}, \quad (1.88)$$

thereby omitting $\boldsymbol{\theta}_0$ from the notations $\boldsymbol{\epsilon}(\boldsymbol{\theta}_0) = \varsigma_0(\boldsymbol{I}_n + \zeta_0\bar{\boldsymbol{C}}(H))\boldsymbol{u}$ and $\boldsymbol{y}(\boldsymbol{\theta}_0)$. The system of equations (1.88) has the structure of a spatial autoregressive (SAR) process of order one with a row-normalized spatial weights matrix. It is representative for a family of models discussed in theoretical and applied in empirical work in the social interactions literature, with the common assumption that for all $\gamma_0 \in \gamma(\Theta)$,

143. Sommer and Sulger (2012) also discuss the implication of the parameter restriction $\beta + \gamma = 1$ for the system of best reply functions at the NE of their network game and the corresponding statistical model (see p. 85).
144. The assumption $\beta = 1$ does not represent a restriction because the ordinal properties of a utility function are preserved by monotonic transformations, where a monotonic transformation is a transformation by a strictly increasing function. The normalization $\beta = 1$ corresponds to such a transformation, with a subsequent relabelling of the players' preference parameters.
145. For the model derived from (1.87), the analogues of the identifying conditions of Propositions 1.86 and 1.89 involve the subparameter restriction $\gamma\boldsymbol{\phi}_{-1} + \boldsymbol{\psi}_{-1} \neq \boldsymbol{0}_{K-1}$.

$|\gamma_0| < 1$, with varying assumptions on the dependence structure and distribution of the error term $\boldsymbol{\epsilon}$, and usually supplemented with, for example, component specific, fixed effects, in which case the constant term $\mathbf{1}_n\phi_0$ is eliminated from (1.88), in order to account for the presence of unobservable correlated effects.[146] With fixed effects, which are denoted by $\boldsymbol{\iota\eta}$, (1.88) reads as follows:[147]

$$\boldsymbol{y} = \gamma_0\bar{\boldsymbol{C}}(G)\boldsymbol{y} + \boldsymbol{X}_2\boldsymbol{\phi}_{0,-1} + \bar{\boldsymbol{C}}(G)\boldsymbol{X}_2\boldsymbol{\psi}_{0,-1} + \boldsymbol{\iota\eta} + \boldsymbol{\epsilon}. \tag{1.89}$$

In the remainder of this section, I briefly discuss theoretical and empirical work in the social interactions literature that is related to the statistical models of (1.88) and (1.89). It is important to note that the parameter point $\boldsymbol{\theta}_0$ with $\gamma_0 = 0$ and $\boldsymbol{\psi}_{0,-1} = \mathbf{0}_{K-1}$ (the case where both endogenous and exogenous effects are zero) is not identified through the mean by these models.

Theoretical work Bramoullé, Djebbari, and Fortin (2009) discuss identification through the mean of social effects, which subsumes both endogenous and exogenous effects, by two statistical models, of which the first (see (1)) is based on (1.88) and the second (see (9)) is based on (1.89). Their notion of identification is different from that used in this paper. They call "social effects … identified if and only if the … structural parameters can be uniquely recovered from the unrestricted reduced-form parameters" (p. 44). What is notably missing from their discussion is the identification of the reduced-form parameters per se. Lee (2007) considers identification and estimation in a social interactions model that corresponds to (1.89), where G consists solely of complete components, with component specific fixed effects and independent and identically distributed error terms. He discusses estimation of the model by three estimators: a conditional maximum likelihood estimator, a two-stage least squares estimator, and an ordinary least squares estimator for the case where the orders of the components are large. Lee, Liu, and Lin (2010) propose a quasi-maximum likelihood (QML for short) approach for the estimation of a model that is similar to (1.89), but with a row-normalized nonnegative weights matrix, which has only zeros on its main diagonal, in place of $\bar{\boldsymbol{C}}(G)$, with group specific fixed effects, and with a SAR(1) error term, the weights matrix of which may be different from the one in the structural equation.[148] They derive the asymptotic distribution of the QML estimator and provide Monte Carlo evidence on its small sample properties.

Empirical work Bramoullé, Djebbari, and Fortin (2009) complement their theoretical findings with a study of peer effects in recreational activities (artistic, sports, and social activities) using data on secondary schools in the U.S. from the National

146. The matrix $\boldsymbol{I}_n - \gamma_0\bar{\boldsymbol{A}}(G)$ is nonsingular if $|\gamma_0| < 1$.

147. If G consists of two weakly connected components of sizes n_1 and $n_2 := n - n_1$, then

$$\boldsymbol{\iota} = \begin{pmatrix} \mathbf{1}_{n_1} & \mathbf{0}_{n_1} \\ \mathbf{0}_{n_2} & \mathbf{1}_{n_2} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}.$$

148. A group is a complete component.

Longitudinal Survey of Adolescent Health (Add Health for short). Their model (Bramoullé, Djebbari, and Fortin 2009, see (10)) corresponds to (1.89) with school specific fixed effects. Patacchini and Zenou (2012) study peer effects in juvenile delinquency using Add Health data. Their model (see (10)) corresponds to (1.89) with school specific fixed effects. Boucher et al. (2014) investigate peer effects in student achievement in four school subjects (French, history, mathematics, and science) in secondary schools in the Province of Québec (Canada). Their model (see (4)) corresponds to (1.89) with school specific fixed effects. On account of missing information on the students' connections, they assume that the students of every school constitute a complete component of G. Fortin and Yazbeck (2015) study peer effects in adolescent weight gain using Add Health data. They propose a two-equation model, where the first equation (see (2)) agrees with (1.89) with school specific fixed effects and a SAR(1) error term. Lin (2015) studies peer influences in four risky behaviors among adolescents (skipping school, cigarette smoking, alcohol drinking, and physical fighting) using Add Health data. Her model (see (1)) corresponds to (1.89) with group specific fixed effects and a SAR(1) error term.

1.5 Concluding remarks

Statistical models of social interactions come in many different flavors as regards their functional form, their endogenous and exogenous effects matrices, and the dependence structure and distribution of their error term. This paper offers some guidance on the specification of such models in terms of economic theory. It proposes a new class of models that are derived from the system of best reply functions at the Nash equilibrium of a network game, and in consequence, every single aspect of the models is fully accounted for by economic theory. The models feature an endogenous effects matrix that is structurally different from the exogenous effects matrix in the presence of players without out-neighbors, an endogenous effects parameter that is unbounded above, and a first-order moving average error term representing unobserved idiosyncrasy. The network game entails testable parameter restrictions, in particular, the exogenous effects parameter is proportional to the individual effects parameter, and the factor of proportionality is equal to the parameter appearing in the moving average error term. The factor of proportionality has a clear economic interpretation: its magnitude is a measure of the strength of the local externalities of the players' idiosyncrasies. Apart from model specification, this paper also provides guidance as to the design of policy interventions to increase welfare or to decrease or increase to a degree aggregate outcomes for phenomena that are driven by conformism.

An important concern in identifying and measuring the origins and natures of correlated behavior in social interactions is the endogeneity of the social space, that is, the digraph(s) by which the individuals are connected. If individuals self-select into neighborhoods and self-selection is driven by unobservable covariates that affect behavioral outcomes, then the endogenous and exogenous effects matrices may not be strictly exogenous. This endogeneity or selection problem invalidates statistical inference. A widespread type of self-selection is homophily (Lazarsfeld and Merton 1954), where individuals tend to bond with other similar individuals. For this particular type of self-selection, the endogeneity or selection problem may be mitigated by component specific fixed effects. This is, however, at best, only a partial solution. Recent research points towards a more comprehensive solution. As shown by Goldsmith-Pinkham and Imbens (2013) and Hsieh and Lee (2016), one possible solution to take into account of the endogeneity or selection problem in a social interactions model is by means of an approach that is similar in nature to Heckman (1976, 1978, 1979) and the control function approach (for surveys see, for example, Navarro 2008; Wooldridge 2015). In light of the foregoing, the present paper constitutes a first step towards measuring endogenous and exogenous effects in social interactions. The proposed statistical models require adjustments in order to cope with the perils of self-selection. Such adjustments will most likely leave the functional form and the structural features of the models unchanged—and it is these which set the models of this paper apart from existing social interactions models.

Chapter 2

Conformism under Incomplete Information

Abstract

As a large body of literature in sociology and economics has shown, social interaction induces conformism, and—as it has been observed—behaviors deviating from the social norm tend to be punished. Although conformism has been studied in a network setting before, this is one of the first papers to examine conformism under incomplete information, and it is the first to provide and discuss a comprehensive theoretical framework. Social interaction is modelled as a Bayesian network game, which is the natural setting for analyzing decisions whose potential returns or costs are ex ante uncertain (like, for example, in education and crime). We establish existence and uniqueness of the equilibrium, characterize the optimal decisions, and examine conditions under which policy interventions can be welfare-improving.

2.1 Introduction

Social and professional networks are omnipresent in people's everyday lives and directly or indirectly influence their choices and their behavior. The decision of an individual to pursue college education, consume a product, work hard or shirk, and even engage in criminal activities, has been shown to be affected by their social environment.

Games on networks (also referred to as network games) can be used as a tool to analyze a wide variety of situations where players' payoffs and, therefore, their behavior depend on the actions of their peers.[1] According to the standard cost–benefit analysis, undertaking an action yields some benefit to the actor but at the same time imposes also a cost on him. This trade-off is captured by the private component of a player's utility function. The social component of the utility function determines how precisely a player's behavior is affected by the behavior of his peers. Two types of specifications of the social component stand out in the literature on network games; they give rise to the so-called class of *local-aggregate* games (or models) and the so-called class of *local-average* games.[2] A local-aggregate game is a network game with local strategic complements, where a player's marginal benefit from taking an action increases as his peers increase their actions. It is used to model situations in which an individual's return from engaging into an activity depends on the total prevalence of this activity in his social neighborhood, irrespective of how this is distributed among the neighbors. A local-average game, on the other hand, builds on the premise that it is the average rather than the aggregate level of activity that matters; specifically, it formalizes the idea that, due to social pressure, individuals who deviate from a social norm (defined as a local average) suffer some punishment in the form of reputational damage, loss of social status, or even social exclusion; in short, it is a model of conformist behavior.

The class of local-aggregate models, proposed by Calvó-Armengol and Zenou (2004) and Ballester, Calvó-Armengol, and Zenou (2006), has been used to model peer effects in crime (Ballester, Zenou, and Calvó-Armengol 2010; Lindquist and Zenou 2014) and in R&D partnerships (König, Liu, and Zenou 2016). The class of local-average models has received significant empirical attention ever since Bramoullé, Djebbari, and Fortin (2009) discussed identification in such type of models and proposed an instrumental variables estimator. Empirical applications of these models cover a wide spectrum of settings where social norms are suspected to play an important role, for example, crime (Lindquist and Zenou 2014; Liu et al. 2015), education (Calvó-Armengol, Patacchini, and Zenou 2009; De Giorgi, Pellizzari, and Redaelli 2010; Liu and Lee 2010), and consumption (De Giorgi, Frederiksen, and Pistaferri 2016).

Despite the fertile and significant literature in this area, there is still a lot of work that needs to be done in order to gain a deeper understanding of the impact of peers on individual decisions. First, the majority of empirical studies focus on

1. For a review of the literature on games on networks, see Jackson and Zenou (2015).
2. For an up-to-date review of the theoretical and empirical literature on this area, see Topa and Zenou (2015).

the significance and magnitude of peer effects in some setting of interest, assuming the model they use (most commonly the local-average model) is the relevant one. Although the choice made is rigorously motivated and quite often intuitively plausible, the use of an alternative model could also be theoretically justifiable in many cases. In fact, it could be argued that in some settings a hybrid model should be used. Liu, Patacchini, and Zenou (2014) are the first to test the local-aggregate and local-average models against each other. They find that, even within the same network, some choices (for example, study effort) are driven by social norms and, hence, are better described by the local-average model, while others (for example, delinquent behavior) are better accounted for by the amount of exposure to peers' total activity, pointing, thus, towards the local-aggregate model. Second, an important assumption implicitly underlying the existing literature has been that the model's parameters are known, if not by the social planner, at least by the decision makers. In many of the aforementioned empirical applications this may, however, be quite a strong assumption. When deciding whether to pursue further education, high school graduates may not know their financial and non-pecuniary benefits of college education. Similarly, potential criminals do not know with certainty what the probability of getting caught is. Uncertainty may apply to the social component of the utility function as well. Individuals in a society, for example, may not be fully cognisant of the strength and the enforceability of the social norm, that is, how tolerant society is towards non-conformist behaviors, and how harsh social punishment is expected to be.

To the best of our knowledge, the only other paper to examine social interaction in the local-average framework with incomplete information is the one by Blume et al. (2015). Their focus, though, lies more on providing the framework and the tools that will allow identification and estimation of the local-average model under incomplete information, rather than studying the theoretical model per se. Moreover, in their model, incomplete information concerns only the players' private benefits. The present paper, in contrast, allows of incomplete information on all components of the players' utility functions, in particular, private benefits and private and social costs. In addition, it studies the implications that the introduction of uncertainty has for equilibrium behavior. Our work can, therefore, be seen as complementary to theirs. Understanding the mechanics and the channels through which peer effects operate in this setting is of primary importance because the results of the baseline model with complete information do in general not carry over to the case of incomplete information. The present work aims moreover to close a gap that exists in the literature as the local-aggregate model has already been studied in an incomplete information setting by de Martí Beltran and Zenou (2015). This paper intends, together with the econometric model introduced by Blume et al. (2015), to give the researcher the necessary tools to study peer effects in settings where the local-average model has been found to be more relevant.

We model the players' decision problem as a Bayesian network game. Specifically, as mentioned above, we allow uncertainty to creep in through any of the three main parameters of the model: private benefit of an action, private cost, and taste for conformity, the latter being a parameter that captures the network-induced cost due to deviation from the endogenously determined social norm. These parameters are potentially heterogeneous across players and consist of two components: a *global* one, which is common to all players, and an *idiosyncratic* one, which captures their individual characteristics. In our model, players receive some signals about the state of the world and optimally decide their actions. Even though players are affected directly only by their direct neighbors in the network and need to infer their actions through the signals received, in equilibrium, they also need to infer the actions of players in higher order neighborhoods, that is, of players located more than one link away (higher order beliefs). We establish the existence and uniqueness of the Bayesian Nash equilibrium, characterize the optimal decisions, and examine conditions under which policy interventions can be welfare-improving. We also perform different comparative statics and show, in particular, how social environment and players' characteristics affect their actions.

The rest of the paper is structured as follows. Section 2.2 introduces formally the Bayesian network game outlined above. Section 2.3 studies the complete information case, which will serve as a benchmark. Our main results on the incomplete information case are presented in Section 2.4. Section 2.5 concludes. A brief review of basic concepts in graph theory is given in Appendix A. All proofs have been deferred to Appendix E.

2.2 The Bayesian network game

The Bayesian network game described hereinafter is a static, non-cooperative game with incomplete information, where the players take their decisions simultaneously and independently of one another. The players are assumed to be rational in the sense that they seek to maximize their well-being.

There are $n > 1$ players. The set of all players is identified with the set $\mathcal{I} := \{1, \dots, n\}$. In what follows, each mathematical object associated with a particular player will be indexed by an element of $\mathcal{I}$, typically, by i.

Let $(\Omega, \mathfrak{S})$ be a measurable space with $\Omega \neq \emptyset$. The set Ω is called the *state space*; it represents all the possible states (of the world) that are relevant to the game. We assume that the players have a common prior (that is, probability measure) $\mathbb{P}$ on $(\Omega, \mathfrak{S})$. The probabilistic nature of the game is, therefore, represented by the probability space $(\Omega, \mathfrak{S}, \mathbb{P})$.

For all $i \in \mathcal{I}$, let $\alpha_i \colon \Omega \to \mathbb{R}_{++}$, $\beta_i \colon \Omega \to \mathbb{R}_{++}$, and $\gamma_i \colon \Omega \to \mathbb{R}_{+}$ be integrable random variables defined on $(\Omega, \mathfrak{S}, \mathbb{P})$. The triple of functions $(\alpha_i, \beta_i, \gamma_i)$ is referred to as player i's *preference parameters* (see also the utility function (2.1) below).

For all $i \in \mathcal{I}$, let $\boldsymbol{s}_i \colon \Omega \to \mathbb{R}^3$ be a simple random 3-vector defined on $(\Omega, \mathfrak{S}, \mathbb{P})$.[3] The function $\boldsymbol{s}_i$ is called player i's private *signal*, which may convey some information about his preference parameters. A player's signal is said to convey no information about a preference parameter if the signal and the parameter are stochastically independent. We assume that the players' signals have a common support, de-

3. A random variable is called simple if it assumes only a finite number of values. In the present case, this implies that the range of $\boldsymbol{s}_i$, $\boldsymbol{s}_i(\Omega) = \{\boldsymbol{s}_i(\omega) \mid \omega \in \Omega\}$, is a finite set.

noted by Θ, which is also referred to as the players' common *type space*. Note that $0 < |\Theta| < \infty$ because $\Omega \neq \emptyset$ and a simple random variable has a finite support. Hereinafter, we shall write $\Theta = \{\boldsymbol{\theta}_t \mid t \in \{1, \ldots, T\}\}$, where $T := |\Theta| \geq 1$. Player i's *type* is given by an element $\boldsymbol{\vartheta}_i$ of Θ.

Note that, by the very definition of a support, for all $(i, r) \in \mathcal{I} \times \{1, \ldots, T\}$, $\mathbb{P}(\boldsymbol{s}_i = \boldsymbol{\theta}_r) > 0$. For ease of exposition, we assume that, for all $(i, j) \in \mathcal{I}^2$ with $i \neq j$ and for all $(r, q) \in \{1, \ldots, T\}^2$, $\mathbb{P}(\boldsymbol{s}_i = \boldsymbol{\theta}_r, \boldsymbol{s}_j = \boldsymbol{\theta}_q) > 0$.[4]

The set of actions available to player i is equal to $\mathbb{R}_+$. An action of player i is denoted by y_i. The set of all possible action profiles $\boldsymbol{y} := (y_1, \ldots, y_n)$ is equal to $\mathbb{R}_+^n$. A (pure) strategy of player i is a mapping $x_i \colon \Theta \to \mathbb{R}_+$. A strategy is therefore a rule that assigns an action to each possible type. The set of all possible (pure) strategies of player i is denoted by ${\mathbb{R}_+}^{\Theta}$. The set of all possible strategy profiles $(x_1, \ldots, x_n)$ is equal to $\prod_{i \in \mathcal{I}} {\mathbb{R}_+}^{\Theta}$.

As is characteristic for a game, a player's well-being depends not only on his action but may also depend on the actions of other players. This dependence is made explicit by means of a network by which the players are connected. We assume that this network is fixed and common knowledge and that it can be represented by a digraph G on $\mathcal{I}$. The digraph G encodes the information about the identities of all the players who directly affect a player's well-being through their actions. For a particular player, the set of all the players who directly affect his well-being is called his *out-neighborhood*. Player i's out-neighborhood (in G) is denoted by $\mathcal{N}_G^+(i)$ and its cardinality, the so-called *out-degree* of i (in G), by $\deg_G^+(i)$. Note that $i \notin \mathcal{N}_G^+(i)$, which is a consequence of the definition of a digraph. The assumption of a digraph implies that a player is not necessarily an out-neighbor of his out-neighbors, that is, $j \in \mathcal{N}_G^+(i)$ does not necessarily imply that $i \in \mathcal{N}_G^+(j)$. In short, the dependence of a player's well-being on the actions of his out-neighbors is potentially unidirectional.

In many applications, including network games, it is more convenient to represent a digraph by an *adjacency matrix*. To this end, let $\dot{\boldsymbol{A}}(G)$ denote the adjacency matrix of G with respect to the identity mapping on $\mathcal{I}$. The component in row i and column j of $\dot{\boldsymbol{A}}(G)$ is denoted by $\dot{a}_{i,j}(G)$. If player j is an out-neighbor of player i (in G), then $\dot{a}_{i,j}(G) = 1$, and $\dot{a}_{i,j}(G) = 0$ otherwise. The definition of a digraph implies that, for all $i \in \mathcal{I}$, $\dot{a}_{i,i}(G) = 0$. We assume that every player has at least one out-neighbor, as stated by the following assumption.

Assumption G For all $i \in \mathcal{I}, \mathcal{N}_G^+(i) \neq \emptyset$.

We assume that players' well-being can be represented by a family of utility functions $\{u_i \colon \Omega \times \mathbb{R}_+^n \to \mathbb{R}\}_{i \in \mathcal{I}}$ given by

$$u_i(\omega, (y_1, \ldots, y_n)) := \alpha_i(\omega) y_i - \frac{\beta_i(\omega)}{2} y_i^2 - \frac{\gamma_i(\omega)}{2} \left(y_i - \frac{\sum_{j \in \mathcal{N}_G^+(i)} y_j}{\deg_G^+(i)} \right)^2, \quad (2.1)$$

where, as assumed above, $\alpha_i > 0$, $\beta_i > 0$, and $\gamma_i \geq 0$.

The common functional form of the players' utility functions is known to all players. Incomplete information may, however, arise by the players' ignorance about

4. This assumption is relevant to the definition of the prediction matrix $\boldsymbol{\Gamma}_{i,j}$ (see Section 2.4).

the value of some of the preference parameters. A particular preference parameter can hereby give rise to incomplete information only if it is not a constant function.

Some comments on utility function (2.1) are in order. To this end, let ω be a state and $\boldsymbol{y} := (y_1, \ldots, y_n)$ an action profile. Player i's utility function is symmetric in his out-neighbors' actions. It exhibits local strategic complements if $\gamma_i > 0$ (that is, γ_i is a positive function) because, for all $j \in \mathcal{I}$,

$$\frac{\partial^2 u_i(\omega, \boldsymbol{y})}{\partial y_i \partial y_j} = \begin{cases} \dfrac{\gamma_i(\omega)}{\deg^+_G(i)} & \text{if } j \in \mathcal{N}^+_G(i), \\ 0 & \text{if } j \notin \mathcal{N}^+_G(i). \end{cases}$$

It does, however, not exhibit positive or negative local externalities.[5] The utility that player i ascribes to the action profile $\boldsymbol{y}$, $u_i(\omega, \boldsymbol{y})$, consists of two components: the private component and the social component. The *private component* is defined as $\alpha_i(\omega) y_i - (\beta_i(\omega)/2) y_i^2$. Since $\alpha_i > 0$ and $\beta_i > 0$, the point $\alpha_i(\omega)/\beta_i(\omega)$ is defined and lies in the interior of $\mathbb{R}_+$, from which it follows that the private component function $y_i \mapsto \alpha_i(\omega) y_i - (\beta_i(\omega)/2) y_i^2$ is strictly increasing on $(0, \alpha_i(\omega)/\beta_i(\omega))$ and strictly decreasing on $(\alpha_i(\omega)/\beta_i(\omega), +\infty)$, with a global maximum point at $\alpha_i(\omega)/\beta_i(\omega)$. The private component can in turn be decomposed into two parts: the private benefit and the private cost. The *private benefit* is defined as $\alpha_i(\omega) y_i$ and the *private cost* as $-(\beta_i(\omega)/2) y_i^2$. The *social component* is defined as

$$-\frac{\gamma_i(\omega)}{2} \left(y_i - \frac{\sum_{j \in \mathcal{N}^+_G(i)} y_j}{\deg^+_G(i)} \right)^2 .$$

The social component represents player i's social cost, if any ($\gamma_i = 0$ is possible), from deviating from a *social norm* that is given by the arithmetic mean of his out-neighbors' actions, $(1/\deg^+_G(i)) \sum_{j \in \mathcal{N}^+_G(i)} y_j = (1/\deg^+_G(i)) \sum_{j=1}^n \dot{a}_{i,j}(G) y_j$. The parameter γ_i is referred to as the *social conformism parameter* or the *strength of the social norm*; it captures the magnitude of player i's social cost relative to his private preference parameters, α_i and β_i. The distance between player i's action and his social norm is referred to as the *social distance* between player i and his out-neighbors (see also Akerlof 1997). It is important to note that the players' social norms are endogenous and potentially heterogeneous (in equilibrium) because the players may vary in their out-neighborhoods and may choose different actions (in equilibrium).

The following definition is useful in order to give a compact representation of the players' utility functions.

Definition A The *row-normalized adjacency matrix of* G (with respect to the identity mapping on $\mathcal{I}$) is the square matrix $\bar{\boldsymbol{A}}(G)$ of order n whose component in row i and column j, denoted by $\bar{a}_{i,j}(G)$, is defined by $\bar{a}_{i,j}(G) := \dot{a}_{i,j}(G)/\deg^+_G(i)$.

5. In accordance with the terminology introduced by Galeotti et al. (2010, pp. 226–27), player i's utility function is said to exhibit negative (respectively, positive) local externalities if, for all $\omega \in \Omega$, for all $(y_1, \ldots, y_n) \in \mathbb{R}^n_+$, and for all $(\tilde{y}_1, \ldots, \tilde{y}_n) \in \mathbb{R}^n_+$ with $\tilde{y}_i = y_i$ and $\{\tilde{y}_j - y_j \mid j \in \mathcal{N}^+_G(i)\} \subset \mathbb{R}_+$, $u_i(\omega, (\tilde{y}_1, \ldots, \tilde{y}_n)) \leq u_i(\omega, (y_1, \ldots, y_n))$ (respectively, $u_i(\omega, (\tilde{y}_1, \ldots, \tilde{y}_n)) \geq u_i(\omega, (y_1, \ldots, y_n))$).

In the remainder of the paper, the component in row i and column j of $\bar{\boldsymbol{A}}(G)$, $\bar{a}_{i,j}(G)$, is abbreviated to $\bar{a}_{i,j}$. Using Definition A, player i's utility function can be written as

$$u_i(\omega, (y_1, \ldots, y_n)) = \alpha_i(\omega) y_i - \frac{\beta_i(\omega)}{2} y_i^2 - \frac{\gamma_i(\omega)}{2} \Big(y_i - \sum_{j \in \mathcal{I}} \bar{a}_{i,j} y_j \Big)^2 .$$

The timing of the Bayesian network game is as follows:

Step 1: **Nature moves:** A state $\omega \in \Omega$ is realized (but not observed by the players).

Step 2: **Players receive information:** Each player i observes $\boldsymbol{s}_i(\omega)$, the value of his private signal $\boldsymbol{s}_i$ at the state ω, which determines his type $\boldsymbol{\vartheta}_i = \boldsymbol{s}_i(\omega) \in \Theta$.

Step 3: **Players move:** Each player i chooses an action $y_i = x_i(\boldsymbol{\vartheta}_i) \in \mathbb{R}_+$ conditional on his type $\boldsymbol{\vartheta}_i$.

Step 4: **Payoffs are realized:** Each player i receives the utility that corresponds to the realized state (and, hence, the values of the players' signals and their types) and the chosen strategy profile, $u_i(\omega, (x_1(\boldsymbol{\vartheta}_1), \ldots, x_n(\boldsymbol{\vartheta}_n)))$.

2.3 The case of complete information

This section discusses the case where the players have a complete knowledge of all the preference parameters. It is assumed, therefore, that the players know with certainty not only their own preference parameters but also those of all other players. In some cases, this may not be a bad approximation, since individuals may be aware of the preferences or characteristics of others or at least of those of their friends. Even if this assumption may not be very plausible in many settings, it is a case that is still worthwhile examining. Apart from serving as a benchmark, it also provides some basic intuition on how the various forces in the model interact to give rise to an equilibrium, before this is perplexed further by the introduction of uncertainty.

Within the framework introduced in Section 2.2, complete information corresponds to the case $|\Omega| = 1$. It follows that, for all $i \in \mathcal{I}$, the preference parameters α_i, β_i, and γ_i are constant, that is, there exists a triple $(\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i) \in \mathbb{R}^2_{++} \times \mathbb{R}_+$ such that $\alpha_i(\Omega) = \{\bar{\alpha}_i\}$, $\beta_i(\Omega) = \{\bar{\beta}_i\}$, and $\gamma_i(\Omega) = \{\bar{\gamma}_i\}$. It follows also that the players' signals are constant and functionally identical, that is, $T = 1$ and $\Theta = \{\boldsymbol{\theta}_1\}$. Let $\Gamma := (\mathcal{I}, G, (\Omega, \mathfrak{S}, \mathbb{P}), \{(\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i)\}_{i \in \mathcal{I}}, \{\boldsymbol{\theta}_1\})$ denote the Bayesian network game with complete information, where the players' constant preference parameters are common knowledge. Let

$$\bar{\boldsymbol{\alpha}} := \begin{pmatrix} \bar{\alpha}_1 \\ \vdots \\ \bar{\alpha}_n \end{pmatrix}, \quad \bar{\boldsymbol{\beta}} := \begin{pmatrix} \bar{\beta}_1 \\ \vdots \\ \bar{\beta}_n \end{pmatrix}, \quad \text{and} \quad \bar{\boldsymbol{\gamma}} := \begin{pmatrix} \bar{\gamma}_1 \\ \vdots \\ \bar{\gamma}_n \end{pmatrix}$$

denote the (column) vectors of the players' preference parameters. In addition, let $\operatorname{diag}(\bar{\boldsymbol{\beta}} + \bar{\boldsymbol{\gamma}})$ and $\operatorname{diag}(\bar{\boldsymbol{\gamma}})$ denote the diagonal matrices of order n with the components in row i and column i equal to $\bar{\beta}_i + \bar{\gamma}_i$ and $\bar{\gamma}_i$, respectively. Player i's strategy x_i

is defined by the values it assumes on the type space Θ. Since $\Theta = \{\boldsymbol{\theta}_1\}$, $x_i(\boldsymbol{\theta}_1)$ is a complete description of player i's strategy. For any strategy profile $\boldsymbol{x} := (x_1, \ldots, x_n)$, let the (column) vector $\boldsymbol{x}_{\boldsymbol{\theta}_1}$ be defined by

$$\boldsymbol{x}_{\boldsymbol{\theta}_1} := \begin{pmatrix} x_1(\boldsymbol{\theta}_1) \\ \vdots \\ x_n(\boldsymbol{\theta}_1) \end{pmatrix}.$$

Evidently, $\boldsymbol{x}_{\boldsymbol{\theta}_1}$ is a complete description of the strategy profile $\boldsymbol{x}$.

The following result characterizes the unique and interior (Bayesian) Nash equilibrium (or (B)NE for short) in Γ.

Proposition 2.1 (BNE in $\boldsymbol{\Gamma}$) *The network game Γ has a unique and interior (Bayesian) NE $\boldsymbol{x}^\star := (x_1^\star, \ldots, x_n^\star)$, which is given by*

$$\boldsymbol{x}^\star_{\boldsymbol{\theta}_1} = \left(\operatorname{diag}(\bar{\boldsymbol{\beta}} + \bar{\boldsymbol{\gamma}}) - \operatorname{diag}(\bar{\boldsymbol{\gamma}})\bar{\boldsymbol{A}}(G)\right)^{-1}\bar{\boldsymbol{\alpha}}. \tag{2.2}$$

It follows from Proposition 2.1 that the game Γ is strategically equivalent to the (Bayesian) network game $(\mathcal{I}, G, (\Omega, \mathfrak{S}, \mathbb{P}), \{\bar{\alpha}_i/\bar{\beta}_i, 1, \bar{\gamma}_i/\bar{\beta}_i\}_{i\in\mathcal{I}}, \{\boldsymbol{\theta}_1\})$.

A player's equilibrium strategy can also be written as a scalar.

Corollary 2.2 *Player i's strategy in the (Bayesian) NE in Γ is given by*

$$x_i^\star(\boldsymbol{\theta}_1) = \frac{\bar{\alpha}_i}{\bar{\beta}_i + \bar{\gamma}_i} + \frac{\bar{\gamma}_i}{\bar{\beta}_i + \bar{\gamma}_i}\boldsymbol{e}_i^\top \bar{\boldsymbol{A}}(G)\left(\operatorname{diag}(\bar{\boldsymbol{\beta}} + \bar{\boldsymbol{\gamma}}) - \operatorname{diag}(\bar{\boldsymbol{\gamma}})\bar{\boldsymbol{A}}(G)\right)^{-1}\bar{\boldsymbol{\alpha}},$$

where $\boldsymbol{e}_i := (\delta_{1,i}, \ldots, \delta_{n,i})$ denotes the ith (column) basis vector of the canonical basis of $\mathbb{R}^n$.[6]

We now study how equilibrium actions are affected by changes in the players' preference parameters. We consider two types of changes: individual and global. In order to facilitate our analysis, we decompose the players' preference parameters as follows:

$$\forall i \in \mathcal{I} \quad \bar{\alpha}_i = \bar{\alpha}^C + \bar{\alpha}_i^I, \quad \bar{\beta}_i = \bar{\beta}^C + \bar{\beta}_i^I, \quad \bar{\gamma}_i = \bar{\gamma}^C + \bar{\gamma}_i^I,$$

where $\bar{\alpha}^C$ is the *global component* of $\bar{\alpha}_i$, common to all players, for example, $\bar{\alpha}^C := (1/n)\sum_{j=1}^n \bar{\alpha}_j$, and $\bar{\alpha}_i^I := \bar{\alpha}_i - \bar{\alpha}^C$ is the *idiosyncratic component*, which may vary across players. The components $\bar{\beta}^C$, $\bar{\beta}_i^I$, $\bar{\gamma}^C$, and $\bar{\gamma}_i^I$ are defined in a similar way.

The global component of each parameter characterizes some attribute of the activity in question that does not directly depend on the specific characteristics of an individual. Consider, for example, the problem of optimal investment in education. In that case, $\bar{\alpha}^C$ can be interpreted as the expected marginal increase in earnings from an additional year of college attendance, while $\bar{\beta}^C$ captures the additional (pecuniary or not) cost incurred by the average individual (for example, tuition fees, average income foregone in the duration of studies). Yet, some students may

6. Note that $\delta_{k,l}$ denotes Kronecker's delta of k and l.

possess skills that enable them to benefit more than the average, while the opposite may be true for lower-skill students. Costs may vary across agents as well. Students of higher ability are more likely to receive a scholarship compared to low-ability students, and their disutility from studying may be lower as well. Alternatively, the opportunity cost of obtaining a postgraduate degree may be higher for an individual who is already employed compared to an individual who is unemployed or has just finished college. Similarly, $\bar{\gamma}^C$ represents the prevailing strength of the social norm in society, while some individuals may feel more ($\bar{\gamma}_i^I > 0$) or less ($\bar{\gamma}_i^I < 0$) compelled to adhere to that norm.

Proposition 2.3 (Effects of shifts in idiosyncratic components) *Let* $(i, j) \in \mathcal{I}^2$ *with* $i \neq j$.

(2.3.1) *Player i's equilibrium action is strictly increasing in his idiosyncratic private benefit component* ($\bar{\alpha}_i^I$) *and increasing in the idiosyncratic private benefit components of other players.*

(2.3.2) *Player i's equilibrium action is strictly decreasing in his idiosyncratic private cost component* ($\bar{\beta}_i^I$) *and decreasing in the idiosyncratic private cost components of other players.*

(2.3.3) *Player i's equilibrium action is strictly increasing (respectively, strictly decreasing) in his idiosyncratic strength of social norm* ($\bar{\gamma}_i^I$) *if his equilibrium action is lower (respectively, higher) than his social norm. Player i's equilibrium action is increasing (respectively, decreasing) in player j's idiosyncratic strength of social norm* ($\bar{\gamma}_j^I$) *if player j's equilibrium action is equal to or lower than (respectively, equal to or higher than) his social norm.*

Result 2.3.1 is quite intuitive: if exerting effort becomes more beneficial for player i, that player will respond by increasing his effort. At a second-order, this will raise the social norms of player i's in-neighbors, inducing them to increase their efforts as well. By the same token, the latter will induce their own in-neighbors to increase their level of activity, propagating this effect via the network. In many networks, this effect may end up raising the social norm of player i, leading to a further increase in his effort. This chain effect will go on dwindling until the new equilibrium is reached. A similar reasoning, albeit in the opposite direction, applies to an increase in the idiosyncratic cost component (Result 2.3.2). The intuition behind Result 2.3.3 is similar, although the mechanics are slightly different. An increase in the idiosyncratic strength of the social norm of player i causes deviations from the social norm to be more costly for that player. If player i's effort is lower than his social norm, this will cause his effort to increase, giving rise to the chain process described above. If player i's effort is higher than his social norm, this effect will work in the opposite direction.

The following proposition shows how the results of Proposition 2.3 change in case of shifts in the global components of the preference parameters.

Proposition 2.4 (Effects of shifts in global components) (*2.4.1*) *All players' equilibrium actions are strictly increasing in the global private benefit component* ($\bar{\alpha}^C$).

(*2.4.2*) *All players' equilibrium actions are strictly decreasing in the global private cost component* ($\bar{\beta}^C$).

(*2.4.3*) *The effect of an increase in the global strength of the social norm* ($\bar{\gamma}^C$) *is* ex ante *unclear; its sign may vary across players.*

Results 2.4.1 and 2.4.2 are quite intuitive, and the consequences of an increase in a global component are realised in a way similar to the one discussed in the context of Proposition 2.3. Result 2.4.3, though, presents more interest; in this case, unlike Results 2.3.1 and 2.3.2, Result 2.3.3 does not carry over to global shifts. The reason is that an increase in the strength of the social norm will affect players asymmetrically, and even the aggregate effect may be unclear. At a first-order, players will intensify their effort if it is relatively low and reduce it if it is relatively high compared to their social norms. Yet, the higher order effects are indeterminate. Consider, for example, a player who increases his effort following an increase in the strength of the social norms. It may well be the case that his out-neighbors reduce their efforts, reducing thus his social norm. This will offset the first-order increase, changing again the social norm of that player's in-neighbors. At the same time, both his in- and out-neighbors will be affected by other players. The effort of each player in the new equilibrium will depend of the particular values of the preference parameters and the structure of the network.

2.4 The case of incomplete information

The previous section studied how a social network influences individuals' choices and how changes in the environment, or even in the characteristics of a single player, can affect the equilibrium outcomes of all individuals. It can, however, be argued that, in more cases than not, individuals have to make their choices in an uncertain environment, without knowing *ex ante* the exact returns or costs of their actions. This paper admits of three sources of uncertainty: incomplete information about the benefit deriving from increasing the level of own activity (exerting more effort), the cost of that additional effort, and the strength of the social norm. Using the terminology introduced in Section 2.2, there will in general be at least two different types of players, so that $T = |\Theta| > 1$. In the present section, we use this to model uncertainty about each of the three aforementioned parameters.

Let $\Gamma(\alpha, \beta, \gamma) := (\mathcal{I}, G, (\Omega, \mathfrak{S}, \mathbb{P}), \{(\alpha_i, \beta_i, \gamma_i)\}_{i \in \mathcal{I}}, \Theta)$ denote the Bayesian network game with incomplete information about the values of the players' α's, β's, and γ's. Let

$$\boldsymbol{\alpha} := \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} : \Omega \to \mathbb{R}^n_{++}, \quad \boldsymbol{\beta} := \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} : \Omega \to \mathbb{R}^n_{++},$$

and

$$\gamma := \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix} : \Omega \to \mathbb{R}^n_+$$

denote the random n-vectors of the players' α's, β's, and γ's. For all $i \in \mathcal{I}$, let

$$\boldsymbol{\alpha}_i^\mu := \begin{pmatrix} \mathbb{E}(\alpha_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_1) \\ \vdots \\ \mathbb{E}(\alpha_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_T) \end{pmatrix}, \quad \boldsymbol{\beta}_i^\mu := \begin{pmatrix} \mathbb{E}(\beta_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_1) \\ \vdots \\ \mathbb{E}(\beta_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_T) \end{pmatrix},$$

and

$$\gamma_i^\mu := \begin{pmatrix} \mathbb{E}(\gamma_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_1) \\ \vdots \\ \mathbb{E}(\gamma_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_T) \end{pmatrix}.$$

The component in row t of $\boldsymbol{\alpha}_i^\mu$, $\mathbb{E}(\alpha_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_t)$, is player i's prediction of α_i given he observes the value $\boldsymbol{\theta}_t$ for his signal $\boldsymbol{s}_i$. Note that $\mathbb{E}(\alpha_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_t)$ is the value that the conditional expectation $\mathbb{E}(\alpha_i \mid \boldsymbol{s}_i)$ assumes on the event $\{\boldsymbol{s}_i = \boldsymbol{\theta}_t\}$. Note also that $\mathbb{E}(\alpha_i \mid \boldsymbol{s}_i)$ is the best predictor of α_i in terms of mean squared prediction error. We call $\mathbb{E}(\alpha_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_t)$ the *posterior expectation* of player i's α given $\{\boldsymbol{s}_i = \boldsymbol{\theta}_t\}$. In addition, let

$$\boldsymbol{\alpha}^\mu := \begin{pmatrix} \boldsymbol{\alpha}_1^\mu \\ \vdots \\ \boldsymbol{\alpha}_n^\mu \end{pmatrix}, \quad \boldsymbol{\beta}^\mu := \begin{pmatrix} \boldsymbol{\beta}_1^\mu \\ \vdots \\ \boldsymbol{\beta}_n^\mu \end{pmatrix}, \quad \text{and} \quad \gamma^\mu := \begin{pmatrix} \gamma_1^\mu \\ \vdots \\ \gamma_n^\mu \end{pmatrix}.$$

For all $(i, j) \in \mathcal{I}^2$, let $\boldsymbol{\Pi}_{i,j}$ and $\boldsymbol{\Gamma}_{i,j}$ be the square matrices of orders T with the components in row r and column q equal to $\mathbb{P}(\boldsymbol{s}_j = \boldsymbol{\theta}_q \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r)$ and $\mathbb{E}(\gamma_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r, \boldsymbol{s}_j = \boldsymbol{\theta}_q)$, respectively; that is, the rth row of $\boldsymbol{\Pi}_{i,j}$ is essentially the conditional probability mass function of player j's signal $\boldsymbol{s}_j$ given that player i observes $\boldsymbol{\theta}_r$ for his signal $\boldsymbol{s}_i$. Note that the matrix $\boldsymbol{\Pi}_{i,j}$ is row-normalized and, in particular, $\boldsymbol{\Pi}_{i,i} = \boldsymbol{I}_T$. Let $\boldsymbol{\Pi}$ and $\boldsymbol{\Gamma}$ be the square matrices of orders nT defined by

$$\boldsymbol{\Pi} := \begin{pmatrix} \boldsymbol{\Pi}_{1,1} & \dots & \boldsymbol{\Pi}_{1,j} & \dots & \boldsymbol{\Pi}_{1,n} \\ \vdots & & \vdots & & \vdots \\ \boldsymbol{\Pi}_{i,1} & \dots & \boldsymbol{\Pi}_{i,j} & \dots & \boldsymbol{\Pi}_{i,n} \\ \vdots & & \vdots & & \vdots \\ \boldsymbol{\Pi}_{n,1} & \dots & \boldsymbol{\Pi}_{n,j} & \dots & \boldsymbol{\Pi}_{n,n} \end{pmatrix}$$

and

$$\boldsymbol{\Gamma} := \begin{pmatrix} \boldsymbol{\Gamma}_{1,1} & \dots & \boldsymbol{\Gamma}_{1,j} & \dots & \boldsymbol{\Gamma}_{1,n} \\ \vdots & & \vdots & & \vdots \\ \boldsymbol{\Gamma}_{i,1} & \dots & \boldsymbol{\Gamma}_{i,j} & \dots & \boldsymbol{\Gamma}_{i,n} \\ \vdots & & \vdots & & \vdots \\ \boldsymbol{\Gamma}_{n,1} & \dots & \boldsymbol{\Gamma}_{n,j} & \dots & \boldsymbol{\Gamma}_{n,n} \end{pmatrix}.$$

A strategy of player i, x_i, is defined by the values it assumes on the type space Θ. It follows that the (column) vector $\boldsymbol{x}_{\Theta,i}$ defined by

$$\boldsymbol{x}_{\Theta,i} := \begin{pmatrix} x_i(\boldsymbol{\theta}_1) \\ \vdots \\ x_i(\boldsymbol{\theta}_T) \end{pmatrix}$$

is a complete description of player i's strategy. For any given strategy profile $\boldsymbol{x} := (x_1, \dots, x_n)$, let the (column) vector $\boldsymbol{x}_\Theta$ be defined by

$$\boldsymbol{x}_\Theta := \begin{pmatrix} \boldsymbol{x}_{\Theta,1} \\ \vdots \\ \boldsymbol{x}_{\Theta,n} \end{pmatrix}.$$

It goes without saying that $\boldsymbol{x}_\Theta$ is a complete description of the strategy profile $\boldsymbol{x}$. Note that $\boldsymbol{x}_\Theta$ is a constant (column) vector with nT components, whereas $\boldsymbol{x}$ is a random n-vector.

Having laid the requisite notational groundwork, we state the three main results of this paper: Proposition 2.5 shows that the network game $\Gamma(\alpha, \beta, \gamma)$ has a unique and interior Bayesian Nash equilibrium (or BNE for short). A characterization of the BNE strategy profile $\boldsymbol{x}^\star := (x_1^\star, \dots, x_n^\star)$ in terms of the values it assumes on Θ^n, that is, in terms of $\boldsymbol{x}_\Theta^\star$, is given by the system of equations (2.3). A characterization of $\boldsymbol{x}^\star$ in terms of first moments is given in Proposition 2.6. A characterization of equilibrium welfare in $\Gamma(\alpha, \beta, \gamma)$, denoted by $w^\star(\Gamma(\alpha, \beta, \gamma))$ and defined as the sum of the players' expected equilibrium utilities,

$$w^\star\big(\Gamma(\alpha, \beta, \gamma)\big) := \sum_{i=1}^{n} \mathbb{E}\big(u_i\big(\mathrm{id}_\Omega, (x_1^\star \circ \boldsymbol{s}_1, \dots, x_n^\star \circ \boldsymbol{s}_n)\big)\big),$$

is the subject of Proposition 2.7.

Proposition 2.5 *The network game $\Gamma(\alpha, \beta, \gamma)$ has a unique and interior BNE $\boldsymbol{x}^\star$, which is given by*

$$\boldsymbol{x}_\Theta^\star = \Big(\mathrm{diag}\big(\boldsymbol{\beta}^\mu + \boldsymbol{\gamma}^\mu\big) - \big(\bar{A}(G) \otimes \mathbf{1}_T \mathbf{1}_T^\top\big) \circ \boldsymbol{\Pi} \circ \boldsymbol{\Gamma}\Big)^{-1} \boldsymbol{\alpha}^\mu. \tag{2.3}$$

Proposition 2.6 *The BNE strategy profile $\boldsymbol{x}^\star$ in $\Gamma(\alpha, \beta, \gamma)$ satisfies*

$$\mathbb{E}\left(\operatorname{diag}(\boldsymbol{\beta}+\gamma)\begin{pmatrix} x_1^\star \circ \boldsymbol{s}_1 \\ \vdots \\ x_n^\star \circ \boldsymbol{s}_n \end{pmatrix}\right) = \mathbb{E}(\boldsymbol{\alpha}) + \mathbb{E}\left(\operatorname{diag}(\gamma)\bar{\boldsymbol{A}}(G)\begin{pmatrix} x_1^\star \circ \boldsymbol{s}_1 \\ \vdots \\ x_n^\star \circ \boldsymbol{s}_n \end{pmatrix}\right).$$

Proposition 2.7 *Equilibrium welfare in $\Gamma(\alpha, \beta, \gamma)$ is given by*

$$\begin{aligned} w^\star\left(\Gamma(\alpha,\beta,\gamma)\right) = &\ \frac{1}{2}\sum_{i=1}^{n} \mathbb{E}\left((\beta_i+\gamma_i)(x_i^\star \circ \boldsymbol{s}_i)^2\right) \\ &- \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \bar{a}_{i,j}\bar{a}_{i,k}\, \mathbb{E}\left(\gamma_i(x_j^\star \circ \boldsymbol{s}_j)(x_k^\star \circ \boldsymbol{s}_k)\right). \end{aligned}$$

In order to discuss the above results, we consider three cases where the players have only incomplete information about a single type of preference parameter. Section 2.4.1 covers the case where the players have incomplete information about their private benefits, Section 2.4.2 the case of incomplete information about private costs, and Section 2.4.3 the case of incomplete information about social costs.

2.4.1 Incomplete information about private benefits

We begin with the case where the private benefit of their actions is unknown to the players. A college student, for example, has to decide on the amount of effort to put down in studying, without knowing how this will reflect on her performance in the exams, or even how obtaining a degree will affect her future labour market outcomes. Similarly, an employee at a firm may be uncertain on whether additional effort and longer working hours will translate into a higher output and potentially a higher salary or a promotion.

To formally model this, assume that both $\boldsymbol{\beta}$ and γ are constant, that is, for all $i \in \mathcal{I}$, there exists a pair $(\bar{\beta}_i, \bar{\gamma}_i) \in \mathbb{R}_{++} \times \mathbb{R}_+$ such that $\beta_i(\Omega) = \{\bar{\beta}_i\}$ and $\gamma_i(\Omega) = \{\bar{\gamma}_i\}$, and there exists a player k whose marginal private benefit parameter α_k is not constant with probability one. Let $\Gamma(\alpha) := (\mathcal{I}, G, (\Omega, \mathfrak{S}, \mathbb{P}), \{(\alpha_i, \bar{\beta}_i, \bar{\gamma}_i)\}_{i\in\mathcal{I}}, \Theta)$ denote the Bayesian network game with incomplete information about the values of the players' α's only, where the players' constant preference parameters are common knowledge. Let $\bar{\boldsymbol{\beta}}$, $\bar{\gamma}$, $\operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\gamma})$, and $\operatorname{diag}(\bar{\gamma})$ be defined as in Section 2.3.

With the preliminaries in order, we can now proceed to state the main results of this section. A characterization of the unique and interior BNE strategy profile is given in Corollary 2.8.

Corollary 2.8 (BNE in $\boldsymbol{\Gamma(\alpha)}$) *The network game $\Gamma(\alpha)$ has a unique and interior BNE $\boldsymbol{x}^\star$, which is given by*

$$\boldsymbol{x}_\Theta^\star = \left(\operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\gamma}) \otimes \boldsymbol{I}_T - \left(\left(\operatorname{diag}(\bar{\gamma})\bar{\boldsymbol{A}}(G)\right) \otimes \mathbf{1}_T\mathbf{1}_T^\intercal\right) \circ \boldsymbol{\Pi}\right)^{-1} \boldsymbol{\alpha}^\mu. \tag{2.4}$$

The system of equations (2.4) is similar in structure to the one derived under complete information (see Proposition 2.1). There are, however, three important differences. The first difference is rather mechanical: Players' BNE strategies are no longer constant functions; in general, they assume different values on Θ, the set of all possible signal values. Player i's BNE strategy as represented by the vector $\boldsymbol{x}^\star_{\Theta,i}$ will therefore in general not be symmetric.[7] This increases the dimension of the equilibrium by the number T of different signal values (the cardinality of the type space Θ), and hence the appearance of Kronecker products in (2.4). The second difference is that the BNE strategies are, unsurprisingly, functions of the players' predictions of the marginal private benefit parameters in the form of posterior expectations ($\boldsymbol{\alpha}^\mu$). This is rather intuitive since the exact values of the marginal private benefit parameters depend on the state of the world, which is no longer known. Note that, based on their private signals, players need to form a posterior expectation not only of their own marginal private benefit parameters ($\mathbb{E}(\alpha_i \mid \boldsymbol{s}_i)$) but also of those of other players ($\mathbb{E}(\alpha_j \mid \boldsymbol{s}_i)$). This remark gives rise to yet another difference compared to the complete information case. Forming expectations of other players' α's is not enough. Players need to form expectations of their neighbors' expectations since the strategies of the latter will be based on their expectations. The same argument goes for the expectations of the expectations of their neighbors, and so on, ad infinitum. This interdependence is formally introduced in the model through the matrix $\boldsymbol{\Pi}$, which is interacted with the row-normalized adjacency matrix $\bar{\boldsymbol{A}}(G)$ to channel this effect only through the links stipulated by the network.

In order to see more clearly that a player's equilibrium action depends on the entire network structure, we give an explicit representation of his BNE strategy as a function on Θ (Corollary 2.9).

Corollary 2.9 *Player i's BNE strategy $x_i^\star$ in $\Gamma(\alpha)$ is given by*

$$
\begin{aligned}
x_i^\star(\boldsymbol{\theta}_r) = {} & \frac{\mathbb{E}(\alpha_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r)}{\bar{\beta}_i + \bar{\gamma}_i} \\
& + \frac{\bar{\gamma}_i}{\bar{\beta}_i + \bar{\gamma}_i} \boldsymbol{c}_i(\boldsymbol{\theta}_r)^{\mathsf{T}} \Big(\operatorname{diag}(\bar{\boldsymbol{\beta}} + \bar{\boldsymbol{\gamma}}) \otimes \boldsymbol{I}_T - \Big(\big(\operatorname{diag}(\bar{\boldsymbol{\gamma}}) \bar{\boldsymbol{A}}(G)\big) \otimes \boldsymbol{1}_T \boldsymbol{1}_T^{\mathsf{T}} \Big) \circ \boldsymbol{\Pi} \Big)^{-1} \boldsymbol{\alpha}^\mu,
\end{aligned}
$$

where $\boldsymbol{c}_i(\boldsymbol{\theta}_r) := (\bar{a}_{i,1}\boldsymbol{\pi}_{i,1,r}, \ldots, \bar{a}_{i,j}\boldsymbol{\pi}_{i,j,r}, \ldots, \bar{a}_{i,n}\boldsymbol{\pi}_{i,n,r})$ is a (column) vector with nT components and $\boldsymbol{\pi}_{i,j,r}$ denotes the rth row of $\boldsymbol{\Pi}_{i,j}$ whose qth component is $\mathbb{P}(\boldsymbol{s}_j = \boldsymbol{\theta}_q \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r)$.

The similarity of the complete information case and the present case of incomplete information is also reflected in the system of equations that the first moments of the players' BNE strategies satisfy, as it can be seen by a comparison of (2.2) (Proposition 2.1) and (2.5) (Corollary 2.10).

Corollary 2.10 *The BNE strategy profile $\boldsymbol{x}^\star$ in $\Gamma(\alpha)$ satisfies*

$$
\mathbb{E}\left(\begin{pmatrix} x_1^\star \circ \boldsymbol{s}_1 \\ \vdots \\ x_n^\star \circ \boldsymbol{s}_n \end{pmatrix} \right) = \big(\operatorname{diag}(\bar{\boldsymbol{\beta}} + \bar{\boldsymbol{\gamma}}) - \operatorname{diag}(\bar{\boldsymbol{\gamma}}) \bar{\boldsymbol{A}}(G) \big)^{-1} \mathbb{E}(\boldsymbol{\alpha}). \tag{2.5}
$$

7. A vector $\boldsymbol{a}$ in $\mathbb{R}^T$ is called symmetric if it is a scalar multiple of $\boldsymbol{1}_T$.

To discuss Corollary 2.10, suppose the players' signals are stochastically independent (Assumption S-I). It follows that the players' BNE strategies are uncorrelated: for all $(j,k) \in \mathcal{I}^2$ with $j \neq k$, $\operatorname{cov}(x_j^\star \circ \boldsymbol{s}_j, x_k^\star \circ \boldsymbol{s}_k) = 0$. This result does, however, not imply that the players' BNE strategies are unrelated. Indeed, the players are connected by the network G, which induces a dependence in their BNE strategies. This dependence is reflected in a functional relation between the first moments of their strategies, as given by the system of equations (2.5).

Assumption S-I The players' signals are stochastically independent.

The assumption of stochastically independent signals is also instrumental in analyzing and discussing equilibrium welfare in $\Gamma(\alpha)$, of which a characterization is given in Corollary 2.12, which follows from Proposition 2.7 and Lemma 2.11.

Lemma 2.11 *If Assumption S-I is satisfied, then, for all $i \in \mathcal{I}$,*

$$\operatorname{var}(x_i^\star \circ \boldsymbol{s}_i) = \frac{\operatorname{var}\big(\mathbb{E}(\alpha_i \mid \boldsymbol{s}_i)\big)}{(\bar{\beta}_i + \bar{\gamma}_i)^2}.$$

Corollary 2.12 *If Assumption S-I is satisfied, then equilibrium welfare in $\Gamma(\alpha)$ is given by*

$$\begin{aligned} w^\star\big(\Gamma(\alpha)\big) = {} & \frac{1}{2}\sum_{i=1}^{n} \frac{\operatorname{var}\big(\mathbb{E}(\alpha_i \mid \boldsymbol{s}_i)\big)}{\bar{\beta}_i + \bar{\gamma}_i} - \frac{1}{2}\sum_{i=1}^{n} \bar{\gamma}_i \sum_{j=1}^{n} \bar{a}_{i,j}^2 \frac{\operatorname{var}\big(\mathbb{E}(\alpha_j \mid \boldsymbol{s}_j)\big)}{(\bar{\beta}_j + \bar{\gamma}_j)^2} \\ & + \frac{1}{2}\sum_{i=1}^{n}(\bar{\beta}_i + \bar{\gamma}_i)\,\mathbb{E}(x_i^\star \circ \boldsymbol{s}_i)^2 - \frac{1}{2}\sum_{i=1}^{n} \bar{\gamma}_i \Bigg(\sum_{j=1}^{n} \bar{a}_{i,j}\,\mathbb{E}(x_j^\star \circ \boldsymbol{s}_j)\Bigg)^2 \qquad (2.6) \end{aligned}$$

and, for all $l \in \mathcal{I}$, $\partial w^\star(\Gamma(\alpha))/\partial \operatorname{var}(\mathbb{E}(\alpha_l \mid \boldsymbol{s}_l)) > 0$ if and only if $\bar{\beta}_l + \bar{\gamma}_l > \sum_{i=1}^{n} \bar{a}_{i,l}^2 \bar{\gamma}_i$.

Let us consider two extreme cases. First, suppose player k's marginal private benefit parameter α_k and his private signal $\boldsymbol{s}_k$ are stochastically independent; in other words, the private signal conveys no information about the private benefit of own action—it is essentially useless in predicting the value of the marginal private benefit parameter, which depends on the unknown state of the world. The best predictor (in terms of mean squared prediction error) of α_k is therefore its mean: $\mathbb{E}(\alpha_k \mid \boldsymbol{s}_k) = \mathbb{E}(\alpha_k)$. It follows that $\operatorname{var}(\mathbb{E}(\alpha_k \mid \boldsymbol{s}_k)) = \operatorname{var}(\mathbb{E}(\alpha_k)) = 0$, which in turn implies that $\operatorname{var}(x_k^\star \circ \boldsymbol{s}_k) = 0$ (Lemma 2.11), that is, player k's BNE strategy $x_k^\star$ is constant. The reason for this result is clear: since player k's prediction of α_k is constant across all states of the world, namely, it is equal to $\mathbb{E}(\alpha_k)$, he will choose the same action in every state. Second, suppose player k's marginal private benefit parameter α_k and his private signal $\boldsymbol{s}_k$ are almost surely functionally dependent, that is, $\alpha_k = f(\boldsymbol{s}_k)$ with probability one for some nonconstant function $f\colon \mathbb{R} \to \mathbb{R}_{++}$; in other words, the value of the marginal private benefit parameter can be observed by the player. The best predictor of α_k is therefore α_k itself: $\mathbb{E}(\alpha_k \mid \boldsymbol{s}_k) = \mathbb{E}(f(\boldsymbol{s}_k) \mid \boldsymbol{s}_k) = f(\boldsymbol{s}_k) = \alpha_k$. It follows that $\operatorname{var}(\mathbb{E}(\alpha_k \mid \boldsymbol{s}_k)) = \operatorname{var}(\alpha_k) > 0$, which in turn implies that $\operatorname{var}(x_k^\star \circ \boldsymbol{s}_k) > 0$ (Lemma 2.11), that is, player k's BNE strategy $x_k^\star$ is not constant. Intuition suggests that player k will favor the second extreme case over the first when confronted with the choice between the two alternatives, that is, to be informed is better than being uninformed about the true state of the world. This is indeed true in the present setting if moving from one case to the other leaves player k's expectation of α_k unchanged: according to Corollaries 2.10 and 2.12, the difference in player k's expected equilibrium utility between the second and the first case is positive and equal to $(1/(2(\bar{\beta}_k + \bar{\gamma}_k)) \operatorname{var}(\alpha_k)$ if $\mathbb{E}(\alpha_k)$ remains constant, in which case also $\mathbb{E}(x_k \circ \boldsymbol{s}_k)$ remains constant. This difference is in fact maximal because $\operatorname{var}(\mathbb{E}(\alpha_k \mid \boldsymbol{s}_k)) \leq \operatorname{var}(\alpha_k)$.[8] The above result does not necessarily hold true if $\mathbb{E}(x_k \circ \boldsymbol{s}_k)$ decreases when moving from one extreme case to the other, as can be seen from (2.6); specifically, the third summand on the right hand side of (2.6) is strictly increasing in $\mathbb{E}(x_k \circ \boldsymbol{s}_k)$ because $\bar{\beta}_k > 0$ and $\bar{\gamma}_k \geq 0$. Moreover, as implied by (2.5), other players' means of their BNE strategies are likely to change in response to a change in $\mathbb{E}(\alpha_k)$. The sign of the overall effect will depend on the players' preference parameters and the structure of the network. This applies also to the case where the players' signals are stochastically dependent.

Corollary 2.12 has direct implications for the design of welfare-improving policy measures. In a stylized setting in which the players' signals are stochastically independent, a policy measure that increases the accuracy of a single player's prediction of his marginal private benefit parameter is welfare-improving if the players' expectations are unaffected by the measure and the sum of the single player's private and social cost parameters exceeds a certain threshold, which depends on his in-neighbors' social cost parameters. In case of stochastically dependent signals, a welfare-improving policy measure must not only be tailored to the players' preference parameters and the network but also to the nature of the signals' interdependencies.

2.4.2 Incomplete information about private costs

A different potential source of uncertainty for the individuals lies in the cost entailed by undertaking some action. One such example is the decision of an individual to smoke. Smoking is an activity whose intensity is largely affected by social networks, especially among adolescents (see, for example, Bisin, Moro, and Topa 2011). The impact of smoking on health is, however, *ex ante* unknown to the smoker. Similarly, individuals engaging in a wide range of delinquent or illegal activities, such as bullying, petty crime (Patacchini and Zenou 2012), hooliganism, or tax evasion, do not know with certainty the probability of getting caught or the exact punishment they would face if caught.

In terms of technical analysis, this case is similar to the one with incomplete information about private benefits (Section 2.4.1). Assume that both $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ are constant, that is, for all $i \in \mathcal{I}$, there exists a pair $(\bar{\alpha}_i, \bar{\gamma}_i) \in \mathbb{R}_{++} \times \mathbb{R}_+$ such that $\alpha_i(\Omega) = \{\bar{\alpha}_i\}$ and $\gamma_i(\Omega) = \{\bar{\gamma}_i\}$. Let $\Gamma(\beta) := (\mathcal{I}, G, (\Omega, \mathfrak{S}, \mathbb{P}), \{(\bar{\alpha}_i, \beta_i, \bar{\gamma}_i)\}_{i \in \mathcal{I}}, \Theta)$ denote the Bayesian network game with incomplete information about the values of

8. According to the *law of total variance*, $\operatorname{var}(\alpha_k) = \mathbb{E}(\operatorname{var}(\alpha_k \mid \boldsymbol{s}_k)) + \operatorname{var}(\mathbb{E}(\alpha_k \mid \boldsymbol{s}_k))$, which implies that $\operatorname{var}(\alpha_k) \geq \operatorname{var}(\mathbb{E}(\alpha_k \mid \boldsymbol{s}_k))$.

the players' β's only, where the players' constant preference parameters are common knowledge. Let $\bar{\boldsymbol{\alpha}}$, $\bar{\boldsymbol{\gamma}}$, and $\operatorname{diag}(\bar{\boldsymbol{\gamma}})$ be defined as in Section 2.3.

A characterization of the unique and interior BNE strategy profile in $\Gamma(\beta)$ is given in the following result.

Corollary 2.13 (BNE in $\boldsymbol{\Gamma(\beta)}$) *The network game $\Gamma(\beta)$ has a unique and interior BNE $\boldsymbol{x}^\star$, which is given by*

$$\boldsymbol{x}^\star_\Theta = \Big(\operatorname{diag}(\boldsymbol{\beta}^\mu + \bar{\boldsymbol{\gamma}} \otimes \mathbf{1}_T) - \Big(\big(\operatorname{diag}(\bar{\boldsymbol{\gamma}})\bar{\boldsymbol{A}}(G)\big) \otimes \mathbf{1}_T\mathbf{1}_T^\top\Big) \circ \boldsymbol{\Pi}\Big)^{-1} (\bar{\boldsymbol{\alpha}} \otimes \mathbf{1}_T).$$

The main intuition behind the above formula is similar to the one for Corollary 2.8 presented in Section 2.4.1. Uncertainty in this case, though, concerns the cost rather than the benefit parameters. A representation of a player's BNE strategy as a function on Θ is given below.

Corollary 2.14 *Player i's BNE strategy $x_i^\star$ in $\Gamma(\beta)$ is given by*

$$\begin{aligned} x_i^\star(\boldsymbol{\theta}_r) &= \frac{\bar{\alpha}_i}{\mathbb{E}(\beta_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r) + \bar{\gamma}_i} + \frac{\bar{\gamma}_i}{\mathbb{E}(\beta_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r) + \bar{\gamma}_i} \\ &\quad \times \boldsymbol{c}_i(\boldsymbol{\theta}_r)^\top \Big(\operatorname{diag}(\boldsymbol{\beta}^\mu + \bar{\boldsymbol{\gamma}} \otimes \mathbf{1}_T) - \Big(\big(\operatorname{diag}(\bar{\boldsymbol{\gamma}})\bar{\boldsymbol{A}}(G)\big) \otimes \mathbf{1}_T\mathbf{1}_T^\top\Big) \circ \boldsymbol{\Pi}\Big)^{-1} (\bar{\boldsymbol{\alpha}} \otimes \mathbf{1}_T), \end{aligned}$$

where $\boldsymbol{c}_i(\boldsymbol{\theta}_r)$ is defined as in Corollary 2.9.

A comparison of Corollaries 2.9 and 2.14 shows that player i's BNE strategies in $\Gamma(\alpha)$ and $\Gamma(\beta)$ have a common structural form. There is, however, an important difference. Whereas his strategy is a linear function of the posterior expectations of his and other players' α's in $\Gamma(\alpha)$, it is a nonlinear function of the posterior expectations of his and other players' β's in $\Gamma(\beta)$. This nonlinearity implies that the welfare results of Section 2.4.1 for the case of stochastically independent signals do not carry over to the present case. A characterization of equilibrium welfare in $\Gamma(\beta)$ is, nevertheless, given in Corollary 2.15. A welfare-improving policy measure must be tailored to the players' preference parameters, the network by which the players are connected, and the nature of the signals' interdependencies. The aforementioned nonlinearity is also reflected in the system of equations obeyed by the first moments of the players BNE strategies in $\Gamma(\beta)$. Note that there does not exist a closed-form solution of $\mathbb{E}((x_1^\star \circ \boldsymbol{s}_1, \ldots, x_n^\star \circ \boldsymbol{s}_n))$, as is evident from Corollary 2.16. By contrast, $\mathbb{E}((x_1^\star \circ \boldsymbol{s}_1, \ldots, x_n^\star \circ \boldsymbol{s}_n)) = (\operatorname{diag}(\bar{\boldsymbol{\beta}} + \bar{\boldsymbol{\gamma}}) - \operatorname{diag}(\bar{\boldsymbol{\gamma}})\bar{\boldsymbol{A}}(G))^{-1}\,\mathbb{E}(\boldsymbol{\alpha})$ in $\Gamma(\alpha)$ (Corollary 2.10).

Corollary 2.15 *Equilibrium welfare in $\Gamma(\beta)$ is given by*

$$\begin{aligned} w^\star(\Gamma(\beta)) &= \frac{1}{2}\sum_{i=1}^n \mathbb{E}\big(\beta_i (x_i^\star \circ \boldsymbol{s}_i)^2\big) + \frac{1}{2}\sum_{i=1}^n \bar{\gamma}_i\,\mathbb{E}\big((x_i^\star \circ \boldsymbol{s}_i)^2\big) \\ &\quad - \frac{1}{2}\sum_{i=1}^n \bar{\gamma}_i \sum_{j=1}^n \sum_{k=1}^n \bar{a}_{i,j}\bar{a}_{i,k}\,\mathbb{E}\big((x_j^\star \circ \boldsymbol{s}_j)(x_k^\star \circ \boldsymbol{s}_k)\big). \end{aligned}$$

Corollary 2.16 *The BNE strategy profile $\boldsymbol{x}^\star$ in $\Gamma(\beta)$ satisfies*

$$\mathbb{E}\left(\operatorname{diag}(\boldsymbol{\beta})\begin{pmatrix} x_1^\star \circ \boldsymbol{s}_1 \\ \vdots \\ x_n^\star \circ \boldsymbol{s}_n \end{pmatrix}\right) + \operatorname{diag}(\bar{\gamma})\big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)\,\mathbb{E}\left(\begin{pmatrix} x_1^\star \circ \boldsymbol{s}_1 \\ \vdots \\ x_n^\star \circ \boldsymbol{s}_n \end{pmatrix}\right) = \bar{\boldsymbol{\alpha}}.$$

2.4.3 Incomplete information about social costs

Finally, we discuss the case where uncertainty stems from ignorance concerning the social costs of the activity in question. In many cases it may be reasonable to treat the strength of the social norm as unknown to the individuals. Consider, for example, a group of freshmen in a college, families moving to a new neighborhood, or immigrants settling down to a country with different culture. Individuals in the aforementioned environments most likely understand that behaving very disparately from their peers may entail consequences, ranging from failure to create a positive image to as far as social marginalization. It may be less clear, however, how important conformist behavior is regarded in each situation or which aspects of everyday life it applies to.

Although the formal setup is similar to the cases discussed above, the mechanism through which uncertainty affects individuals' behavior is slightly different, as it will be seen. Assume that both $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are constant, that is, for all $i \in \mathcal{I}$, there exists a pair $(\bar{\alpha}_i, \bar{\beta}_i) \in \mathbb{R}_{++} \times \mathbb{R}_{++}$ such that $\alpha_i(\Omega) = \{\bar{\alpha}_i\}$ and $\beta_i(\Omega) = \{\bar{\beta}_i\}$. Let $\Gamma(\gamma) := (\mathcal{I}, G, (\Omega, \mathfrak{S}, \mathbb{P}), \{(\bar{\alpha}_i, \bar{\beta}_i, \gamma_i)\}_{i\in\mathcal{I}}, \Theta)$ denote the Bayesian network game with incomplete information about the values of the players' γ's only, where the players' constant preference parameters are common knowledge. Let $\bar{\boldsymbol{\alpha}}$ and $\bar{\boldsymbol{\beta}}$ be defined as in Section 2.3.

A characterization of the unique and interior BNE strategy profile in $\Gamma(\gamma)$ is given in the following result.

Corollary 2.17 (BNE in $\boldsymbol{\Gamma(\gamma)}$) *The network game $\Gamma(\gamma)$ has a unique and interior BNE $\boldsymbol{x}^\star$, which is given by*

$$\boldsymbol{x}_\Theta^\star = \Big(\operatorname{diag}\big(\bar{\boldsymbol{\beta}} \otimes \mathbf{1}_T + \boldsymbol{\gamma}^\mu\big) - \big(\bar{\boldsymbol{A}}(G) \otimes \mathbf{1}_T\mathbf{1}_T^\intercal\big) \circ \boldsymbol{\Pi} \circ \boldsymbol{\Gamma}\Big)^{-1}(\bar{\boldsymbol{\alpha}} \otimes \mathbf{1}_T).$$

As expected, uncertainty in this case has been shifted to the social conformism parameter $\boldsymbol{\gamma}$. The above formula, however, includes an additional term compared to the cases with incomplete information about the private preference parameters. The matrix $\boldsymbol{\Gamma}$ implies that players need to form their expectation about the unknown parameter based not only on their signal, but also considering every possible signal of their peers.

Similar to the case of incomplete information about private costs, a player's BNE strategy in $\Gamma(\gamma)$ is a nonlinear function of the posterior expectations of his and other players' γ's. The implication of this nonlinearity for the design of welfare-improving policy measures is therefore the same as in Section 2.4.2. For the sake of completeness, a characterization of the first moments of the players' BNE strategies in $\Gamma(\gamma)$ is given in Corollary 2.18 and a characterization of equilibrium welfare in $\Gamma(\gamma)$ in Corollary 2.19.

Corollary 2.18 *The BNE strategy profile $\boldsymbol{x}^\star$ in $\Gamma(\gamma)$ satisfies*

$$\operatorname{diag}(\bar{\boldsymbol{\beta}})\,\mathbb{E}\left(\begin{pmatrix} x_1^\star \circ \boldsymbol{s}_1 \\ \vdots \\ x_n^\star \circ \boldsymbol{s}_n \end{pmatrix}\right) + \mathbb{E}\left(\operatorname{diag}(\gamma)\big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)\begin{pmatrix} x_1^\star \circ \boldsymbol{s}_1 \\ \vdots \\ x_n^\star \circ \boldsymbol{s}_n \end{pmatrix}\right) = \bar{\boldsymbol{\alpha}}.$$

Corollary 2.19 *Equilibrium welfare in $\Gamma(\gamma)$ is given by*

$$\begin{aligned} w^\star\big(\Gamma(\gamma)\big) = {} & \frac{1}{2}\sum_{i=1}^{n} \bar{\beta}_i\,\mathbb{E}\big((x_i^\star \circ \boldsymbol{s}_i)^2\big) + \frac{1}{2}\sum_{i=1}^{n} \mathbb{E}\big(\gamma_i (x_i^\star \circ \boldsymbol{s}_i)^2\big) \\ & - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \bar{a}_{i,j}\bar{a}_{i,k}\,\mathbb{E}\big(\gamma_i (x_j^\star \circ \boldsymbol{s}_j)(x_k^\star \boldsymbol{s}_k)\big). \end{aligned}$$

2.5 Concluding remarks

This paper provides a comprehensive theoretical framework for studying the effects of social conformism when the parameters of the model are unknown to the individuals. Specifically, uncertainty enters an individual's decision problem through three different channels: private benefit of an action, private costs, and costs due to socially divergent behavior. There is a wide range of potential applications of our model, including the study of decisions on education, work effort, the practice of religion, socially delinquent behaviors, and crime. Although the role of social networks in influencing behavior in these areas has been studied in the recent literature, it would be interesting to revisit existing results in the light of our findings since a game with incomplete information appears to constitute a more natural approach to these kind of decision problems.

We find that an increase in social pressure can have ambiguous effects on the level of activity in a society, which is in line with findings from the empirical string of the networks literature (see, for example, Bisin, Moro, and Topa 2011). A reason for this indeterminacy is that the direction of these effects is contingent on the structure of the network. Based on our analysis, we conjecture that in networks with strongly prevalent homophily an increase in the social conformism parameter reinforces the existing differences in behavior among different groups, while in networks with milder homophily it may actually dampen them. We believe, moreover, that the magnitude of these effects is more likely to be larger under higher uncertainty. Corroborating these conjectures requires further research.

Chapter 3

Local Key Player Analysis

Abstract

This paper introduces formally the concept of a local key player in the context of a network game where the social planner's objective is to reduce aggregate activity of only those players who reside in a certain local area or part of the network. The players are potentially heterogeneous with respect to the marginal private benefit from taking an action and the network effects exerted on own behavior by the actions of connected players, of which the magnitude and sign can vary across players. Potential areas of applications of local key player analysis include crime and worker productivity. In the context of crime, where networks of criminals spread across different police areas, local key player analysis provides a means to identify and neutralize the key criminal in each police area in order to reduce criminal activity locally (that is, in each police area) most, thereby taking the criminals' cross-area connections into account. In the context of worker productivity, in a team based organization where co-worker ties are dense among members of the same team but ties exist also between members of different teams (that is, the workers of the organization form a universe of interconnected islands), local key player analysis enables to identify the key worker of a team who contributes most to overall team productivity, thereby taking interteam ties and their effects on worker productivity into consideration.

3.1 Introduction

Identifying key players is an important aspect in network analysis. Key players are vertices in a network who are considered important in a certain sense. A precise definition depends on the context and the purpose to which their identification is put. The problem of identifying key players arises, for example, in the context of public health, criminal justice, and marketing. In the public health context, identification of individuals with higher odds of spreading a disease plays an important role in targeted immunization schemes to prevent major epidemic outbreaks.[1] In the criminal justice context, one of many possible police strategies to fight crime is to identify and neutralize (for example, by arresting, exposing, discrediting, or imposing and enforcing time and location or even residency restrictions) a small number of offenders in a network to maximally reduce criminal activity. In the context of marketing, seeding strategies for viral marketing campaigns involve identifying individuals in a social network, for example, opinion leaders or well-connected individuals, who ensure rapid diffusion of information about a new product, technology, or behavior.[2]

The problem of identifying key players in a social network has a long history in sociology. Early research has focused on vertex centrality measures to quantify the importance of individuals in a social network, for example, degree centrality (Freeman 1978-1979), betweenness centrality (Freeman 1980), closeness centrality (Bavelas 1950), eigenvector centrality (Bonacich 1972a, 1972b), Katz's (1953) status index, or Bonacich's (1987) measure of power and centrality.[3,4] The extent to which a centrality measure is appropriate to quantify an individual's importance in a social network depends on the context and the purpose of identification; for example, individuals in a communication network with high betweenness are considered important because they can facilitate or inhibit the communication of others, individuals in a friendship network with high degree are typically considered popular, and individuals are often considered influential if they are connected to other influential individuals and, therefore, have a high eigenvector centrality. More recent

1. For a discussion of different population immunization strategies, including targeted immunization, see, for example, Pastor-Satorras and Vespignani (2002) and Madar et al. (2004).

2. For a discussion of viral marketing strategies see, for example, Hinz et al. (2011).

3. For an overview and precise definition of different centrality measures see, for example, Wasserman and Faust (1994, chapter 5), Koschützki et al. (2005, chapter 3), or Borgatti and Everett (2006). In an undirected graph, degree centrality is defined as the number of edges incident to a given vertex. The idea is that centrality increases with the number of neighboring vertices, for example, friends in a friendship network. In a strongly connected undirected graph, closeness centrality is defined as the reciprocal of the sum of the distances between a given vertex and all other vertices in the graph. The intuition is that more central vertices have on average a short distance to other vertices, resulting in a larger centrality. In a strongly connected undirected graph, betweenness centrality is defined as the sum of the fractions of the number of shortest paths between any pair of distinct vertices that pass through a given vertex. The idea is that a vertex's importance as a junction point increases with the fractions of shortest paths that pass through it. For a definition of eigenvector centrality, Katz's (1953) status index, and Bonacich's (1987) measure of power and centrality see Section 3.2.2.

4. Although different in nature, all aforementioned centrality measures have in common that they are based only on information contained in the graph's edge set and ignore all other aspects that may affect the importance of an individual.

research by Borgatti (2003, 2006) has focused on the very problem of identifying key players. He defines two types of key player problems that are based on measuring the contribution of a set of vertices to the cohesion or connectedness of a network: the *Key Player Problem/Negative* (or KPP-Neg for short) is defined in terms of the "extent to which the network depends on its key players to maintain its cohesiveness" (Borgatti 2006, p. 22), and the *Key Player Problem/Positive* (or KPP-Pos for short) is defined in terms of the "extent to which key players are connected to and embedded in the network around them" (p. 22); specifically, a set of k vertices is a solution of KPP-Neg if they cause a maximal reduction in the network's cohesiveness once removed from the network, and it is a solution of KPP-Pos if they are maximally connected to all other vertices. Two examples of KPP-Neg are given in the opening paragraph in the context of public health and criminal justice; the example given in the context of marketing is a KPP-Pos. The KPP-Neg is defined relative to a *graph property*, that is, a function that assigns values to graphs such that they are preserved under graph isomorphisms. The difference between the value of a graph property on the graph with and without the set of k vertices is called *vitality index* (Koschützki et al. 2005, Defintion 3.6.1) and is a measure of the importance of the k vertices in the graph. The mapping that assigns to every set of k vertices a vitality index is called *vitality measure* (section 3.6). Every graph property yields therefore a vitality measure, which can be regarded as a new meaningful centrality measure.

The economics approach to identifying key players (see Zenou 2016 for a comprehensive survey) is different from the sociological approach in at least one important respect, namely, the microeconomic foundation of the graph property relative to which a key player problem is defined, which gives rise to an economically meaningful vitality or centrality measure. Ballester, Calvó-Armengol, and Zenou (2006) pioneered the identification of key players in the context of network games, that is, strategic games on networks. They introduce a static, noncooperative network game with complete information and interplayer dependent payoff functions that are linear-quadratic in nature, exhibit local complementarity effects, and have a global, uniform (in players) substitutability component. The beauty of their theory lies in an appealing characterization of the game's unique and interior Nash equilibrium (for the existence of which they give sufficient conditions); it is related to a variant of Bonacich's (1987) measure of power and centrality. Apart from comparative statics, they discuss a policy that consists of targeting the key player defined as the player who, once removed from the game and thereby removed from the network, leads to a maximal decrease in aggregate equilibrium action, provided that action has a negative connotation (like, for example, crime). They give an explicit characterization of the key player in terms of a vitality measure, which they call *intercentrality measure*; specifically, the player with the highest intercentrality is the key player.[5] Subsequent research has extended Ballester, Calvó-Armengol, and Zenou's (2006) network game and key player analysis to allow for contextual effects (Ballester and Zenou 2014), incomplete information (de Martí Beltran and

5. For a definition of the intercentrality measure see Ballester, Calvó-Armengol, and Zenou (2006, Definition 2).

Zenou 2015), and network formation (Liu et al. 2015). Empirical studies of key player based policies have been conducted in a variety of contexts including criminal networks (Lindquist and Zenou 2014; Liu et al. 2015), co-worker networks (Lindquist, Sauermann, and Zenou 2016), education (Hahn et al. 2015), networks of R&D collaborations (König, Liu, and Zenou 2016), financial networks (Denbee et al. 2016), and diffusion of microfinance loan programs (Banerjee et al. 2013). It is fair to say that existing empirical evidence is in favor of key player based policies when compared to other reasonable policies; for example, in the context of criminal networks, key player based policing policies have proved superior to other policing policies, such as targeting the most active or prolific criminal, in reducing criminal activity (Lindquist and Zenou 2014). The reason for this is that key player based policies take explicitly into account the social multiplier (Glaeser, Scheinkman, and Sacerdote 2003) emanating from the social network and the interdependent behavior of its members.

An important feature of Ballester, Calvó-Armengol, and Zenou's (2006) key player analysis and of its variants mentioned above is that the smallest (in terms of set inclusion) part of a network for which a key player can be identified is a weakly connected component. This paper aims at overcoming this limitation by introducing formally the concept of the local key player; more specifically, it discusses identification of local key players in the context of a static, noncooperative network game with complete information where the players are connected by a digraph. The players are potentially heterogeneous with respect to their marginal private benefits from own action and with respect to network effects exerted on own behavior by the actions taken by connected players; specifically, a player's utility or payoff function exhibits either local strategic complements or substitutes. The notion of a key player is defined relative to a social planner's objective function. The focus in this paper is on weighted aggregate equilibrium action defined as a weighted sum of the players' actions at the unique Nash equilibrium of the network game. The qualifier weighted refers thereby to the nonnegative weights the planner assigns to the players of the game. The profile of weights reflects the importance the planner attaches to a particular subset of players. A profile of weights where all components are positive gives rise to a global key player because every player is accounted for when computing the value of the objective function. A profile of weights for which only the components corresponding to players residing in a certain subdigraph are positive gives rise to a local key player, thereby taking into account that the players in the subdigraph are affected by the actions of neighboring players not residing in the same subdigraph.

Potential areas of applications of local key player analysis include crime and worker productivity. As regards crime, sovereign states and other political entities usually organize their jurisdictions into geographical districts to provide law enforcement. For example, there are over forty police areas in England and Wales for each of which a territorial police force is responsible for policing; there are twenty-six cantonal police agencies in the Swiss Confederation, which are responsible for law enforcement in their jurisdictions; the city of Los Angeles is subdivided into eighteen policing districts, called law enforcement reporting districts. Given that networks of criminals spread across different jurisdictions, police areas, or other geographical areas of political importance or interest, local key player analysis provides a means to identify and neutralize the key criminal in each area in order to reduce criminal activity locally (that is, in each geographical area) most, thereby taking the criminals' cross-area connections into account. An example of a network that spreads across different geographical areas, namely, neighborhoods, is the youth co-offending network analyzed by Schaefer (2012). His analysis draws upon a random sample of 10,629 youth who were arrested in Maricopa County, Arizona, in 2000. Neighborhoods are identified with census tracts, which are small geographical areas with population sizes ranging from 1,500 to 8,000.[6] The home address at the time of arrest was used to assign each youth to one of the census tracts in Maricopa County. He finds that cross-tract offending is the norm among youth in the sample: 72 per cent of the 3,058 co-offending relationships between youth involved co-offenders residing in different census tracts. As regards worker productivity, in a team based organization where co-worker ties are dense among members of the same team but ties exist also between members of different teams (that is, the workers of the organization form a universe of interconnected islands), local key player analysis enables to identify the key worker of a team who contributes most to overall team productivity, thereby taking interteam ties and their effects on worker productivity into consideration. An example of a co-worker network that has the structure of a universe of interconnected islands is the network of TV production teams analyzed by Zaheer and Soda (2009). More specifically, they analyze data on a co-membership network among 501 production teams in the Italian TV production industry tracked over a period of 12 years. They find that production teams are interconnected by virtue of co-memberships of industry specialists in production teams.[7]

The Nash equilibrium of the network game discussed in the present paper is given by a variant or, more precisely, generalization of Katz's (1953) status index and Bonacich's (1987) measure of power and centrality, similar to the network game of Ballester, Calvó-Armengol, and Zenou (2006); it is for this reason called the *generalized Katz-Bonacich* (or GKB for short) *centrality measure*. Similar to its cognate measures, the GKB measure is a self-referential centrality measure where a vertex's centrality is related to the centralities of neighboring vertices, which in turn are related to the centralities of their vertices, and so on, possibly ad infinitum. The generalization concerns two points. First, the GKB measure is not only based on network encoded information but accounts also for exogenously given, vertex-specific information, referred to as vertex idiosyncrasy.[8] Second, the relation between a vertex's centrality and the centralities of its neighbors is potentially heterogeneous across vertices, in terms of both sign and magnitude (or intensity). Bonacich (1987)

6. Schaefer (2012, p. 143) points out that census tracts do not necessarily correspond to conventional definitions of neighborhoods; they have, however, been used frequently to study the ecological settings of crime and delinquency.

7. See, in particular, Zaheer and Soda (2009, Figure 1).

8. The notion of vertex idiosyncrasy is similar in nature to Hubbell's (1965) *exogenous inputs* or *boundary conditions* that enter the definition of his *status score* for clique identification.

was the first to propose a centrality measure that admits of a negative relation between a vertex's centrality and the centralities of its neighbors, where the sign and magnitude of the relation is homogeneous across vertices.[9] Despite the prevalence of Bonacich's (1987) measure of power and centrality in sociological and economic research, conditions that ensure its nonnegativity or positivity—either of them a property that a meaningful centrality measure should exhibit—in case of a negative relation between a vertex's centrality and the centralities of its neighboring vertices have not been discussed in the literature yet. One aim of this paper is to close this gap, in particular, to state such conditions for the GKB measure, of which the centrality measure of Bonacich (1987) is a special case. In addition, little is known about the properties of Bonacich's (1987) centrality measure. Another aim is, therefore, to provide and discuss comparative statics results for the GKB measure. The definition of the GKB measure, the discussion of sufficient conditions for its nonnegativity or positivity, and an exposition of related comparative statics results are the subject of a self-contained section (Section 3.2) because these results are of interest outside the context of (local) key player analysis. Almost all results are, though, relevant for the local key player analysis (Section 3.3), which constitutes the main contribution of this paper.

The rest of the paper is structured as follows. Section 3.2 introduces the centrality measure and provides a discussion of its properties, including conditions for nonnegativity and the main results on comparative statics. Section 3.3 introduces the network game in the context of which the concept of a local key player is defined. Section 3.4 concludes. A brief review of basic concepts in graph theory is given in Appendix A. Some basic results in matrix analysis are collected in Appendix B. The proofs of all main results can be found in Appendix F.

3.2 Centrality with vertex idiosyncrasy

This section is organized in the following way. After laying some notational groundwork, Section 3.2.1 introduces the *generalized Katz-Bonacich centrality measure*. It is a centrality measure with vertex idiosyncrasy because it accounts for exogenously given information not encoded in the network on which it is defined. Section 3.2.2 gives a brief overview of related centrality measures. Section 3.2.3 discusses sufficient conditions for the nonnegativity and positivity of the centrality measure. Section 3.2.4 studies comparative statics results. Section 3.2.5 introduces measures of the degree of idiosyncrasy of the centrality measure.

3.2.1 Definition

Let D be a nonempty digraph of order at least two, and let $\mathcal{I}(|\mathcal{V}(D)|)$ denote the set of all integers that are at least one and not larger than the order of D, $|\mathcal{V}(D)|$, that is,

9. Bonacich (1987) argues that a negative relation is, for example, appropriate in bargaining situations where "it is advantageous to be connected to those who have few options; power comes from being connected to those who are powerless. Being connected to powerful others who have many potential trading partners reduces one's bargaining power." (p. 1171).

$\mathcal{I}(|\mathcal{V}(D)|) := \{1, \ldots, |\mathcal{V}(D)|\}$. Let $h\colon \mathcal{V}(D) \to \mathcal{I}(|\mathcal{V}(D)|)$ be a bijection, that is, h is an enumeration of the vertices of D. The vertex set $\mathcal{V}(D)$ can be identified with the set $\mathcal{I}(|\mathcal{V}(D)|)$ by means of h, the elements of which shall also be called vertices. The adjacency matrix of D with respect to h is denoted by $\dot{\boldsymbol{A}}_h(D)$ and the component in row i and column j of $\dot{\boldsymbol{A}}_h(D)$ by $\dot{a}_{h,i,j}(D)$. Note that $\dot{\boldsymbol{A}}_h(D)$ is different from the zero matrix $\boldsymbol{O}_{|\mathcal{V}(D)|}$ because D is not empty.

Every vertex in D (and via h the corresponding vertex in $\mathcal{I}(|\mathcal{V}(D)|)$) is equipped with two characteristics that are represented by real numbers: a vertex idiosyncrasy and a localness parameter. Specifically, let α and λ denote two mappings that assign to every digraph E (of any order) the functions $\alpha(E)\colon \mathcal{V}(E) \to \mathbb{R}$ and $\lambda(E)\colon \mathcal{V}(E) \to \mathbb{R}$. The two mappings α and λ satisfy the following consistency condition: if E and F are two digraphs that are isomorphic by means of the digraph isomorphism $f\colon \mathcal{V}(E) \to \mathcal{V}(F)$, then $\alpha(E) = \alpha(F) \circ f$ and $\lambda(E) = \lambda(F) \circ f$. The functions $\alpha(D)\colon \mathcal{V}(D) \to \mathbb{R}$ and $\lambda(D)\colon \mathcal{V}(D) \to \mathbb{R}$ assign to every vertex $v \in \mathcal{V}(D)$ a *vertex idiosyncrasy* $\alpha(D)(v)$ and a *localness parameter* $\lambda(D)(v)$. For all $i \in \mathcal{I}(|\mathcal{V}(D)|)$, let $\alpha_{h,i}(D)$ denote the image of vertex i under the composition $\alpha(D) \circ h^{-1}\colon \mathcal{I}(|\mathcal{V}(D)|) \to \mathbb{R}$, that is, $\alpha_{h,i}(D)$ is the idiosyncrasy of vertex i. The (column) vector $\boldsymbol{\alpha}_h(D) := (\alpha_{h,1}(D), \ldots, \alpha_{h,|\mathcal{V}(D)|}(D)) \in \mathbb{R}^{|\mathcal{V}(D)|}$ is referred to as the *profile of vertex idiosyncrasies* (of D with respect to h). The *profile of localness parameters* (of D with respect to h), $\boldsymbol{\lambda}_h(D) := (\lambda_{h,1}(D), \ldots, \lambda_{h,|\mathcal{V}(D)|}(D)) \in \mathbb{R}^{|\mathcal{V}(D)|}$, is defined similarly.

Example 3.1 Let $\bar{\alpha} \in \mathbb{R}_{++}$. Suppose α satisfies, for all digraphs E, for all $v \in \mathcal{V}(E)$, $\alpha(E)(v) = \bar{\alpha}$. It follows that $\boldsymbol{\alpha}_h(D) = \bar{\alpha}\mathbf{1}_{|\mathcal{V}(D)|}$. ◇

Example 3.2 Suppose α satisfies, for all digraphs E, for all $v \in \mathcal{V}(E)$, $\alpha(E)(v) = \deg_E^+(v)$. It follows that $\boldsymbol{\alpha}_h(D) = \dot{\boldsymbol{A}}_h(D)\mathbf{1}_{|\mathcal{V}(D)|} \geq_c \mathbf{0}_{|\mathcal{V}(D)|}$. If all vertices of D have at least one out-neighbor, then $\boldsymbol{\alpha}_h(D) >_c \mathbf{0}_{|\mathcal{V}(D)|}$. If D is d-regular, that is, every vertex in D has in-degree d and out-degree d, then $\boldsymbol{\alpha}_h(D) = d\mathbf{1}_{|\mathcal{V}(D)|}$. ◇

Example 3.3 Suppose α satisfies, for all digraphs E, for all $v \in \mathcal{V}(E)$, $\alpha(E)(v) = \deg_E^-(v)$. It follows that $\boldsymbol{\alpha}_h(D) = \dot{\boldsymbol{A}}_h(D)^\mathsf{T}\mathbf{1}_{|\mathcal{V}(D)|} \geq_c \mathbf{0}_{|\mathcal{V}(D)|}$. If all vertices of D have at least one in-neighbor, then $\boldsymbol{\alpha}_h(D) >_c \mathbf{0}_{|\mathcal{V}(D)|}$. ◇

Example 3.4 considers the case where the vertices are endowed with K characteristics, each represented by a real number, and a vertex's idiosyncrasy is defined as the exponential of a linear combination of these characteristics.

Example 3.4 Let K be a positive integer, $\boldsymbol{\beta} := (\beta_1, \ldots, \beta_K)$ a (column) vector in $\mathbb{R}^K$, and $\boldsymbol{\chi}$ a mapping that assigns to every digraph E a function $\boldsymbol{\chi}(E)\colon \mathcal{V}(E) \to \mathbb{R}^K$. The function $\boldsymbol{\chi}(D)\colon \mathcal{V}(D) \to \mathbb{R}^K$ assigns to every vertex $v \in \mathcal{V}(D)$ a (column) vector $\boldsymbol{\chi}(D)(v)$ of K characteristics. For all $i \in \mathcal{I}(|\mathcal{V}(D)|)$, let $\boldsymbol{\chi}_{h,i}(D)$ denote the image of vertex i under the composition $\boldsymbol{\chi}(D) \circ h^{-1}\colon \mathcal{I}(|\mathcal{V}(D)|) \to \mathbb{R}^K$. Suppose α satisfies, for all digraphs E, for all $v \in \mathcal{V}(E)$, $\alpha(E)(v) = \exp(\langle \boldsymbol{\chi}(E)(v), \boldsymbol{\beta}\rangle)$. It follows that $\boldsymbol{\alpha}_h(D) = (\exp(\langle \boldsymbol{\chi}_{h,1}(D), \boldsymbol{\beta}\rangle), \ldots, \exp(\langle \boldsymbol{\chi}_{h,|\mathcal{V}(D)|}(D), \boldsymbol{\beta}\rangle)) >_c \mathbf{0}_{|\mathcal{V}(D)|}$. ◇

Example 3.5 Let $\bar{\lambda} \in \mathbb{R}$. Suppose λ satisfies, for all digraphs E, for all $v \in \mathcal{V}(E)$, $\lambda(E)(v) = \bar{\lambda}$. It follows that $\boldsymbol{\lambda}_h(D) = \bar{\lambda}\mathbf{1}_{|\mathcal{V}(D)|}$. ◇

Example 3.6 Suppose λ satisfies, for all digraphs E, for all $v \in \mathcal{V}(E)$,

$$\lambda(E)(v) = \begin{cases} 0 & \text{if } \deg_E^+(v) = 0, \\ \dfrac{\phi(E)(v)}{\deg_E^+(v)} & \text{if } \deg_E^+(v) > 0, \end{cases}$$

where ϕ is a mapping that assigns to every digraph E a function $\phi(E)\colon \mathcal{V}(E) \to \mathbb{R}$. Suppose $\phi(D)$ is positive. It follows that $\boldsymbol{\lambda}_h(D) \geq_c \mathbf{0}_{|\mathcal{V}(D)|}$. If all vertices of D have at least one out-neighbor, then $\boldsymbol{\lambda}_h(D) >_c \mathbf{0}_{|\mathcal{V}(D)|}$. ◇

Example 3.7 Let L be a positive integer, $\boldsymbol{\gamma} := (\gamma_1, \dots, \gamma_L)$ a (column) vector in $\mathbb{R}^L$, and $\boldsymbol{\psi}$ a mapping that assigns to every digraph E a function $\boldsymbol{\psi}(E)\colon \mathcal{V}(E) \to \mathbb{R}^L$. The function $\boldsymbol{\psi}(D)\colon \mathcal{V}(D) \to \mathbb{R}^L$ assigns to every vertex $v \in \mathcal{V}(D)$ a (column) vector $\boldsymbol{\psi}(D)(v)$ of L characteristics. For all $i \in \mathcal{I}(|\mathcal{V}(D)|)$, let $\boldsymbol{\psi}_{h,i}(D)$ denote the image of vertex i under the composition $\boldsymbol{\psi}(D) \circ h^{-1}\colon \mathcal{I}(|\mathcal{V}(D)|) \to \mathbb{R}^L$. Let $\boldsymbol{\Psi}_h(D)$ denote the $|\mathcal{V}(D)| \times L$ matrix whose ith row is equal to $\boldsymbol{\psi}_{h,i}(D)^{\mathsf{T}}$. Suppose λ satisfies, for all digraphs E, for all $v \in \mathcal{V}(E)$, $\lambda(E)(v) = \langle \boldsymbol{\psi}(E)(v), \boldsymbol{\gamma} \rangle$. It follows that $\boldsymbol{\lambda}_h(D) = \boldsymbol{\Psi}_h(D)\boldsymbol{\gamma}$. ◇

Based on the foregoing definitions, I introduce the following (point or vertex) centrality.

Definition C A *generalized Katz-Bonacich* (*GKB*) centrality (measure) in D with respect to h with profile of vertex idiosyncrasies $\boldsymbol{\alpha}_h(D)$ and profile of localness parameters $\boldsymbol{\lambda}_h(D)$ is any (column) vector $\boldsymbol{b} := (b_1, \dots, b_{|\mathcal{V}(D)|}) \in \mathbb{R}^{|\mathcal{V}(D)|}$ such that

$$\forall i \in \mathcal{I}(|\mathcal{V}(D)|) \quad b_i = \alpha_{h,i}(D) + \lambda_{h,i}(D) \sum_{j=1, j \neq i}^{|\mathcal{V}(D)|} \dot{a}_{h,i,j}(D)\, b_j \,. \tag{3.1}$$

Note that (3.1) is equivalent to

$$\big(\boldsymbol{I}_{|\mathcal{V}(D)|} - \operatorname{diag}\big(\boldsymbol{\lambda}_h(D)\big)\dot{\boldsymbol{A}}_h(D)\big)\, \boldsymbol{b} = \boldsymbol{\alpha}_h(D) \tag{3.2}$$

because all main diagonal elements of $\dot{\boldsymbol{A}}_h(D)$ vanish (D has no self-loops), where $\operatorname{diag}(\boldsymbol{\lambda}_h(D))$ denotes the diagonal matrix of order $|\mathcal{V}(D)|$ with component in row i and column i equal to the ith component of $\boldsymbol{\lambda}_h(D)$.

In order to advance the reader's understanding of the idea underlying Definition C, conceive of the vertices in $\mathcal{I}(|\mathcal{V}(D)|)$ as agents in an economy or as players participating in a particular game, where the agents or players are connected by a (social) network represented by the adjacency matrix of D with respect to h, in particular, there exists a directed (social) connection from player i to player j if and only if $\dot{a}_{h,i,j}(D) = 1$. Within this interpretational framework, Definition C embodies the idea that every player's centrality is composed of two parts: an idiosyncratic part and a part that takes a player's position within the network into account. Consider player $i \in \mathcal{I}(|\mathcal{V}(D)|)$. If she has no out-neighbors (in D), that is, $\mathcal{N}_D^+(h^{-1}(i)) = \varnothing$ or, equivalently, $\{j \in \mathcal{I}(|\mathcal{V}(D)|) \mid \dot{a}_{h,i,j}(D) = 1\} = \varnothing$, then her centrality is equal to the idiosyncratic part: $b_i = \alpha_{h,i}(D)$. If she has at least one out-neighbor, then her centrality is affected by those of her out-neighbors, where the direction and extent of the effect depend on the sign and magnitude of $\lambda_{h,i}(D)$, respectively. In case $\lambda_{h,i}(D) = 0$, there is no effect, that is, $b_i = \alpha_{h,i}(D)$. In case $\lambda_{h,i}(D) > 0$ (respectively, $\lambda_{h,i}(D) < 0$), her centrality depends positively (respectively, negatively) on the centralities of her out-neighbors, provided that, for all $j \in \mathcal{I}(|\mathcal{V}(D)|)$, $b_j \geq 0$. The magnitude of $\lambda_{h,i}(D)$ reflects therefore the degree to which b_i is a local or global (in the sense that the network position is taken into account) measure of centrality (similar to Bonacich's (1987) measure of centrality).

Within the foregoing interpretational framework, if connections among agents have a positive connotation, which is, for example, the case with friendship relationships, agents with a high out-degree are often considered influential and agents with a high in-degree are said to be prominent or to have high prestige. Example 3.2 is therefore relevant when centrality is associated with influence, and Example 3.3 is of interest when centrality refers to some kind of status.

A GKB centrality in D with respect to h need neither exist nor be unique. The following result gives sufficient conditions that ensure both existence and uniqueness.

Proposition 3.8 *There exists a unique GKB centrality in D with respect to h with profile of vertex idiosyncrasies $\boldsymbol{\alpha}_h(D)$ and profile of localness parameters $\boldsymbol{\lambda}_h(D)$, denoted by*

$$\boldsymbol{b}_h\big(\boldsymbol{\alpha}_h(D), \boldsymbol{\lambda}_h(D), D\big) := \Big(b_{h,1}\big(\boldsymbol{\alpha}_h(D), \boldsymbol{\lambda}_h(D), D\big), \ldots, b_{h,|\mathcal{V}(D)|}\big(\boldsymbol{\alpha}_h(D), \boldsymbol{\lambda}_h(D), D\big)\Big),$$

if any of the following conditions is satisfied: (3.8.1) $1 \notin \sigma(\operatorname{diag}(\boldsymbol{\lambda}_h(D))\dot{\boldsymbol{A}}_h(D))$ or (3.8.2) $\rho(\operatorname{diag}(\boldsymbol{\lambda}_h(D))\dot{\boldsymbol{A}}_h(D)) < 1$.[10] *If Condition 3.8.1 or Condition 3.8.2 are satisfied, then*

$$\boldsymbol{b}_h\big(\boldsymbol{\alpha}_h(D), \boldsymbol{\lambda}_h(D), D\big) = \big(\boldsymbol{I}_{|\mathcal{V}(D)|} - \operatorname{diag}\big(\boldsymbol{\lambda}_h(D)\big)\dot{\boldsymbol{A}}_h(D)\big)^{-1}\boldsymbol{\alpha}_h(D).$$

As regards the notation $\boldsymbol{b}_h(\boldsymbol{\alpha}_h(D), \boldsymbol{\lambda}_h(D), D)$, the third argument of $\boldsymbol{b}_h$, D, and the subscript h of $\boldsymbol{b}_h$ refer to the adjacency matrix of D with respect to h, $\dot{\boldsymbol{A}}_h(D)$. Note that Condition 3.8.2 is stronger than Condition 3.8.1. If Condition 3.8.2 is satisfied and $\boldsymbol{\lambda}_h(D) \neq \boldsymbol{0}_{|\mathcal{V}(D)|}$, then $\boldsymbol{b}_h(\boldsymbol{\alpha}_h(D), \boldsymbol{\lambda}_h(D), D)$ has a series representation, namely,

$$\sum_{k=0}^{\infty} \big(\operatorname{diag}\big(\boldsymbol{\lambda}_h(D)\big)\dot{\boldsymbol{A}}_h(D)\big)^k \boldsymbol{\alpha}_h(D), \tag{3.3}$$

which will prove useful in discussing conditions for its nonnegativity or positivity.[11] It is worth pointing out that the series (3.3) may converge even if Condition 3.8.2 is not satisfied, that is, $\rho(\operatorname{diag}(\boldsymbol{\lambda}_h(D))\dot{\boldsymbol{A}}_h(D)) \geq 1$. By contrast, the Neumann series $\sum_{k=0}^{\infty}(\operatorname{diag}(\boldsymbol{\lambda}_h(D))\dot{\boldsymbol{A}}_h(D))^k$ converges (strongly) if and only if Condition 3.8.2 is satisfied (Lemma B.2). A necessary and sufficient condition for the convergence of the series (3.3) is given in Lemma 3.9. If the series (3.3) converges, then its limit is a (not necessarily unique) solution to the system of equations (3.2).

10. Note that $\sigma(\operatorname{diag}(\boldsymbol{\lambda}_h(D))\dot{\boldsymbol{A}}_h(D))$ and $\rho(\operatorname{diag}(\boldsymbol{\lambda}_h(D))\dot{\boldsymbol{A}}_h(D))$ denote the spectrum and the spectral radius of $\operatorname{diag}(\boldsymbol{\lambda}_h(D))\dot{\boldsymbol{A}}_h(D)$, respectively.

11. The expressions 0^0 and, for all $k \in \mathbb{Z}_{++}$, $\boldsymbol{O}_k^0$, are left undefined.

$$
\begin{tikzcd}
\mathbb{R} \arrow[rr, leftrightarrow, "\mathrm{id}_{\mathbb{R}}"] & & \mathbb{R} \\
D = (\mathcal{V}(D), \mathcal{A}(D)) \arrow[u, "\alpha(D) \text{ or } \lambda(D)"] \arrow[rr, "f"] \arrow[d, "h"'] & & E = (\mathcal{V}(E), \mathcal{A}(E)) \arrow[u, "\alpha(E) \text{ or } \lambda(E)"'] \arrow[d, "k"] \\
\mathcal{I}(|\mathcal{V}(D)|) & & \mathcal{I}(|\mathcal{V}(D)|) \arrow[ll, "\pi"]
\end{tikzcd}
$$

Figure 3.1. Commutative diagram (Proposition 3.11)

Lemma 3.9 (Suzuki 1976) *Suppose $\boldsymbol{\lambda}_h(D) \neq \mathbf{0}_{|\mathcal{V}(D)|}$. The series (3.3) converges (respectively, converges strongly) if and only if $(\operatorname{diag}(\boldsymbol{\lambda}_h(D))\dot{\boldsymbol{A}}_h(D))^k \boldsymbol{\alpha}_h(D)$ converges (respectively, converges strongly) to $\mathbf{0}_{|\mathcal{V}(D)|}$ as $k \to \infty$.*

I conclude this section by making two important observations. First, the GKB centrality is invariant under different enumerations of $\mathcal{V}(D)$, that is, it is invariant under different choices for h. A formal statement is given in Proposition 3.10. Second, the GKB centralities in two isomorphic digraphs are equal up to permutation. The corresponding result is stated in Proposition 3.11 and the underlying structure is illustrated in Figure 3.1.

Proposition 3.10 *Suppose Condition 3.8.1 is satisfied. Let $k\colon \mathcal{V}(D) \to \mathcal{I}(|\mathcal{V}(D)|)$ be a bijection. The GKB centrality in D with respect to h with profile of vertex idiosyncrasies $\boldsymbol{\alpha}_h(D)$ and profile of localness parameters $\boldsymbol{\lambda}_h(D)$ and the GKB centrality in D with respect to k with profile of vertex idiosyncrasies $\boldsymbol{\alpha}_k(D)$ and profile of localness parameters $\boldsymbol{\lambda}_k(D)$ are equal up to permutation; specifically, there exists a unique permutation π of $\mathcal{I}(|\mathcal{V}(D)|)$ such that $\boldsymbol{\alpha}_k(D) = \boldsymbol{P}_\pi \boldsymbol{\alpha}_h(D)$, $\boldsymbol{\lambda}_k(D) = \boldsymbol{P}_\pi \boldsymbol{\lambda}_h(D)$, $\dot{\boldsymbol{A}}_k(D) = \boldsymbol{P}_\pi \dot{\boldsymbol{A}}_h(D) \boldsymbol{P}_\pi^{-1}$, and $\boldsymbol{b}_k(\boldsymbol{\alpha}_k(D), \boldsymbol{\lambda}_k(D), D) = \boldsymbol{P}_\pi \boldsymbol{b}_h(\boldsymbol{\alpha}_h(D), \boldsymbol{\lambda}_h(D), D)$, where $\boldsymbol{P}_\pi$ is the permutation matrix of π.*

Proposition 3.11 *Suppose Condition 3.8.1 is satisfied. Let E be a nonempty digraph of order $|\mathcal{V}(D)|$ that is isomorphic to D by means of the digraph isomorphism $f\colon \mathcal{V}(D) \to \mathcal{V}(E)$. Let $k\colon \mathcal{V}(E) \to \mathcal{I}(|\mathcal{V}(D)|)$ be a bijection. There exists a unique permutation π of $\mathcal{I}(|\mathcal{V}(D)|)$ such that $\boldsymbol{\alpha}_k(E) = \boldsymbol{P}_\pi \boldsymbol{\alpha}_h(D)$, $\boldsymbol{\lambda}_k(E) = \boldsymbol{P}_\pi \boldsymbol{\lambda}_h(D)$, $\dot{\boldsymbol{A}}_k(E) = \boldsymbol{P}_\pi \dot{\boldsymbol{A}}_h(D) \boldsymbol{P}_\pi^{-1}$, and $\boldsymbol{b}_k(\boldsymbol{\alpha}_k(E), \boldsymbol{\lambda}_k(E), E) = \boldsymbol{P}_\pi \boldsymbol{b}_h(\boldsymbol{\alpha}_h(D), \boldsymbol{\lambda}_h(D), D)$, where $\boldsymbol{P}_\pi$ is the permutation matrix of π.*

Propositions 3.10 and 3.11 imply that the GKB centrality is well-defined. They also imply that one may assume without loss of generality that $\mathcal{V}(D) = \mathcal{I}(|\mathcal{V}(D)|)$ and h is the identity mapping on $\mathcal{I}(|\mathcal{V}(D)|)$, which will be done for the rest of the paper. As regards notation, the order of D will be denoted by n, that is $n := |\mathcal{V}(D)|$, and the subscript h will be dropped from $\boldsymbol{\lambda}_h(D)$, $\boldsymbol{\alpha}_h(D)$, $\dot{\boldsymbol{A}}_h(D)$, and $\boldsymbol{b}_h$ and their components. The (column) vector $\boldsymbol{b}(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D)$ is referred to as the GKB centrality (measure) in D with profile of vertex idiosyncrasies $\boldsymbol{\alpha}(D)$ and profile of localness parameters $\boldsymbol{\lambda}(D)$, provided that it exists and it is unique. Often D is omitted from the notations $\boldsymbol{\alpha}(D)$ and $\boldsymbol{\lambda}(D)$, and they are simply written as $\boldsymbol{\alpha}$ and $\boldsymbol{\lambda}$. In addition, for the rest of the paper, with one exception, the vertex set of a digraph E of order k is identified with the set $\mathcal{I}(k) = \{1, \ldots, k\}$, that is, $\mathcal{V}(E) = \mathcal{I}(k)$.[12]

3.2.2 Related centrality measures

The GKB centrality (Definition C) is related to various centrality measures discussed in sociology and economics.

Katz's (1953) status index (in D) is a scalar multiple of $\boldsymbol{b}(\bar{\lambda}\dot{\boldsymbol{A}}(D^\mathsf{T})\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D^\mathsf{T}) = \bar{\lambda}(\boldsymbol{I}_n - \bar{\lambda}\dot{\boldsymbol{A}}(D)^\mathsf{T})^{-1}\dot{\boldsymbol{A}}(D)^\mathsf{T}\mathbf{1}_n$, where $\bar{\lambda}$ is a scalar (probability) in the interval $(0, 1]$ such that $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$.[13] Note the dependence of the profile of vertex idiosyncrasies $\bar{\lambda}\dot{\boldsymbol{A}}(D^\mathsf{T})\mathbf{1}_n$ and the profile of localness parameters $\bar{\lambda}\mathbf{1}_n$ on $\bar{\lambda}$.

Bonacich's (1987) measure of centrality corresponds to $\boldsymbol{b}(\bar{\alpha}\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D)$, where $\bar{\lambda} \in \mathbb{R}$ is such that $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$.[14] It is important to note that $\bar{\lambda}$ can be of any sign. Bonacich (1987) does not make an assumption about the sign of $\bar{\alpha}$. In his analysis, $\bar{\alpha}$ is chosen such that $\|\boldsymbol{b}(\bar{\alpha}\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D)\|_2^2 = n$ (p. 1173), that is, $|\bar{\alpha}| = \sqrt{n}/\|\boldsymbol{b}(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D)\|_2$, thereby assuming that $\boldsymbol{b}(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D) \neq \mathbf{0}_n$.

The vertex idiosyncrasies implied by Katz's (1953) and Bonacich's (1987) measures are closely linked to the adjacency matrix $\dot{\boldsymbol{A}}(D)$. Definition C is less restrictive in this respect. In the absence of information on the vertices' idiosyncracies beyond that contained in D, $\dot{\boldsymbol{A}}(D)\mathbf{1}_n$ or $\dot{\boldsymbol{A}}(D)^\mathsf{T}\mathbf{1}_n$ are natural choices for $\boldsymbol{\alpha}$.

Ballester, Calvó-Armengol, and Zenou (2006) introduce the *weighted Bonacich centrality measure* (Remark 1) to characterize the interior Nash equilibrium of their network game; it corresponds to $\boldsymbol{b}(\boldsymbol{\alpha}, \bar{\lambda}\mathbf{1}_n, D)$, where $\boldsymbol{\alpha} >_c \mathbf{0}_n$ and $\bar{\lambda} > 0$ is such that $\bar{\lambda}\rho(\dot{\boldsymbol{A}}(D)) < 1$.[15]

The GKB centrality accommodates eigenvector centrality (Bonacich 1972a, 1972b). Indeed, if $\boldsymbol{\alpha} = \mathbf{0}_n$ and $\boldsymbol{\lambda} = \bar{\lambda}\mathbf{1}_n$ for some $\bar{\lambda} \in \mathbb{R} \setminus \{0\}$, then (3.2) reduces to

$$\dot{\boldsymbol{A}}(D)\,\boldsymbol{b} = \frac{1}{\bar{\lambda}}\,\boldsymbol{b}\,. \tag{3.4}$$

Apart from $\mathbf{0}_n$, any eigenvector of $\dot{\boldsymbol{A}}(D)$ corresponding to $1/\bar{\lambda}$ solves (3.4). If $1/\bar{\lambda} = \rho(\dot{\boldsymbol{A}}(D))$, then any eigenvector of $\dot{\boldsymbol{A}}(D)$ corresponding to $\rho(\dot{\boldsymbol{A}}(D))$ is a

12. This exception applies to the digraph $D \ominus x$ that emerges from D by removing vertex x (see Section 3.2.4.2); specifically, the vertex set of $D \ominus x$ is not identified with $\mathcal{I}(n-1)$ but equal to $\mathcal{I}(n) \setminus \{x\}$.

13. Katz (1953) does not assume explicitly that $\bar{\lambda}$ is such that $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$. He implicitly assumes that the series $\sum_{k=1}^{\infty} \bar{\lambda}^k \dot{\boldsymbol{A}}(D)^k$ converges with limit $(\boldsymbol{I}_n - \bar{\lambda}\dot{\boldsymbol{A}}(D))^{-1} - \boldsymbol{I}_n$, which is true if and only if $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$ (Lemma B.2).

14. Bonacich (1987) does not explicitly assume that $\bar{\lambda}$ is such that $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$. He assumes that $\bar{\lambda}$ is such that its absolute value is less than the reciprocal of the largest eigenvalue of $\dot{\boldsymbol{A}}(D)$ (p. 1178), thereby implicitly assuming that it is different from zero. His analysis is, however, not explicitly restricted to cases where D is such that all eigenvalues of $\dot{\boldsymbol{A}}(D)$ are real (which is, for example, the case when D is symmetric) and the concept of a largest eigenvalue is defined. The largest eigenvalue of $\dot{\boldsymbol{A}}(D)$ should therefore be interpreted as the spectral radius of $\dot{\boldsymbol{A}}(D)$. In this regard, note that $\dot{\boldsymbol{A}}(D)$ has a nonnegative real eigenvalue that is equal to its spectral radius (Lemma B.5).

15. In contrast to Definition C, the weighted Bonacich centrality measure is defined by $\boldsymbol{b}(\boldsymbol{\alpha}, \bar{\lambda}\mathbf{1}_n, D) := (\boldsymbol{I}_n - \bar{\lambda}\dot{\boldsymbol{A}}(D))^{-1}\boldsymbol{\alpha}$.

nonnegative eigenvector centrality (Lemma B.5), which is positive if D is strongly connected.[16]

An example of a centrality measure that cannot be represented as a GKB centrality is *The PageRank* (Page et al. 1999), which measures the importance of website pages. In order to define this measure (see Langville and Meyer 2006, chapter 4, for details), let c be a scalar in the interval $(0,1)$, $\mathcal{I}_0^+(D)$ denote the set of all vertices in D with zero out-degree, $\bar{D}$ denote the directed pseudograph[17] with vertex set $\mathcal{I}(n)$ and arc set $\mathcal{A}(D) \cup \bigcup_{i \in \mathcal{I}_0^+(D)} \{(i,j) \mid j \in \mathcal{I}(n)\}$, and $\boldsymbol{G}(D)$ be the nonnegative, row-normalized, irreducible, and aperiodic matrix of order n defined by

$$\boldsymbol{G}(D) := c \operatorname{diag}\big(\dot{\boldsymbol{A}}(\bar{D})\mathbf{1}_n\big)^{-1}\dot{\boldsymbol{A}}(\bar{D}) + (1-c)\frac{1}{n}\mathbf{1}_n\mathbf{1}_n^{\mathsf{T}}.$$

The PageRank (in D) is the unique positive solution $\boldsymbol{\pi}$ to the linear homogeneous system of equations $(\boldsymbol{I}_n - \boldsymbol{G}(D)^{\mathsf{T}})\boldsymbol{\pi} = \mathbf{0}_n$ with $\langle \mathbf{1}_n, \boldsymbol{\pi} \rangle = 1$.

An example of a quantity not related to centrality but with a functional form similar to the GKB centrality is the *knowledge index* of Calvó-Armengol and de Martí Beltran (2009, Definition 1):

$$\boldsymbol{k}(r, \boldsymbol{\Sigma}) := (1-r)\left(\boldsymbol{I}_n - \frac{r}{n-1}\boldsymbol{\Omega}(\boldsymbol{\Sigma})\right)^{-1}\mathbf{1}_n,$$

where r is a scalar in the interval $[0,1)$, $\boldsymbol{\Sigma}$ is a covariance matrix of order n with, for all $(i,j) \in \mathcal{I}(n)^2$ with $i \neq j$, $[\boldsymbol{\Sigma}]_{i,j} \leq [\boldsymbol{\Sigma}]_{i,i}$, and $\boldsymbol{\Omega}(\boldsymbol{\Sigma})$ is a square matrix of order n that depends on $\boldsymbol{\Sigma}$ and has a zero main diagonal and components off the main diagonal lying in the interval $[0,1]$ (which together imply that $\rho(\boldsymbol{\Omega}(\boldsymbol{\Sigma})) \leq \|\boldsymbol{\Omega}(\boldsymbol{\Sigma})\|_\infty \leq n-1$).

3.2.3 Conditions for nonnegativity and positivity

There have been a few attempts in the literature to distill axioms for centrality. In his fundamental paper, Sabidussi (1966, section 4) proposes a set of axioms to be satisfied by a centrality index of an undirected graph. The index is defined relative to a given point or vertex centrality (function) that assigns a vector of *nonnegative* real numbers to any undirected graph. Other contributions to an axiomatic foundation of vertex centrality include Harary, Norman, and Cartwright (1966), Nieminen (1973), and Nieminen (1974). Nieminen (1973, section 2.1) requires the centrality of a vertex to be a *nonnegative* integer, thereby referring to a status measure that has been axiomatized in Harary, Norman, and Cartwright (1966, p. 189). Nieminen (1974, p. 332) assumes that vertex centrality is a *nonnegative* real number.[18]

16. The digraph D is strongly connected if and only if $\dot{\boldsymbol{A}}(D)$ is irreducible (see, for example, Berman and Plemmons 1994, Theorem 2.7 on p. 30), in which case $\rho(\dot{\boldsymbol{A}}(D))$ is a simple eigenvalue of $\dot{\boldsymbol{A}}(D)$ to which there corresponds a positive eigenvector (Perron 1907; Frobenius 1912).

17. A directed pseudograph is a graph for which multiple arcs or self-loops are admissible (see, for example, Bang-Jensen and Gutin 2009, Section 1.2).

18. More recent contributions on axiomatic foundations of certain classes of centrality measures include Dequiedt and Zenou (2015) and Bloch, Jackson, and Tebaldi (2016), where the authors of the former paper stipulate that a vertex's centrality is a nonnegative real number and the authors of the latter admit of negative values.

I concur with the views expressed by the aforenamed authors: a reasonable theory of point or vertex centrality should impose a lower bound on the measure; moreover, zero is a natural choice for the lower bound. On account of this, I will discuss sufficient conditions under which the GKB centrality is nonnegative. I shall also discuss conditions for its positivity. The interest in the latter conditions is motivated by the key player analysis (Section 3.3).

The GKB centrality (measure) in D with profile of vertex idiosyncracies $\boldsymbol{\alpha}$ and profile of localness parameters $\boldsymbol{\lambda}$, $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$, is given by $(\boldsymbol{I}_n - \mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D))^{-1}\boldsymbol{\alpha}$ if $\rho(\mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$ (Proposition 3.8). Suppose, for simplicity of exposition, $\boldsymbol{\lambda} = \bar{\lambda}\mathbf{1}_n$ for some $\bar{\lambda} \in \mathbb{R}$. If $\bar{\lambda} = 0$, then $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) = \boldsymbol{\alpha}$, and if $\bar{\lambda} \neq 0$ and vertex $i \in \mathcal{I}(n)$ has no out-neighbors (in D), that is, $\mathcal{N}_D^+(i) = \emptyset$, then $b_i(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) = \alpha_i$. It is, therefore, reasonable to restrict attention to cases where $\boldsymbol{\alpha}$ is either nonnegative or positive in order to establish conditions for the nonnegativity or positivity of $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$. Provided that $\boldsymbol{\alpha}$ is nonnegative, a sufficient condition for the nonnegativity of $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ is that $(\boldsymbol{I}_n - \bar{\lambda}\dot{\boldsymbol{A}}(D))^{-1}$ is nonnegative. Two results of matrix analysis are of interest in this regard (Lemma B.2). First, the Neumann series $\sum_{k=0}^{\infty} \bar{\lambda}^k \dot{\boldsymbol{A}}(D)^k$ converges if and only if $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$. Second, the matrix $\boldsymbol{I}_n - \bar{\lambda}\dot{\boldsymbol{A}}(D)$ is nonsingular with inverse $\sum_{k=0}^{\infty} \bar{\lambda}^k \dot{\boldsymbol{A}}(D)^k$ if $\sum_{k=0}^{\infty} \bar{\lambda}^k \dot{\boldsymbol{A}}(D)^k$ converges. If $\bar{\lambda} \geq 0$ is such that $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$, then every term in the Neumann series $\sum_{k=0}^{\infty} \bar{\lambda}^k \dot{\boldsymbol{A}}(D)^k$ is nonnegative, so that the limit of the series, $(\boldsymbol{I}_n - \bar{\lambda}\dot{\boldsymbol{A}}(D))^{-1}$, is nonnegative. If $\bar{\lambda} < 0$, then $(\boldsymbol{I}_n - \lambda\dot{\boldsymbol{A}}(D))^{-1}$ is not necessarily nonnegative, even if $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$. This is evident from the Neumann series representation of $(\boldsymbol{I}_n - \bar{\lambda}\dot{\boldsymbol{A}}(D))^{-1}$. Note that $\sum_{k=0}^{\infty} \bar{\lambda}^k \dot{\boldsymbol{A}}(D)^k = \sum_{k=0}^{\infty} \bar{\lambda}^{2k} \dot{\boldsymbol{A}}(D)^{2k} + \sum_{k=0}^{\infty} \bar{\lambda}^{2k+1} \dot{\boldsymbol{A}}(D)^{2k+1}$ if $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$, where the first series on the right-hand side of the equality is nonnegative and the second series is nonpositive. If $\bar{\lambda} < 0$ is too large in magnitude, then the second series dominates the first and their sum, $(\boldsymbol{I}_n - \bar{\lambda}\dot{\boldsymbol{A}}(D))^{-1}$, is nonpositive. We conclude that a lower bound must be imposed on $\bar{\lambda} < 0$ in order that $(\boldsymbol{I}_n - \lambda\dot{\boldsymbol{A}}(D))^{-1}$ is nonnegative. In the general case where $\boldsymbol{\lambda}$ is not a scalar multiple of $\mathbf{1}_n$, a similar result applies; specifically, the negative components of $\boldsymbol{\lambda}$ must not be less than a certain lower bound for $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ to be nonnegative.

Nonsingular M-matrices play an important role in establishing results on the nonnegativity or positivity of the GKB centrality because their inverses are nonnegative and bounded below by the identity matrix. A square matrix of order n over $\mathbb{R}$ is called *M-matrix* if it is of the form $s\boldsymbol{I}_n - \boldsymbol{B}$, where $\boldsymbol{B}$ is a nonnegative matrix (of order n over $\mathbb{R}$) and $s \in \mathbb{R}_+$ is not less than the spectral radius of $\boldsymbol{B}$, that is, $s \geq \rho(\boldsymbol{B})$. An M-matrix $s\boldsymbol{I}_n - \boldsymbol{B}$ is nonsingular if and only if $s > \rho(\boldsymbol{B})$, in which case its inverse is nonnegative and bounded below by $\boldsymbol{I}_n$ (see also Lemma B.6).[19]

Proposition 3.12 gives a representation of the inverse of $\boldsymbol{I}_n - \mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)$ in terms of nonsingular M-matrices. The result is instrumental in establishing conditions for the nonnegativity or positivity of $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$. The representation is based on the following decomposition of the profile of localness parameters $\boldsymbol{\lambda}$: $\boldsymbol{\lambda} = \boldsymbol{\lambda}^+ - \boldsymbol{\lambda}^-$, where $\boldsymbol{\lambda}^+$ and $\boldsymbol{\lambda}^-$ denote the componentwise positive and negative parts of $\boldsymbol{\lambda}$, respectively. Note that $|\boldsymbol{\lambda}| = \boldsymbol{\lambda}^+ + \boldsymbol{\lambda}^-$, where $|\boldsymbol{\lambda}|$ denotes the profile or (column)

19. For a discussion of M-matrices see, for example, Berman and Plemmons (1994, chapter 6).

vector whose components are the absolute values of the components of $\boldsymbol{\lambda}$. The conditions for the representation to exist are stated separately in Condition C-ρ for easy reference. Corollary 3.13 deals with the special cases $\boldsymbol{\lambda} \geq_c \mathbf{0}_n$ (or, equivalently, $\boldsymbol{\lambda}^- = \mathbf{0}_n$) and $\boldsymbol{\lambda} \leq_c \mathbf{0}_n$ (or, equivalently, $\boldsymbol{\lambda}^+ = \mathbf{0}_n$).

Condition C-$\boldsymbol{\rho}$ $\rho(\operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D)) < 1$ (which implies that $\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D)$ is nonsingular) and $\rho(\operatorname{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D))^{-1}) < 1$.

Proposition 3.12 *If Condition C-ρ is satisfied, then $\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)$ is nonsingular with inverse*

$$\Big(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D)\Big)^{-1}\Big(\boldsymbol{I}_n - \Big(\operatorname{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)\big(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D)\big)^{-1}\Big)^2\Big)^{-1} \times \Big(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)\big(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D)\big)^{-1}\Big),$$

where $\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D)$, $\boldsymbol{I}_n - (\operatorname{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D))^{-1})^2$, and $\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D))^{-1}$ are nonsingular M-matrices.

Corollary 3.13 *(3.13.1) If $\boldsymbol{\lambda} \geq_c \mathbf{0}_n$ and $\rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$, then $\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)$ is a nonsingular M-matrix.*

(3.13.2) If $\boldsymbol{\lambda} \leq_c \mathbf{0}_n$ and $\rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$, then $\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)$ is nonsingular with inverse

$$\Big(\boldsymbol{I}_n - \big(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)\big)^2\Big)^{-1}\Big(\boldsymbol{I}_n - \operatorname{diag}(|\boldsymbol{\lambda}|)\dot{\boldsymbol{A}}(D)\Big),$$

where $\boldsymbol{I}_n - (\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D))^2$ and $\boldsymbol{I}_n - \operatorname{diag}(|\boldsymbol{\lambda}|)\dot{\boldsymbol{A}}(D)$ are nonsingular M-matrices.[20]

Proposition 3.14 gives sufficient conditions for the nonnegativity (Result 3.14.1) and positivity (Result 3.14.2) of $\boldsymbol{b}(\boldsymbol{\alpha}, \lambda, D)$. Corollary 3.15 is concerned with the two cases $\boldsymbol{\lambda} \geq_c \mathbf{0}_n$ and $\boldsymbol{\lambda} \leq_c \mathbf{0}_n$.

Proposition 3.14 *(3.14.1) If Condition C-ρ is satisfied,*

$$\operatorname{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)\big(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D)\big)^{-1}\boldsymbol{\alpha} \leq_c \boldsymbol{\alpha}$$

(and $\boldsymbol{\alpha} \neq \mathbf{0}_n$), then $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) \geq_c \mathbf{0}_n$ (and $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) \neq \mathbf{0}_n$).

(3.14.2) If Condition C-ρ is satisfied and

$$\operatorname{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)\big(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D)\big)^{-1}\boldsymbol{\alpha} <_c \boldsymbol{\alpha},$$

then $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) >_c \mathbf{0}_n$.

20. If $\boldsymbol{\lambda} \leq_c \mathbf{0}_n$, then $|\boldsymbol{\lambda}| = \boldsymbol{\lambda}^- = -\boldsymbol{\lambda}$, which implies that $\rho(\operatorname{diag}(|\boldsymbol{\lambda}|)\dot{\boldsymbol{A}}(D)) = \rho(\operatorname{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)) = \rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D))$ (Lemma B.9).

Corollary 3.15 *(3.15.1) If* $\boldsymbol{\lambda} \geq_c \mathbf{0}_n$, $\boldsymbol{\alpha} \geq_c \mathbf{0}_n$ *(and* $\boldsymbol{\alpha} \neq \mathbf{0}_n$*), and* $\rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$*, then* $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) \geq_c \boldsymbol{\alpha} \geq_c \mathbf{0}_n$ *(and* $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) \neq \mathbf{0}_n$*).*

(3.15.2) If $\boldsymbol{\lambda} \geq_c \mathbf{0}_n$, $\boldsymbol{\alpha} >_c \mathbf{0}_n$*, and* $\rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$*, then* $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) \geq_c \boldsymbol{\alpha} >_c \mathbf{0}_n$.

(3.15.3) If $\boldsymbol{\lambda} \leq_c \mathbf{0}_n$, $\operatorname{diag}(|\boldsymbol{\lambda}|)\dot{\boldsymbol{A}}(D)\boldsymbol{\alpha} \leq_c \boldsymbol{\alpha}$ *(and* $\boldsymbol{\alpha} \neq \mathbf{0}_n$*), and* $\rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$*, then* $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) \geq_c \mathbf{0}_n$ *(and* $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) \neq \mathbf{0}_n$*).*

(3.15.4) If $\boldsymbol{\lambda} \leq_c \mathbf{0}_n$, $\operatorname{diag}(|\boldsymbol{\lambda}|)\dot{\boldsymbol{A}}(D)\boldsymbol{\alpha} <_c \boldsymbol{\alpha}$*, and* $\rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$*, then* $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) >_c \mathbf{0}_n$.

The conditions of Result 3.14.1 (respectively, Result 3.14.2) entail a nonnegativity (respectively, positivity) restriction for the profile of vertex idiosyncrasies; namely, if Condition C-ρ is satisfied and $\operatorname{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D))^{-1}\boldsymbol{\alpha} \leq_c \boldsymbol{\alpha}$ (respectively, $\operatorname{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D))^{-1}\boldsymbol{\alpha} <_c \boldsymbol{\alpha}$), then $\boldsymbol{\alpha} \geq_c \mathbf{0}_n$ (respectively, $\boldsymbol{\alpha} >_c \mathbf{0}_n$).[21]

The inequality $\operatorname{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D))^{-1}\boldsymbol{\alpha} <_c \boldsymbol{\alpha}$ imposes a lower bound on the negative components of $\boldsymbol{\lambda}$ if $\boldsymbol{\alpha} >_c \mathbf{0}_n$ and $\rho(\operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D)) < 1$. This can best be seen if $\boldsymbol{\alpha} = \bar{\alpha}\mathbf{1}_n$ and $\boldsymbol{\lambda} = \bar{\lambda}\mathbf{1}_n$ for some $\bar{\alpha} > 0$ and $\bar{\lambda} < 0$. In this case, $\operatorname{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D))^{-1}\boldsymbol{\alpha} <_c \boldsymbol{\alpha}$ reduces to $|\bar{\lambda}|\dot{\boldsymbol{A}}(D)\mathbf{1}_n <_c \mathbf{1}_n$, which implies that $\bar{\lambda} > -1/\max\{\deg_D^+(i) \mid i \in \mathcal{I}(n)\}$.[22] Likewise, the inequality $\rho(\operatorname{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D))^{-1}) < 1$ imposes a lower bound on the negative components of $\boldsymbol{\lambda}$ if $\rho(\operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D)) < 1$. Under the foregoing assumptions about $\boldsymbol{\alpha}$ and $\boldsymbol{\lambda}$, $\rho(\operatorname{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D))^{-1}) < 1$ reduces to $|\bar{\lambda}|\rho(\dot{\boldsymbol{A}}(D)) < 1$, which implies that $\bar{\lambda} > -1/\rho(\dot{\boldsymbol{A}}(D))$ if $\rho(\dot{\boldsymbol{A}}(D)) > 0$. The lower bound imposed on the negative components of $\boldsymbol{\lambda}$ by the inequality $\operatorname{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D))^{-1}\boldsymbol{\alpha} <_c \boldsymbol{\alpha}$ is not less than that imposed by the inequality $\rho(\operatorname{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D))^{-1}) < 1$ under the stated assumptions (Lemma 3.16). Specifically, $-1/\max\{\deg_D^+(i) \mid i \in \mathcal{I}(n)\}$ is not less than $-1/\rho(\dot{\boldsymbol{A}}(D))$.[23]

Lemma 3.16 (Debreu and Herstein 1953) *For all* $\boldsymbol{a} \in \mathbb{R}_+^n$ *and for all* $\boldsymbol{A} \in \mathcal{M}(n, \mathbb{R})$ *nonnegative, if* $\boldsymbol{A}\boldsymbol{a} <_c \boldsymbol{a}$*, then* $\rho(\boldsymbol{A}) < 1$.

There exists by virtue of Lemma 3.16 a set of conditions that is equivalent to the sufficient conditions of Result 3.14.2. The result is stated in Corollary 3.17. A similar result applies to Result 3.15.4 and is stated in Corollary 3.18.

21. The proof of the statement involving strict inequalities is as follows. Suppose Condition C-ρ is satisfied and $\boldsymbol{A}\boldsymbol{\alpha} <_c \boldsymbol{\alpha}$, where $\boldsymbol{A} := \operatorname{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D))^{-1}$. Note that $\boldsymbol{A}\boldsymbol{\alpha} <_c \boldsymbol{\alpha}$ is equivalent to $(\boldsymbol{I}_n - \boldsymbol{A})\boldsymbol{\alpha} >_c \mathbf{0}_n$ and $\boldsymbol{I}_n - \boldsymbol{A}$ is a nonsingular M-matrix whose inverse is bounded below by $\boldsymbol{I}_n$ (Lemma B.6 with $c = 1$). Premultiplying both sides of the inequality $(\boldsymbol{I}_n - \boldsymbol{A})\boldsymbol{\alpha} >_c \mathbf{0}_n$ by the inverse of $\boldsymbol{I}_n - \boldsymbol{A}$ yields $\boldsymbol{\alpha} >_c \mathbf{0}_n$ (Lemma B.1).

22. Note that $\max\{\deg_D^+(i) \mid i \in \mathcal{I}(n)\} \geq 1$ because D is not empty.

23. This result also shows that $\rho(\dot{\boldsymbol{A}}(D))$ is bounded above by $\max\{\deg_D^+(i) \mid i \in \mathcal{I}(n)\}$, which is a well-known result in graph theory.

Corollary 3.17 *The following two sets of conditions are equivalent in the sense that Conditions 3.17.1 are satisfied if and only if Conditions 3.17.2 are satisfied:*

(3.17.1) $\rho(\operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D)) < 1$, $\operatorname{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D))^{-1}\boldsymbol{\alpha} <_c \boldsymbol{\alpha}$, *and* $\rho(\operatorname{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D))^{-1}) < 1$.

(3.17.2) $\rho(\operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D)) < 1$, $\operatorname{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D))^{-1}\boldsymbol{\alpha} <_c \boldsymbol{\alpha}$, *and* $\boldsymbol{\alpha} >_c \boldsymbol{0}_n$.

Corollary 3.18 *The following two sets of conditions are equivalent in the sense that Conditions 3.18.1 are satisfied if and only if Conditions 3.18.2 are satisfied:*

(3.18.1) $\boldsymbol{\lambda} \leq_c \boldsymbol{0}_n$, $\operatorname{diag}(|\boldsymbol{\lambda}|)\dot{\boldsymbol{A}}(D)\boldsymbol{\alpha} <_c \boldsymbol{\alpha}$, *and* $\rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$.

(3.18.2) $\boldsymbol{\lambda} \leq_c \boldsymbol{0}_n$, $\operatorname{diag}(|\boldsymbol{\lambda}|)\dot{\boldsymbol{A}}(D)\boldsymbol{\alpha} <_c \boldsymbol{\alpha}$, *and* $\boldsymbol{\alpha} >_c \boldsymbol{0}_n$.

3.2.4 Comparative statics

This section is concerned with the response of the components of the GKB centrality $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ to changes in the profile of vertex idiosyncrasies $\boldsymbol{\alpha}$, the profile of localness parameters $\boldsymbol{\lambda}$, or the arc set $\mathcal{A}(D)$ of D. General statements about the signs of the responses necessitate conditions that ensure the nonnegativity or positivity of the components of $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$—without such conditions, anything goes.

Some of the comparative statics results stated below refer to Condition I-α-λ to rule out a dependence between the mappings $\boldsymbol{\alpha}(D)$ and $\boldsymbol{\lambda}(D)$ or Conditions C-α and C-λ, which introduce the notions of constant mappings α and λ.

Condition I-$\boldsymbol{\alpha}$-$\boldsymbol{\lambda}$ The mappings α and λ are independent in the following sense: for all digraphs E, the functions $\alpha(E)$ and $\lambda(E)$ are functionally independent.[24]

Katz's (1953) status index is an example of functionally dependent α and λ.

Condition C-$\boldsymbol{\alpha}$ The mapping α is constant in the following sense: for all pairs of digraphs (E, F) of the same order, $\alpha(E) = \alpha(F)$.

Condition C-$\boldsymbol{\lambda}$ The mapping λ is constant in the following sense: for all pairs of digraphs (E, F) of the same order, $\lambda(E) = \lambda(F)$.

The presentation of the results is organized as follows. The first result (Proposition 3.22) is concerned with the change in the GKB centrality in response to changes in the vertex idiosyncrasies. The second result (Proposition 3.23) deals with the effects of changes in the localness parameters. Both results apply only to cases where Condition I-α-λ is satisfied and the vertex idiosyncrasies and localness parameters vary continuously. The two results state conditions under which the components of the GKB centrality are strictly increasing in the vertex idiosyncrasies

24. Suppose the digraph E is of order n. The functions $\alpha(E)\colon \mathcal{I}(n) \to \mathbb{R}$ and $\lambda(E)\colon \mathcal{I}(n) \to \mathbb{R}$ are called *functionally dependent* if there exists a nonzero function $g\colon \mathbb{R}^2 \to \mathbb{R}$ such that $i \mapsto g(\alpha(E)(i), \lambda(E)(i))$ is identically zero on $\mathcal{I}(n)$. If no such g exists, then $\alpha(E)$ and $\lambda(E)$ are called *functionally independent*.

Figure 3.2. A digraph of order 3 and size 4 (Examples 3.20, 3.28, and 3.33)

or localness parameters. The monotonicity results follow from the respective partial derivatives and Lemma 3.19. As regards Lemma 3.19 (and also Propositions 3.22 and 3.23), note that, for all $(i,j) \in \mathcal{I}(n)^2$,

$$b_i\big(\boldsymbol{e}_j, \boldsymbol{\lambda}, D\big) = \boldsymbol{e}_i^\top \big(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)\big)^{-1} \boldsymbol{e}_j = \big[\big(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)\big)^{-1}\big]_{i,j},$$

where $\{\boldsymbol{e}_k\}_{k=1}^n$ denotes the canonical basis of $\mathbb{R}^n$ with $\boldsymbol{e}_k := (\delta_{k,1}, \ldots, \delta_{k,n})$, where $\delta_{k,l}$ is Kronecker's delta of k and l. The third result (Proposition 3.24) is about the change in the GKB centrality in response to a change in the digraph's arc set or, equivalently, in its adjacency matrix. The remainder of the section covers two special cases that involve changes in the digraph's arc set and changes in the digraph's vertex set and analyzes their effects on the GKB centrality; specifically, Section 3.2.4.1 discusses the effects resulting from isolating a vertex from all over vertices in D and Section 3.2.4.2 those resulting from completely removing a vertex from D.

Lemma 3.19 *Let $(i,j) \in \mathcal{I}(n)^2$ with $i \neq j$.*

(3.19.1) *If $\boldsymbol{\lambda} \geq_c \mathbf{0}_n$ and $\rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$, then $b_i(\boldsymbol{e}_i, \boldsymbol{\lambda}, D) \geq 1$ and $b_i(\boldsymbol{e}_j, \boldsymbol{\lambda}, D) \geq 0$.*

(3.19.2) *If $\boldsymbol{\lambda} \geq_c \mathbf{0}_n$, $\rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$, and there exists a walk $(i_0, \ldots, i_p)$ in D of length p from i to j such that $(\lambda_{i_0}, \ldots, \lambda_{i_{p-1}}) >_c \mathbf{0}_p$, then $b_i(\boldsymbol{e}_j, \boldsymbol{\lambda}, D) > 0$.*

(3.19.3) *If $\boldsymbol{\lambda} \geq_c \mathbf{0}_n$, $\rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$, and there does not exist a walk in D from i to j, then $b_i(\boldsymbol{e}_j, \boldsymbol{\lambda}, D) = 0$.*

Examples 3.20 and 3.21 demonstrate that the inequality $\boldsymbol{\lambda} \geq_c \mathbf{0}_n$ is a key condition for Results 3.19.1 and 3.19.2, respectively.

Example 3.20 Suppose $D = \{\{1,2,3\}, \{(1,2), (2,1), (2,3), (3,2)\}\}$. See Figure 3.2 for an illustration of D. Suppose $\boldsymbol{\lambda} = (-\bar{\lambda}, \bar{\lambda}, \bar{\lambda})$ for some $\bar{\lambda} \in \mathbb{R} \setminus \{0\}$. Note that $\boldsymbol{I}_3 - \operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)$ is nonsingular with inverse

$$\begin{pmatrix} 1-\bar{\lambda}^2 & -\bar{\lambda} & -\bar{\lambda}^2 \\ \bar{\lambda} & 1 & \bar{\lambda} \\ \bar{\lambda}^2 & \bar{\lambda} & 1+\bar{\lambda}^2 \end{pmatrix}$$

because $\rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) = 0 < 1$ (Lemma B.3). We find $b_1(\boldsymbol{e}_1, \boldsymbol{\lambda}, D) < 0$ if $\bar{\lambda} \in (-\infty, -1) \cup (1, +\infty)$, $b_1(\boldsymbol{e}_1, \boldsymbol{\lambda}, D) = 0$ if $\bar{\lambda} \in \{-1, 1\}$, and $0 < b_1(\boldsymbol{e}_1, \boldsymbol{\lambda}, D) < 1$ if $\bar{\lambda} \in (-1,1) \setminus \{0\}$. $\diamond$

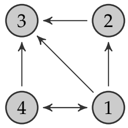


Figure 3.3. A digraph of order 4 and size 6 (Examples 3.21 and 3.53)

Example 3.21 Suppose $D = \{\{1,2,3,4\},\{(1,2),(1,3),(1,4),(2,3),(4,1),(4,3)\}\}$. See Figure 3.3 for an illustration of D. Suppose $\boldsymbol{\lambda} = (\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ is such that $\lambda_1 > 0$, $\lambda_2 < 0$, $\lambda_4 > 0$, and $\lambda_1\lambda_4 < 1$. Note that $\boldsymbol{I}_4 - \operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)$ is nonsingular with inverse

$$\frac{1}{1-\lambda_1\lambda_4}\begin{pmatrix} 1 & \lambda_1 & \lambda_1(1+\lambda_2+\lambda_4) & \lambda_1 \\ 0 & 1-\lambda_1\lambda_4 & \lambda_2(1-\lambda_1\lambda_4) & 0 \\ 0 & 0 & 1-\lambda_1\lambda_4 & 0 \\ \lambda_4 & \lambda_1\lambda_4 & \lambda_4(1+\lambda_1+\lambda_1\lambda_2) & 1 \end{pmatrix}$$

because $\rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) = \sqrt{\lambda_1\lambda_4} < 1$ (Lemma B.3). We find $b_1(\boldsymbol{e}_3,\boldsymbol{\lambda},D) = 0$ if $\lambda_2 = -(1+\lambda_4)$, $b_4(\boldsymbol{e}_3,\boldsymbol{\lambda},D) = 0$ if $\lambda_2 = -(1+\lambda_1)/\lambda_1$, and $b_1(\boldsymbol{e}_3,\boldsymbol{\lambda},D) = b_4(\boldsymbol{e}_3,\boldsymbol{\lambda},D) = 0$ if $\lambda_2 = -(1+\lambda_1)/\lambda_1$ and $\lambda_4 = 1/\lambda_1$. Note that there exist walks in D from 1 to 3 and from 4 to 3. ◇

Proposition 3.22 *Suppose Condition I-α-λ is satisfied. If* $1 \notin \sigma(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D))$, *then*

$$\forall (i,j) \in \mathcal{I}(n)^2 \quad \frac{\partial\, b_i(\boldsymbol{\alpha},\boldsymbol{\lambda},D)}{\partial \alpha_j} = b_i(\boldsymbol{e}_j,\boldsymbol{\lambda},D). \tag{3.5}$$

Let $(i,j) \in \mathcal{I}(n)^2$ *with* $i \neq j$.

(3.22.1) *If* $\boldsymbol{\lambda} \geq_c \boldsymbol{0}_n$ *and* $\rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$, *then* $b_i(\boldsymbol{\alpha},\boldsymbol{\lambda},D)$ *is strictly increasing in* α_i.

(3.22.2) *If* $\boldsymbol{\lambda} \geq_c \boldsymbol{0}_n$, $\rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$, *and there exists a walk* $(i_0,\ldots,i_p)$ *in* D *of length* p *from* i *to* j *such that* $(\lambda_{i_0},\ldots,\lambda_{i_{p-1}}) >_c \boldsymbol{0}_p$, *then* $b_i(\boldsymbol{\alpha},\boldsymbol{\lambda},D)$ *is strictly increasing in* α_j.

The statement of the second result calls for some extra notation: for any $i \in \mathcal{I}(n)$, let the function $g_i(\boldsymbol{\lambda},D)\colon \mathbb{R}_+ \to \mathbb{R}_+$ be defined by

$$g_i(\boldsymbol{\lambda},D)(t) := \rho\big(\operatorname{diag}\big((\lambda_1,\ldots,\lambda_{i-1},t,\lambda_{i+1},\ldots,\lambda_n)\big)\dot{\boldsymbol{A}}(D)\big).$$

Note that $g_i(\boldsymbol{\lambda},D)$ is continuous. If $[\boldsymbol{\lambda}]_{-i} \geq_c \boldsymbol{0}_{n-1}$, then $g_i(\boldsymbol{\lambda},D)$ is increasing and the following set is an (empty or proper) interval:[25]

$$\Gamma_i(\boldsymbol{\lambda},D) := g_i(\boldsymbol{\lambda},D)^{-1}\big([0,1)\big) = \{t \in \mathbb{R}_+ \mid 0 \leq g_i(\boldsymbol{\lambda},D)(t) < 1\}.$$

25. The spectral radius of a nonnegative matrix is increasing in any of its components (see, for example, Varga 2000, Theorem 2.20).

Proposition 3.23 *Suppose Condition I-α-λ is satisfied. If* $1 \notin \sigma(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D))$, *then*

$$\forall(i,j) \in \mathcal{I}(n)^2 \quad \frac{\partial\, b_i(\boldsymbol{\alpha},\boldsymbol{\lambda},D)}{\partial \lambda_j} = b_i(\boldsymbol{e}_j,\boldsymbol{\lambda},D) \sum_{k \in \mathcal{N}_D^+(j)} b_k(\boldsymbol{\alpha},\boldsymbol{\lambda},D). \tag{3.6}$$

Let $(i,j) \in \mathcal{I}(n)^2$ *with* $i \neq j$.

(3.23.1) *If* $[\boldsymbol{\lambda}]_{-i} \geq_c \mathbf{0}_{n-1}$, $\boldsymbol{\alpha} \geq_c \mathbf{0}_n$, *and there exists a* $k \in \mathcal{N}_D^+(i)$ *with* $\alpha_k > 0$, *then* $b_i(\boldsymbol{\alpha},\boldsymbol{\lambda},D)$ *is strictly increasing in* λ_i *on* $\Gamma_i(\boldsymbol{\lambda},D)$.

(3.23.2) *If* $[\boldsymbol{\lambda}]_{-j} \geq_c \mathbf{0}_{n-1}$, $\boldsymbol{\alpha} \geq_c \mathbf{0}_n$, *there exists a* $k \in \mathcal{N}_D^+(j)$ *with* $\alpha_k > 0$, *and there exists a walk* $(i_0,\dots,i_p)$ *in* D *of length* p *from* i *to* j *such that* $(\lambda_{i_0},\dots,\lambda_{i_{p-1}}) >_c \mathbf{0}_p$, *then* $b_i(\boldsymbol{\alpha},\boldsymbol{\lambda},D)$ *is strictly increasing in* λ_j *on* $\Gamma_j(\boldsymbol{\lambda},D)$.

Proposition 3.24 *Let* E *be any digraph of order* n *that is different from* D.[26] *If* $1 \notin \sigma(\operatorname{diag}(\boldsymbol{\lambda}(D))\dot{\boldsymbol{A}}(D)) \cup \sigma(\operatorname{diag}(\boldsymbol{\lambda}(E))\dot{\boldsymbol{A}}(E))$, *then*

$$\begin{aligned}\boldsymbol{b}\big(\boldsymbol{\alpha}(E),\boldsymbol{\lambda}(E),E\big) - \boldsymbol{b}\big(\boldsymbol{\alpha}(D),\boldsymbol{\lambda}(D),D\big) &= \boldsymbol{b}\big(\boldsymbol{\alpha}(E)-\boldsymbol{\alpha}(D),\boldsymbol{\lambda}(D),D\big)\\ &+ \boldsymbol{b}\Big(\big(\operatorname{diag}\big(\boldsymbol{\lambda}(E)\big)\dot{\boldsymbol{A}}(E) - \operatorname{diag}\big(\boldsymbol{\lambda}(D)\big)\dot{\boldsymbol{A}}(D)\big)\boldsymbol{b}\big(\boldsymbol{\alpha}(E),\boldsymbol{\lambda}(E),E\big),\boldsymbol{\lambda}(D),D\Big).\end{aligned} \tag{3.7}$$

(3.24.1) *If* E *is a superdigraph of* D, $\boldsymbol{\alpha}(E) \geq_c \boldsymbol{\alpha}(D)$, $\boldsymbol{\alpha}(E) \geq_c \mathbf{0}_n$, $\boldsymbol{\lambda}(E) \geq_c \boldsymbol{\lambda}(D) \geq_c \mathbf{0}_n$, *and* $\rho(\operatorname{diag}(\boldsymbol{\lambda}(E))\dot{\boldsymbol{A}}(E)) < 1$, *then* $\boldsymbol{b}(\boldsymbol{\alpha}(E),\boldsymbol{\lambda}(E),E) \geq_c \boldsymbol{b}(\boldsymbol{\alpha}(D),\boldsymbol{\lambda}(D),D)$.

(3.24.2) *If* E *is a superdigraph of* D *with, for all* $i \in \mathcal{I}(n)$, $\deg_E^+(i) > \deg_D^+(i)$, $\boldsymbol{\alpha}(E) \geq_c \boldsymbol{\alpha}(D)$, $\boldsymbol{\alpha}(E) >_c \mathbf{0}_n$, $\boldsymbol{\lambda}(E) \geq_c \boldsymbol{\lambda}(D) \geq_c \mathbf{0}_n$, *and* $\rho(\operatorname{diag}(\boldsymbol{\lambda}(E))\dot{\boldsymbol{A}}(E)) < 1$, *then* $\boldsymbol{b}(\boldsymbol{\alpha}(E),\boldsymbol{\lambda}(E),E) >_c \boldsymbol{b}(\boldsymbol{\alpha}(D),\boldsymbol{\lambda}(D),D)$.

The key insight from Proposition 3.24 is that the change in the GKB centrality in response to a change in the digraph's arc set or, equivalently, adjacency matrix, can be decomposed into two summands (see the right-hand side of (3.7)). The first summand represents the change in response to the change in idiosyncracies that is induced by the change in the arc set from $\mathcal{A}(D)$ to $\mathcal{A}(E)$. This summand is zero if $\boldsymbol{\alpha}(E) = \boldsymbol{\alpha}(D)$, which is true if Condition C-α is satisfied. Without specific assumptions about the mapping α, the components of $\boldsymbol{\alpha}(E) - \boldsymbol{\alpha}(D)$ can be of any sign, in which case no general statement about the sign of any of the components of $\boldsymbol{b}(\boldsymbol{\alpha}(E)-\boldsymbol{\alpha}(D),\boldsymbol{\lambda}(D),D)$ is possible. The second summand represents the change in response to a ceteris paribus change (that is, the idiosyncracies are fixed at $\boldsymbol{\alpha}(E) = \boldsymbol{\alpha}(D)$) in the localness parameters and the adjacency matrix. Predictions about the signs of its components are possible if, inter alia, E is a superdigraph of D; specifically, Result 3.24.1 (respectively, Result 3.24.2) states conditions under which the components of the GKB centrality are increasing (respectively, strictly increasing) in the digraph's arc set (in terms of set inclusion).

26. It follows that the digraphs D and E have the same vertex set, namely, $\mathcal{I}(n)$, but different arc sets, that is, $\mathcal{A}(D) \neq \mathcal{A}(E)$, and therefore different adjacency matrices, that is, $\dot{A}(D) \neq \dot{A}(E)$.

3.2.4.1 Effects from isolating a vertex

This section analyzes the effects of a particular change in the digraph's arc set on the GKB centrality: the isolation of a vertex. To this end, let x be an arbitrary vertex in D, and let $D \boxminus x$ denote the digraph of order n that emerges from D by removing all arcs incident to x, that is, $D \boxminus x$ is the subdigraph of D with vertex set $\mathcal{I}(n)$ and arc set $\{(u,v) \in \mathcal{A}(D) \mid \mathbb{1}_{\{u,v\}}(x) = 0\}$. The effects of a change in the digraph's arc set from $\mathcal{A}(D)$ to $\mathcal{A}(D \boxminus x)$ on the components of the GKB centrality are given by (3.7). A discussion of the directions and magnitudes of the effects, that is, the signs and magnitudes of the components of $\boldsymbol{b}(\boldsymbol{\alpha}(D \boxminus x), \boldsymbol{\lambda}(D \boxminus x), D \boxminus x) - \boldsymbol{b}(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D)$, calls for additional assumptions. The focus is on two cases. The first case (Proposition 3.25) assumes that α and λ are constant (Conditions C-α and C-λ). The second case (Proposition 3.31) assumes that α is as in Example 3.2 and λ as in Example 3.5 and, therefore, constant.

I introduce the following notation to facilitate a compact representation of the results: $\boldsymbol{\Delta}(D \boxminus x, D) := \boldsymbol{b}(\boldsymbol{\alpha}(D \boxminus x), \boldsymbol{\lambda}(D \boxminus x), D \boxminus x) - \boldsymbol{b}(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D)$, and, for all $i \in \mathcal{I}(n)$, $\Delta_i(D \boxminus x, D)$ denotes the ith component of $\boldsymbol{\Delta}(D \boxminus x, D)$.

Proposition 3.25 *Suppose Conditions C-α and C-λ are satisfied. If $1 \notin \sigma(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) \cup \sigma(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D \boxminus x))$, then $b_x(\boldsymbol{e}_x, \boldsymbol{\lambda}, D) \neq 0$ and*

$$\boldsymbol{\Delta}(D \boxminus x, D) = \alpha_x \boldsymbol{e}_x - \frac{b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)}{b_x(\boldsymbol{e}_x, \boldsymbol{\lambda}, D)} \boldsymbol{b}(\boldsymbol{e}_x, \boldsymbol{\lambda}, D), \tag{3.8}$$

specifically, for all $i \in \mathcal{I}(n)$,

$$\Delta_i(D \boxminus x, D) = \delta_{i,x}\big(\alpha_x - b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)\big) - (1 - \delta_{i,x}) \frac{b_i(\boldsymbol{e}_x, \boldsymbol{\lambda}, D)}{b_x(\boldsymbol{e}_x, \boldsymbol{\lambda}, D)} b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D). \tag{3.9}$$

Let $i \in \mathcal{I}(n) \setminus \{x\}$.

(3.25.1) If $\boldsymbol{\lambda} \geq_c \boldsymbol{0}_n$, $\rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$, and $\boldsymbol{\alpha} \geq_c \boldsymbol{0}_n$, then $\Delta_x(D \boxminus x, D) \leq 0$ and $\Delta_i(D \boxminus x, D) \leq 0$.

(3.25.2) If $\lambda_x > 0$, $[\boldsymbol{\lambda}]_{-x} \geq_c \boldsymbol{0}_{n-1}$, $\rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$, there exists a $k \in \mathcal{N}_D^+(x)$ with $\alpha_k > 0$, and $[\boldsymbol{\alpha}]_{-k} \geq_c \boldsymbol{0}_{n-1}$, then $\Delta_x(D \boxminus x, D) < 0$.

(3.25.3) If $\boldsymbol{\lambda} \geq_c \boldsymbol{0}_n$, there exists a walk $(i_0, \ldots, i_p)$ in D of length p from i to x such that $(\lambda_{i_0}, \ldots, \lambda_{i_{p-1}}) >_c \boldsymbol{0}_p$, $\rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$, $\alpha_x > 0$, and $[\boldsymbol{\alpha}]_{-x} \geq_c \boldsymbol{0}_{n-1}$, then $\Delta_i(D \boxminus x, D) < 0$.

A sufficient condition for $1 \notin \sigma(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) \cup \sigma(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D \boxminus x))$ is given in Lemma 3.26.

Lemma 3.26 *Suppose Condition C-λ is satisfied. If $\rho(\operatorname{diag}(|\boldsymbol{\lambda}|)\dot{\boldsymbol{A}}(D)) < 1$, then $1 \notin \sigma(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) \cup \sigma(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D \boxminus x))$.*

In what follows, I discuss Proposition 3.25. Certain changes in the GKB centrality are obvious, without referring to (3.8) or (3.9), and are related to three simple cases.

First, if $\boldsymbol{\lambda} = \mathbf{0}_n$, then $\boldsymbol{\Delta}(D \boxminus x, D) = \mathbf{0}_n$ because $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) = \boldsymbol{\alpha} = \boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D \boxminus x)$. Second, if x is isolated in D, then $\boldsymbol{\Delta}(D \boxminus x, D) = \mathbf{0}_n$ because $\deg_D^-(x) = \deg_D^+(x) = 0$ implies that $D = D \boxminus x$. Third, $\Delta_x(D \boxminus x, D) = \alpha_x - b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$.[27] This difference is nonpositive if $\boldsymbol{\lambda} \geq_c \mathbf{0}_n$, $\rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$, and $\boldsymbol{\alpha} \geq_c \mathbf{0}_n$ (Result 3.25.1); it is negative if $\lambda_x > 0$, $[\boldsymbol{\lambda}]_{-x} \geq_c \mathbf{0}_{n-1}$, $\rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$, there exists a $k \in \mathcal{N}_D^+(x)$ with $\alpha_k > 0$, and $[\boldsymbol{\alpha}]_{-k} \geq_c \mathbf{0}_{n-1}$ (Result 3.25.2). As regards the general case, according to (3.9),

$$\forall i \in \mathcal{I}(n) \setminus \{x\} \quad \Delta_i(D \boxminus x, D) = -\frac{b_i(\boldsymbol{e}_x, \boldsymbol{\lambda}, D)}{b_x(\boldsymbol{e}_x, \boldsymbol{\lambda}, D)}\, b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D). \tag{3.10}$$

The change in the GKB centrality of vertex $i \neq x$ is proportional to the centrality of vertex x, $b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$, where the factor of proportionality is given by the additive inverse of the quotient with dividend $b_i(\boldsymbol{e}_x, \boldsymbol{\lambda}, D)$ and divisor $b_x(\boldsymbol{e}_x, \boldsymbol{\lambda}, D)$. In light of Lemma 3.19 and Examples 3.20 and 3.21, this quotient can be of any sign without assuming $\boldsymbol{\lambda} \geq_c \mathbf{0}_n$ and $\rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$. The subsequent discussion is for this reason confined to the case where $\boldsymbol{\lambda} \geq_c \mathbf{0}_n$ is such that $\rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$. In order to interpret the aforementioned quotient, note that if $\boldsymbol{\lambda} = \bar{\lambda}\mathbf{1}_n$ for some $\bar{\lambda} \geq 0$ with $\bar{\lambda}\rho(\dot{\boldsymbol{A}}(D)) < 1$, then

$$\forall i \in \mathcal{I}(n) \setminus \{x\} \quad \frac{b_i(\boldsymbol{e}_x, \boldsymbol{\lambda}, D)}{b_x(\boldsymbol{e}_x, \boldsymbol{\lambda}, D)} = \begin{cases} 0 & \text{if } \bar{\lambda} = 0, \\ \dfrac{\sum_{k=0}^{\infty} \bar{\lambda}^k d_{D,i\to x}^k}{\sum_{k=0}^{\infty} \bar{\lambda}^k d_{D,x\to x}^k} & \text{if } \bar{\lambda} > 0, \end{cases} \tag{3.11}$$

where $d_{D,i\to j}^0 := \delta_{i,j}$ and, for all $k \in \mathbb{Z}_{++}$, $d_{D,i\to j}^k$ denotes the number of walks in D of length k from i to j.[28] Note also that the more walks (in D) of any length from i to x exist, the larger is the limit of the series of weighted walks, $\sum_{k=0}^{\infty} \bar{\lambda}^k d_{D,i\to x}^k$. The larger the limit $\sum_{k=0}^{\infty} \bar{\lambda}^k d_{D,i\to x}^k$, the stronger is vertex i said to be connected (in D) to vertex x. The quotient with dividend $\sum_{k=0}^{\infty} \bar{\lambda}^k d_{D,i\to x}^k$ and divisor $\sum_{k=0}^{\infty} \bar{\lambda}^k d_{D,x\to x}^k$ may therefore be interpreted as a measure of how strongly vertex i is connected (in D) to vertex x relative to how strongly vertex x is connected (in D) to itself. The foregoing considerations motivate the following definition.

Definition CS Let $i \in \mathcal{I}(n) \setminus \{x\}$. The *connection strength* of (i, x) in D is the function $S_{D,i,x}\colon \{\tilde{\boldsymbol{\lambda}} \in \mathbb{R}_+^n \mid \rho(\operatorname{diag}(\tilde{\boldsymbol{\lambda}})\dot{\boldsymbol{A}}(D)) < 1\} \to \mathbb{R}$ defined by[29]

$$S_{D,i,x}(\tilde{\boldsymbol{\lambda}}) := \frac{b_i\big(\boldsymbol{e}_x, \tilde{\boldsymbol{\lambda}}, D\big)}{b_x\big(\boldsymbol{e}_x, \tilde{\boldsymbol{\lambda}}, D\big)}.$$

27. Indeed, $b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D \boxminus x) = \alpha_x + \lambda_x \sum_{j\in\mathcal{I}(n)} \dot{a}_{x,j}(D \boxminus x)\, b_j(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D \boxminus x) = \alpha_x$ because $\deg_{D\boxminus x}^+(x) = 0$ implies that, for all $j \in \mathcal{I}(n)$, $\dot{a}_{x,j}(D \boxminus x) = 0$.

28. Indeed, if $\boldsymbol{\lambda} = \bar{\lambda}\mathbf{1}_n$ for some $\bar{\lambda} \geq 0$ with $\bar{\lambda}\rho(\dot{\boldsymbol{A}}(D)) < 1$, then

$$\frac{b_i(\boldsymbol{e}_x, \boldsymbol{\lambda}, D)}{b_x(\boldsymbol{e}_x, \boldsymbol{\lambda}, D)} = \frac{\left[\left(\boldsymbol{I}_n - \bar{\lambda}\dot{\boldsymbol{A}}(D)\right)^{-1}\right]_{i,x}}{\left[\left(\boldsymbol{I}_n - \bar{\lambda}\dot{\boldsymbol{A}}(D)\right)^{-1}\right]_{x,x}} = \frac{\sum_{k=0}^{\infty} \bar{\lambda}^k \left[\dot{\boldsymbol{A}}(D)^k\right]_{i,x}}{\sum_{k=0}^{\infty} \bar{\lambda}^k \left[\dot{\boldsymbol{A}}(D)^k\right]_{x,x}} = \frac{\sum_{k=0}^{\infty} \bar{\lambda}^k d_{D,i\to x}^k}{\sum_{k=0}^{\infty} \bar{\lambda}^k d_{D,x\to x}^k},$$

where $b_x(\boldsymbol{e}_x, \boldsymbol{\lambda}, D) \geq 1$ (Result 3.19.1).

29. If $\tilde{\boldsymbol{\lambda}} \in \mathbb{R}_+^n$ is such that $\rho(\operatorname{diag}(\tilde{\boldsymbol{\lambda}})\dot{\boldsymbol{A}}(D)) < 1$, then $b_x(\boldsymbol{e}_x, \tilde{\boldsymbol{\lambda}}, D) \geq 1$ (Result 3.19.1).

The range of the connection strength of (i,x) in D is a subset of $\mathbb{R}_+$ (Result 3.19.1) containing zero ($b_i(\boldsymbol{e}_x, \mathbf{0}_n, D) = \delta_{i,x} = 0$ because $i \neq x$). The range of its restriction to $\{\tilde{\boldsymbol{\lambda}} \in \mathbb{R}^n_+ \mid \operatorname{diag}(\tilde{\boldsymbol{\lambda}})\dot{\boldsymbol{A}}(D)\mathbf{1}_n \leq_c \mathbf{1}_n\}$ is a subset of $[0,1]$ (Proposition 3.27), which renders connection strength easy to interpret. The connection strength of (i,x) in D at the point $(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n)$ is positive if there exists a walk $(i_0, \ldots, i_p)$ in D of length p from i to x such that $(\tilde{\lambda}_{i_0}, \ldots, \tilde{\lambda}_{i_{p-1}}) >_c \mathbf{0}_p$ (Results 3.19.1 and 3.19.2), and it is zero if there does not exist a walk in D from i to x (Result 3.19.3), which is, for example, the case when i and x lie in two different weakly connected components of D.

Proposition 3.27 *For all $i \in \mathcal{I}(n) \setminus \{x\}$ and for all $\tilde{\boldsymbol{\lambda}} \in \mathbb{R}^n_+$ with $\rho(\operatorname{diag}(\tilde{\boldsymbol{\lambda}})\dot{\boldsymbol{A}}(D)) < 1$ and $\operatorname{diag}(\tilde{\boldsymbol{\lambda}})\dot{\boldsymbol{A}}(D)\mathbf{1}_n \leq_c \mathbf{1}_n$, $S_{D,i,x}(\tilde{\boldsymbol{\lambda}}) \in [0,1]$.*

I make three comments on the inequality $\operatorname{diag}(\tilde{\boldsymbol{\lambda}})\dot{\boldsymbol{A}}(D)\mathbf{1}_n \leq_c \mathbf{1}_n$ of Proposition 3.27, where $\tilde{\boldsymbol{\lambda}} := (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n) \in \mathbb{R}^n_+$. First, note that $\operatorname{diag}(\tilde{\boldsymbol{\lambda}})\dot{\boldsymbol{A}}(D)\mathbf{1}_n \leq_c \mathbf{1}_n$ is equivalent to $(\boldsymbol{I}_n - \operatorname{diag}(\tilde{\boldsymbol{\lambda}})\dot{\boldsymbol{A}}(D))\mathbf{1}_n \geq_c \mathbf{0}_n$, which is a nonnegativity condition for the row sums of $\boldsymbol{I}_n - \operatorname{diag}(\tilde{\boldsymbol{\lambda}})\dot{\boldsymbol{A}}(D)$. Second, note that the strict inequality $\operatorname{diag}(\tilde{\boldsymbol{\lambda}})\dot{\boldsymbol{A}}(D)\mathbf{1}_n <_c \mathbf{1}_n$ implies that $\rho(\operatorname{diag}(\tilde{\boldsymbol{\lambda}})\dot{\boldsymbol{A}}(D)) < 1$ (Lemma 3.16). Third, note that $\operatorname{diag}(\tilde{\boldsymbol{\lambda}})\dot{\boldsymbol{A}}(D)\mathbf{1}_n \leq_c \mathbf{1}_n$ imposes an upper bound on a vertex's out-degree in terms of the corresponding localness parameter, presuming it is positive; specifically, for all $i \in \mathcal{I}(n)$ with $\tilde{\lambda}_i > 0$, $\operatorname{diag}(\tilde{\boldsymbol{\lambda}})\dot{\boldsymbol{A}}(D)\mathbf{1}_n \leq_c \mathbf{1}_n$ implies that $\deg^+_D(i) \leq 1/\tilde{\lambda}_i$.

Example 3.28 illustrates Definition CS and Proposition 3.27.

Example 3.28 Consider the digraph D of Example 3.20. Suppose $\boldsymbol{\lambda}(D) = \bar{\lambda}\mathbf{1}_3$ for some $\bar{\lambda} > 0$. Note that $\{\tilde{\lambda} \in \mathbb{R}_+ \mid \rho(\tilde{\lambda}\dot{\boldsymbol{A}}(D)) < 1\} = [0, 1/\sqrt{2})$ because $\rho(\dot{\boldsymbol{A}}(D)) = \sqrt{2}$, and $\{\tilde{\lambda} \in \mathbb{R}_+ \mid \tilde{\lambda}\dot{\boldsymbol{A}}(D)\mathbf{1}_3 \leq_c \mathbf{1}_3\} = [0, 1/2]$ (cf. Proposition 3.27). If $\bar{\lambda} = 1/2$, then

$$\left(\boldsymbol{I}_3 - \operatorname{diag}\left(\boldsymbol{\lambda}(D)\right)\dot{\boldsymbol{A}}(D)\right)^{-1} = \frac{1}{2}\begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix},$$

from which

$$\begin{aligned}
&S_{D,1,1}(\boldsymbol{\lambda}(D)) = 1, && S_{D,1,2}(\boldsymbol{\lambda}(D)) = 1/2, && S_{D,1,3}(\boldsymbol{\lambda}(D)) = 1/3, \\
&S_{D,2,1}(\boldsymbol{\lambda}(D)) = 2/3, && S_{D,2,2}(\boldsymbol{\lambda}(D)) = 1, && S_{D,2,3}(\boldsymbol{\lambda}(D)) = 2/3, \\
&S_{D,3,1}(\boldsymbol{\lambda}(D)) = 1/3, && S_{D,3,2}(\boldsymbol{\lambda}(D)) = 1/2, && S_{D,3,3}(\boldsymbol{\lambda}(D)) = 1
\end{aligned}$$

follow. If $\bar{\lambda} = 2/3$, then

$$\left(\boldsymbol{I}_3 - \operatorname{diag}\left(\boldsymbol{\lambda}(D)\right)\dot{\boldsymbol{A}}(D)\right)^{-1} = \begin{pmatrix} 5 & 6 & 4 \\ 6 & 9 & 6 \\ 4 & 6 & 5 \end{pmatrix},$$

from which

$$\begin{aligned}
&S_{D,1,1}(\boldsymbol{\lambda}(D)) = 1, && S_{D,1,2}(\boldsymbol{\lambda}(D)) = 2/3, && S_{D,1,3}(\boldsymbol{\lambda}(D)) = 4/5, \\
&S_{D,2,1}(\boldsymbol{\lambda}(D)) = 6/5, && S_{D,2,2}(\boldsymbol{\lambda}(D)) = 1, && S_{D,2,3}(\boldsymbol{\lambda}(D)) = 6/5, \\
&S_{D,3,1}(\boldsymbol{\lambda}(D)) = 4/5, && S_{D,3,2}(\boldsymbol{\lambda}(D)) = 2/3, && S_{D,3,3}(\boldsymbol{\lambda}(D)) = 1
\end{aligned}$$

follow (note that $S_{D,2,1}(\boldsymbol{\lambda}(D)) = S_{D,2,3}(\boldsymbol{\lambda}(D)) > 1$). ◇

Propositions 3.29 and 3.30 state results about the monotonicity of the connection strength of (i, x) in D.

Proposition 3.29 *For all $i \in \mathcal{I}(n) \setminus \{x\}$, if there exists a walk in D from i to x, then the restriction of the connection strength of (i, x) in D to $\{\tilde{\lambda}\mathbf{1}_n \mid \tilde{\lambda} \in \mathbb{R}_{++}\}$ is strictly increasing.*

Proposition 3.30 *Let $i \in \mathcal{I}(n) \setminus \{x\}$.*

(3.30.1) *The connection strength of (i, x) in D is constant in its xth argument; specifically, for all $\tilde{\boldsymbol{\lambda}} := (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n)$ in the domain of $S_{D,i,x}$, the function $t \mapsto S_{D,i,x}(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{x-1}, t, \tilde{\lambda}_{x+1}, \ldots, \tilde{\lambda}_n)$ is constant on $\Gamma_x(\tilde{\boldsymbol{\lambda}}, D)$.*

(3.30.2) *For all $j \in \mathcal{I}(n) \setminus \{x\}$, the connection strength of (i, x) in D is increasing in its jth argument; specifically, for all $\tilde{\boldsymbol{\lambda}} := (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n)$ in the domain of $S_{D,i,x}$, the function $t \mapsto S_{D,i,x}(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{j-1}, t, \tilde{\lambda}_{j+1}, \ldots, \tilde{\lambda}_n)$ is increasing on $\Gamma_j(\tilde{\boldsymbol{\lambda}}, D)$.*

(3.30.3) *If there exists a walk $(i_0, \ldots, i_p)$ in D of length p from i to x, then, for all $\tilde{\boldsymbol{\lambda}} := (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n)$ in the domain of $S_{D,i,x}$ with $(\tilde{\lambda}_{i_1}, \ldots, \tilde{\lambda}_{i_{p-1}}) >_c \mathbf{0}_{p-1}$, the function $t \mapsto S_{D,i,x}(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{i-1}, t, \tilde{\lambda}_{i+1}, \ldots, \tilde{\lambda}_n)$ is strictly increasing on $\Gamma_i(\tilde{\boldsymbol{\lambda}}, D)$.*

I conclude the discussion of Proposition 3.25 by interpreting (3.10) in light of Definition CS. Suppose $\boldsymbol{\alpha} \geq_c \mathbf{0}_n$ (in addition to $\boldsymbol{\lambda} \geq_c \mathbf{0}_n$ and $\rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$). The GKB centrality of vertex $i \neq x$ does not increase in response to isolating vertex x from all other vertices in D (Result 3.25.1): $\Delta_i(D \boxminus x, D) = -S_{D,i,x}(\boldsymbol{\lambda})\, b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) \leq 0$. This is also true for any other vertex $j \neq x$, from which it follows that the effect from isolating vertex x cannot be positive for one vertex and negative for another. The GKB centrality of vertex i decreases if, in addition to the foregoing assumptions, there exists a walk $(i_0, \ldots, i_p)$ in D of length p from i to x such that $(\lambda_{i_0}, \ldots, \lambda_{i_{p-1}}) >_c \mathbf{0}_p$ and $\alpha_x > 0$ (Result 3.25.3). In this case, the decrease is proportional to the centrality of vertex x, $b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$, which is positive (Result 3.15.1 and $\alpha_x > 0$). The magnitude of the change depends on how strongly vertex i is connected (in D) to vertex x, as measured by the connection strength of (i, x) in D, $S_{D,i,x}(\boldsymbol{\lambda})$, which is positive (Results 3.19.1 and 3.19.2). Specifically, the stronger vertex i is connected (in D) to vertex x, the larger the magnitude of the change. Under certain (additional) assumptions (see Results 3.23.2 and 3.30.3), $S_{D,i,x}(\boldsymbol{\lambda})$ and $b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ are strictly increasing in λ_i on $\Gamma_i(\boldsymbol{\lambda}, D)$, in which case the magnitude of the decrease in the GKB centrality of vertex i is increasing in λ_i on $\Gamma_i(\boldsymbol{\lambda}, D)$.

Next, I discuss the second case where α is as in Example 3.2 and λ as in Example 3.5. The result is stated as Proposition 3.31.

Proposition 3.31 *Suppose α is as in Example 3.2, that is, $\boldsymbol{\alpha}(D) = \dot{\boldsymbol{A}}(D)\mathbf{1}_n$ and $\boldsymbol{\alpha}(D \boxminus x) = \dot{\boldsymbol{A}}(D \boxminus x)\mathbf{1}_n$, λ is as in Example 3.5, that is, $\boldsymbol{\lambda}(D) = \boldsymbol{\lambda}(D \boxminus x) = \bar{\lambda}\mathbf{1}_n$ for some $\bar{\lambda} \in \mathbb{R}$, and $1 \notin \sigma(\bar{\lambda}\dot{\boldsymbol{A}}(D)) \cup \sigma(\bar{\lambda}\dot{\boldsymbol{A}}(D \boxminus x))$. If $\bar{\lambda} = 0$, then*

$$\boldsymbol{\Delta}(D \boxminus x, D) = \big(\dot{\boldsymbol{A}}(D \boxminus x) - \dot{\boldsymbol{A}}(D)\big)\mathbf{1}_n, \tag{3.12}$$

and if $\bar{\lambda} \neq 0$, then

$$\boldsymbol{\Delta}(D \boxminus x, D) = \frac{1}{\bar{\lambda}}\left(\boldsymbol{e}_x - \frac{1}{b_x(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)}\boldsymbol{b}(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)\right) - \frac{b_x\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D\big)}{b_x(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)}\boldsymbol{b}(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D). \tag{3.13}$$

Specifically, if $\bar{\lambda} = 0$, then, for all $i \in \mathcal{I}(n)$,

$$\Delta_i(D \boxminus x, D) = \begin{cases} -\deg_D^+(x) & \text{if } i = x, \\ -\mathbb{1}_{\mathcal{N}_D^+(i)}(x) & \text{if } i \neq x, \end{cases} \tag{3.14}$$

and if $\bar{\lambda} \neq 0$, then, for all $i \in \mathcal{I}(n)$,

$$\Delta_i(D \boxminus x, D) = -\delta_{i,x}\, b_x\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D\big) - (1 - \delta_{i,x})\frac{b_i(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)}{b_x(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)}\left(\frac{1}{\bar{\lambda}} + b_x\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D\big)\right). \tag{3.15}$$

Let $i \in \mathcal{I}(n) \setminus \{x\}$.

(3.31.1) If $\bar{\lambda} \geq 0$ and $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$, then $\Delta_x(D \boxminus x, D) \leq 0$ and $\Delta_i(D \boxminus x, D) \leq 0$.

(3.31.2) If $\bar{\lambda} \geq 0$, $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$, and $\deg_D^+(x) > 0$, then $\Delta_x(D \boxminus x, D) < 0$.

(3.31.3) If $\bar{\lambda} > 0$, $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$, and there exists a walk in D from i to x, then $\Delta_i(D \boxminus x, D) < 0$.

I discuss the effects of isolating vertex x in the context of Proposition 3.31. First, consider the case $\bar{\lambda} = 0$. The centrality of vertex x falls to zero if $\deg_D^+(x) > 0$. The centrality of vertex $i \neq x$ decreases by 1 if vertex x is an out-neighbor of i in D, otherwise it remains unchanged. The decrease by 1 corresponds to the loss in vertex i's out-degree by isolating vertex x. Second, consider the case $\bar{\lambda} \neq 0$. The centrality of vertex x changes by $-\,b_x(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D)$. This change is negative, that is, the centrality decreases, if $\bar{\lambda} > 0$, $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$, and $\deg_D^+(x) > 0$ (Result 3.15.1). The corresponding change in the context of Proposition 3.25 is given by $\alpha_x - b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$. The difference arises from the fact that vertex x's idiosyncrasy is constant in the context of Proposition 3.25, whereas it drops to zero in the context of Proposition 3.31. The change in vertex i's centrality is given by

$$-\frac{1}{\bar{\lambda}}\frac{b_i(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)}{b_x(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)} - \frac{b_i(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)}{b_x(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)}\, b_x\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D\big). \tag{3.16}$$

The first summand of (3.16) represents the change in response to a ceteris paribus change (that is, the adjacency matrix is fixed at $\dot{\boldsymbol{A}}(D \boxminus x)$) in the profile of idiosyncracies from $\dot{\boldsymbol{A}}(D)\mathbf{1}_n$ to $\dot{\boldsymbol{A}}(D \boxminus x)\mathbf{1}_n$ (see Proposition 3.24). This summand is nonpositive if $\bar{\lambda} > 0$ and $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$ (Result 3.19.1), it is negative if $\bar{\lambda} > 0$, $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$, and there exists a walk in D from i to x (Result 3.19.2), and it is zero if $\bar{\lambda} > 0$, $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$, and there does not exist a walk in D from i to x (Result 3.19.3). The second summand of (3.16) represents the change in response to a ceteris paribus change (that is, the idiosyncracies are fixed at $\dot{\boldsymbol{A}}(D)\mathbf{1}_n$) in the adjacency matrix from $\dot{\boldsymbol{A}}(D)$ to $\dot{\boldsymbol{A}}(D \boxminus x)$. This summand is nonpositive if $\bar{\lambda} > 0$ and $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$ (Result 3.19.1), it is negative if $\bar{\lambda} > 0$, $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$, there exists a walk in D from i to x, and $\deg^+_D(x) > 0$ (Results 3.15.1 and 3.19.2), and it is zero if $\bar{\lambda} > 0$, $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$, and there does not exist a walk in D from i to x (Results 3.15.1 and 3.19.3). Note that the second summand is similar to the expression for $\Delta_i(D \boxminus x, D)$ in (3.10). Thus, if the centrality of vertex i decreases in the context of Proposition 3.31, then the magnitude of the decrease is larger than that in the context of Proposition 3.25, provided that the values of $\boldsymbol{\alpha}$ and $\boldsymbol{\lambda}$ match those of $\dot{\boldsymbol{A}}(D)\mathbf{1}_n$ and $\bar{\lambda}\mathbf{1}_n$, respectively, and $\bar{\lambda} > 0$ and $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$. The reason for this is clear: the components of the GKB centrality are increasing in the digraph's arc set (Result 3.24.1) and they are increasing in the profile of vertex idiosyncrasies. The change in vertex i's centrality is proportional to the connection strength of (i, x) in D:

$$\Delta_i(D \boxminus x, D) = -S_{D,i,x}(\bar{\lambda}\mathbf{1}_n)\left(\frac{1}{\bar{\lambda}} + b_x\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D\big)\right).$$

Vertex i's centrality does not increase if $\bar{\lambda} \geq 0$ and $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$ (Result 3.31.1), and it decreases if $\bar{\lambda} > 0$, $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$, and there exists a walk in D from i to x (Result 3.31.3). If there exists a walk in D from i to x and $\rho(\dot{\boldsymbol{A}}(D))^2 < \sum_{j \in \mathcal{N}^+_D(x)} \deg^+_D(j)$, then there exists a positive constant $c(D, x)$ such that the magnitude of the decrease is increasing in $\bar{\lambda}$ on the nonempty interval $\{\tilde{\lambda} \in \mathbb{R}_{++} \mid \rho(\tilde{\lambda}\dot{\boldsymbol{A}}(D)) < 1\} \cap (c(D, x), +\infty)$ (Proposition 3.29 and Lemma 3.32).

Lemma 3.32 *If $\rho(\dot{\boldsymbol{A}}(D))^2 < \sum_{j \in \mathcal{N}^+_D(x)} \deg^+_D(j)$, then there exists a constant $c(D, x) > 0$ such that the function $t \mapsto (1/t) + b_x(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, t\mathbf{1}_n, D)$ is strictly increasing on the nonempty interval $\{\tilde{\lambda} \in \mathbb{R}_{++} \mid \rho(\tilde{\lambda}\dot{\boldsymbol{A}}(D)) < 1\} \cap (c(D, x), +\infty)$.*[30]

Sufficient conditions for $\Delta_i(D \boxminus x, D)$ to be of a particular sign are difficult to obtain if $\bar{\lambda} < 0$. If $\bar{\lambda} < 0$, $|\bar{\lambda}|\dot{\boldsymbol{A}}(D)^2\mathbf{1}_n <_c \dot{\boldsymbol{A}}(D)\mathbf{1}_n$, and $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$, then $b_x(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D) > 0$ (Result 3.15.4). The sum $(1/\bar{\lambda}) + b_x\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D\big)$ or the connection strength of (i, x) in D may, however, be negative under the foregoing conditions. This is illustrated in the following example.

Example 3.33 Consider the digraph D of Example 3.20. Note that

$$\dot{\boldsymbol{A}}(D) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

with $\rho(\dot{\boldsymbol{A}}(D)) = \sqrt{2}$. Suppose α is as in Example 3.2, that is, $\boldsymbol{\alpha}(D) = \dot{\boldsymbol{A}}(D)\mathbf{1}_3$ and, for all $i \in \mathcal{I}(3)$, $\boldsymbol{\alpha}(D \boxminus i) = \dot{\boldsymbol{A}}(D \boxminus i)\mathbf{1}_3$, and λ is as in Example 3.5, that

30. If $\rho(\dot{\boldsymbol{A}}(D)) = 0$, then $\{\tilde{\lambda} \in \mathbb{R}_{++} \mid \rho(\tilde{\lambda}\dot{\boldsymbol{A}}(D)) < 1\} \cap (c(D,x), +\infty) = (c(D,x), +\infty)$, and if $\rho(\dot{\boldsymbol{A}}(D)) > 0$, then $\{\tilde{\lambda} \in \mathbb{R}_{++} \mid \rho(\tilde{\lambda}\dot{\boldsymbol{A}}(D)) < 1\} \cap (c(D,x), +\infty) = (c(D,x), 1/\rho(\dot{\boldsymbol{A}}(D)))$.

is, $\boldsymbol{\lambda}(D) = \boldsymbol{\lambda}(D \boxminus 1) = \boldsymbol{\lambda}(D \boxminus 2) = \boldsymbol{\lambda}(D \boxminus 3) = \bar{\lambda}\mathbf{1}_n$ for some $\bar{\lambda} \in \mathbb{R}$. Note that $\{\tilde{\lambda} \in \mathbb{R} \mid \rho(\tilde{\lambda}\dot{\boldsymbol{A}}(D)) < 1\} = (-1/\sqrt{2}, 1/\sqrt{2})$ and $\{\tilde{\lambda} \in \mathbb{R} \mid |\tilde{\lambda}|\dot{\boldsymbol{A}}(D)^2\mathbf{1}_n <_c \dot{\boldsymbol{A}}(D)\mathbf{1}_n\}$ $= (-1/2, +\infty)$. Suppose $\bar{\lambda} = -1/3$, which implies that $\boldsymbol{b}(\dot{\boldsymbol{A}}(D)\mathbf{1}_3, \bar{\lambda}\mathbf{1}_3, D) >_c \mathbf{0}_3$ (Result 3.15.4). We find

$$\left(\boldsymbol{I}_3 - \bar{\lambda}\dot{\boldsymbol{A}}(D)\right)^{-1} = \frac{1}{7}\begin{pmatrix} 8 & -3 & 1 \\ -3 & 9 & -3 \\ 1 & -3 & 8 \end{pmatrix}$$

and

$$\boldsymbol{b}\left(\dot{\boldsymbol{A}}(D)\mathbf{1}_3, \bar{\lambda}\mathbf{1}_3, D\right) = \left(\boldsymbol{I}_3 - \bar{\lambda}\dot{\boldsymbol{A}}(D)\right)^{-1}\dot{\boldsymbol{A}}(D)\mathbf{1}_3 = \frac{1}{7}\begin{pmatrix} 3 \\ 12 \\ 3 \end{pmatrix}.$$

Consequently, according to (3.13), if $x = 1$, then

$$\begin{aligned}
&\boldsymbol{\Delta}(D \boxminus 1, D) \\
&\quad = \frac{1}{\bar{\lambda}}\left(\boldsymbol{e}_1 - \frac{1}{b_1(\boldsymbol{e}_1, \bar{\lambda}\mathbf{1}_3, D)}\boldsymbol{b}(\boldsymbol{e}_1, \bar{\lambda}\mathbf{1}_3, D)\right) - \frac{b_1\left(\dot{\boldsymbol{A}}(D)\mathbf{1}_3, \bar{\lambda}\mathbf{1}_3, D\right)}{b_1(\boldsymbol{e}_1, \bar{\lambda}\mathbf{1}_3, D)}\boldsymbol{b}(\boldsymbol{e}_1, \bar{\lambda}\mathbf{1}_3, D) \\
&\quad = -3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{7}{8}\cdot\frac{1}{7}\begin{pmatrix} 8 \\ -3 \\ 1 \end{pmatrix} - \frac{3}{8}\cdot\frac{1}{7}\begin{pmatrix} 8 \\ -3 \\ 1 \end{pmatrix} = \frac{1}{28}\begin{pmatrix} -12 \\ -27 \\ 9 \end{pmatrix},
\end{aligned}$$

if $x = 2$, then

$$\begin{aligned}
&\boldsymbol{\Delta}(D \boxminus 2, D) \\
&\quad = \frac{1}{\bar{\lambda}}\left(\boldsymbol{e}_2 - \frac{1}{b_2(\boldsymbol{e}_2, \bar{\lambda}\mathbf{1}_3, D)}\boldsymbol{b}(\boldsymbol{e}_2, \bar{\lambda}\mathbf{1}_3, D)\right) - \frac{b_2\left(\dot{\boldsymbol{A}}(D)\mathbf{1}_3, \bar{\lambda}\mathbf{1}_3, D\right)}{b_2(\boldsymbol{e}_2, \bar{\lambda}\mathbf{1}_3, D)}\boldsymbol{b}(\boldsymbol{e}_2, \bar{\lambda}\mathbf{1}_3, D) \\
&\quad = -3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \frac{7}{9}\cdot\frac{1}{7}\begin{pmatrix} -3 \\ 9 \\ -3 \end{pmatrix} - \frac{4}{3}\cdot\frac{1}{7}\begin{pmatrix} -3 \\ 9 \\ -3 \end{pmatrix} = \frac{1}{28}\begin{pmatrix} -12 \\ -48 \\ -12 \end{pmatrix},
\end{aligned}$$

and if $x = 3$, then

$$\begin{aligned}
&\boldsymbol{\Delta}(D \boxminus 3, D) \\
&\quad = \frac{1}{\bar{\lambda}}\left(\boldsymbol{e}_3 - \frac{1}{b_3(\boldsymbol{e}_3, \bar{\lambda}\mathbf{1}_3, D)}\boldsymbol{b}(\boldsymbol{e}_3, \bar{\lambda}\mathbf{1}_3, D)\right) - \frac{b_3\left(\dot{\boldsymbol{A}}(D)\mathbf{1}_3, \bar{\lambda}\mathbf{1}_3, D\right)}{b_3(\boldsymbol{e}_3, \bar{\lambda}\mathbf{1}_3, D)}\boldsymbol{b}(\boldsymbol{e}_3, \bar{\lambda}\mathbf{1}_3, D) \\
&\quad = -3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{7}{8}\cdot\frac{1}{7}\begin{pmatrix} 1 \\ -3 \\ 8 \end{pmatrix} - \frac{3}{8}\cdot\frac{1}{7}\begin{pmatrix} 1 \\ -3 \\ 8 \end{pmatrix} = \frac{1}{28}\begin{pmatrix} 9 \\ -27 \\ -12 \end{pmatrix}.
\end{aligned}$$

Note that, for all $i \in \mathcal{I}(3)$, $(1/\bar{\lambda}) + b_i(\dot{\boldsymbol{A}}(D)\mathbf{1}_3, \bar{\lambda}\mathbf{1}_3, D)$ is negative. Note also that $S_{D,2,1}(\bar{\lambda}\mathbf{1}_n)$, $S_{D,1,2}(\bar{\lambda}\mathbf{1}_n)$, $S_{D,3,2}(\bar{\lambda}\mathbf{1}_n)$, and $S_{D,2,3}(\bar{\lambda}\mathbf{1}_n)$ are negative. $\diamond$

3.2.4.2 Effects from removing a vertex

This section analyzes the effects of a particular change in the digraph's vertex set on the GKB centrality: the removal of a single vertex. To this end, let x be an arbitrary vertex in D, and let $D \ominus x$ denote the digraph of order $n-1$ that emerges from D by removing vertex x (and thereby all arcs incident to vertex x), that is, $D \ominus x$ is the subdigraph of D with vertex set $\mathcal{I}(n) \setminus \{x\}$ and arc set $\{(u,v) \in \mathcal{A}(D) \mid \mathbb{1}_{\{u,v\}}(x) = 0\}$. Note that $D \ominus x$ is similar to but quite different from the digraph that emerges from D by isolating vertex x from all other vertices in D, $D \boxminus x$. Both $D \ominus x$ and $D \boxminus x$ are subdigraphs of D that have the same arc set but different vertex sets. Under certain conditions, this common feature implies that the two digraphs entail equal changes in the components of the GKB centrality other than x. Note that the GKB centrality $\boldsymbol{b}(\boldsymbol{\alpha}(D \ominus x), \boldsymbol{\lambda}(D \ominus x), D \ominus x)$ has $n-1$ components, whereas the GKB centrality $\boldsymbol{b}(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D)$ has n components. In order to analyze the effect from removing vertex x from D on the centrality of vertex $i \neq x$, we need a means to localize the ith component of the (column) vector $\boldsymbol{b}(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D)$ in the (column) vector $\boldsymbol{b}(\boldsymbol{\alpha}(D \ominus x), \boldsymbol{\lambda}(D \ominus x), D \ominus x)$. To this end, let the mapping $\hat{\imath}_x \colon \mathcal{I}(n) \setminus \{x\} \to \mathcal{I}(n-1)$ be defined by

$$\hat{\imath}_x(i) := \begin{cases} i & \text{if } i < x, \\ i-1 & \text{if } i > x, \end{cases}$$

that is, $\hat{\imath}_x$ is the unique order isomorphism from $\mathcal{I}(n) \setminus \{x\}$ to $\mathcal{I}(n-1)$. For example, if $n = 5$ and $x = 3$, then $\hat{\imath}_3 \colon \{1,2,4,5\} \to \{1,2,3,4\}$ is given by $\hat{\imath}_3(1) = 1$, $\hat{\imath}_3(2) = 2$, $\hat{\imath}_3(4) = 3$, and $\hat{\imath}_3(5) = 4$. For any (column) vector $\boldsymbol{v} \in \mathbb{R}^n$, the mapping $\hat{\imath}_x$ enables to localize the component in row $i \neq x$ of $\boldsymbol{v}$ in its subvector $[\boldsymbol{v}]_{-x}$, in particular,

$$[\boldsymbol{v}]_i = \big[[\boldsymbol{v}]_{-x}\big]_{\hat{\imath}_x(i)}.$$

Finally, let $\dot{\boldsymbol{A}}(D \ominus x)$ denote the adjacency matrix of $D \ominus x$ with respect to $\hat{\imath}_x$ and $\boldsymbol{\alpha}(D \ominus x)$ (respectively, $\boldsymbol{\lambda}(D \ominus x)$) the (column) vector or profile of vertex idiosyncrasies (respectively, localness parameters) of $D \ominus x$ with respect to $\hat{\imath}_x$. Note that $\dot{\boldsymbol{A}}(D \ominus x)$ is of order $n-1$, whereas $\dot{\boldsymbol{A}}(D \boxminus x)$ is of order n. Note also that $\dot{\boldsymbol{A}}(D \ominus x)$ and $\dot{\boldsymbol{A}}(D \boxminus x)$ are related to each other by $\dot{\boldsymbol{A}}(D \ominus x) = \boldsymbol{E}_x^\mathsf{T} \dot{\boldsymbol{A}}(D \boxminus x) \boldsymbol{E}_x$, where $\boldsymbol{E}_x := (\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{x-1}, \boldsymbol{e}_{x+1}, \ldots, \boldsymbol{e}_n)$.

Proposition 3.35 shows that the GKB centralities $\boldsymbol{b}(\boldsymbol{\alpha}(D \ominus x), \boldsymbol{\lambda}(D \ominus x), D \ominus x)$ and $\boldsymbol{b}(\boldsymbol{\alpha}(D \boxminus x), \boldsymbol{\lambda}(D \boxminus x), D \boxminus x)$ are closely related if the mappings α and λ are such that $\boldsymbol{\alpha}(D \ominus x) = [\boldsymbol{\alpha}(D \boxminus x)]_{-x}$ and $\boldsymbol{\lambda}(D \ominus x) = [\boldsymbol{\lambda}(D \boxminus x)]_{-x}$ or, equivalently, for all $i \in \mathcal{I}(n) \setminus \{x\}$, $\alpha(D \ominus x)(i) = \alpha(D \boxminus x)(i)$ and $\lambda(D \ominus x)(i) = \lambda(D \boxminus x)(i)$. The equality $\boldsymbol{\alpha}(D \ominus x) = [\boldsymbol{\alpha}(D \boxminus x)]_{-x}$ is true for the mappings α of Examples 3.1, 3.2, 3.3, and 3.4; it is not true for the mapping α of Example 3.34. The equality $\boldsymbol{\lambda}(D \ominus x) = [\boldsymbol{\lambda}(D \boxminus x)]_{-x}$ is true for the mappings λ of Examples 3.5, 3.6 (if, for example, $\phi(E)$ is constant for all digraphs E), and 3.7.

Example 3.34 Suppose α satisfies, for all digraphs E, for all $v \in \mathcal{V}(E)$, $\alpha(E)(v) = (1/\text{order of } E)\deg_E^+(v)$. It follows that $\boldsymbol{\alpha}(D \ominus x) = (1/(n-1))\dot{\boldsymbol{A}}(D \ominus x)\mathbf{1}_{n-1}$ and

$\boldsymbol{\alpha}(D \boxminus x) = (1/n)\dot{\boldsymbol{A}}(D \boxminus x)\mathbf{1}_n$. Note that $\boldsymbol{\alpha}(D \ominus x) \neq [\boldsymbol{\alpha}(D \boxminus x)]_{-x}$ if $D \boxminus x$ is not empty. ◇

Proposition 3.35 *Suppose* $\boldsymbol{\alpha}(D \ominus x) = [\boldsymbol{\alpha}(D \boxminus x)]_{-x}$ *and* $\boldsymbol{\lambda}(D \ominus x) = [\boldsymbol{\lambda}(D \boxminus x)]_{-x}$.

(3.35.1) $\sigma(\operatorname{diag}(\boldsymbol{\lambda}(D \ominus x))\dot{\boldsymbol{A}}(D \ominus x)) \setminus \{0\} = \sigma(\operatorname{diag}(\boldsymbol{\lambda}(D \boxminus x))\dot{\boldsymbol{A}}(D \boxminus x)) \setminus \{0\}$.

(3.35.2) *If* $1 \notin \sigma(\operatorname{diag}(\boldsymbol{\lambda}(D \ominus x))\dot{\boldsymbol{A}}(D \ominus x)) \cup \sigma(\operatorname{diag}(\boldsymbol{\lambda}(D))\dot{\boldsymbol{A}}(D))$, *then*

$$\boldsymbol{b}\big(\boldsymbol{\alpha}(D \ominus x), \boldsymbol{\lambda}(D \ominus x), D \ominus x\big) = \big[\boldsymbol{b}\big(\boldsymbol{\alpha}(D \boxminus x), \boldsymbol{\lambda}(D \boxminus x), D \boxminus x\big)\big]_{-x} \tag{3.17}$$

and

$$\begin{aligned}\boldsymbol{b}\big(\boldsymbol{\alpha}(D \ominus x), \boldsymbol{\lambda}(D \ominus x), D \ominus x\big) - \big[\boldsymbol{b}\big(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D\big)\big]_{-x} \\ = \big[\boldsymbol{b}\big(\boldsymbol{\alpha}(D \boxminus x), \boldsymbol{\lambda}(D \boxminus x), D \boxminus x\big) - \boldsymbol{b}\big(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D\big)\big]_{-x}.\end{aligned} \tag{3.18}$$

It is important to note that Results 3.35.1 and 3.35.2 do not rest on Conditions C-α and C-λ. Result 3.35.1 is relevant to Result 3.35.2 because it implies that $\boldsymbol{b}(\boldsymbol{\alpha}(D \boxminus x), \lambda, D \boxminus x)$ exists and is unique if and only if $\boldsymbol{b}(\boldsymbol{\alpha}(D \ominus x), \lambda, D \ominus x)$ exists and is unique. Provided that both equalities $\boldsymbol{\alpha}(D \ominus x) = [\boldsymbol{\alpha}(D \boxminus x)]_{-x}$ and $\boldsymbol{\lambda}(D \ominus x) = [\boldsymbol{\lambda}(D \boxminus x)]_{-x}$ are true, all vertices other than x have the same GKB centrality in $D \ominus x$ and $D \boxminus x$ (see (3.17)): for all $i \in \mathcal{I}(n) \setminus \{x\}$,

$$b_i\big(\boldsymbol{\alpha}(D \boxminus x), \boldsymbol{\lambda}(D \boxminus x), D \boxminus x\big) = b_{\hat{\imath}_x(i)}\big(\boldsymbol{\alpha}(D \ominus x), \boldsymbol{\lambda}(D \ominus x), D \ominus x\big).$$

Equation (3.18) is a direct consequence of (3.17) and states that the change in the GKB centrality in response to removing vertex x is identical to the change in response to isolating vertex x for all vertices other than vertex x: for all $i \in \mathcal{I}(n) \setminus \{x\}$,

$$\begin{aligned}b_{\hat{\imath}_x(i)}\big(\boldsymbol{\alpha}(D \ominus x), \boldsymbol{\lambda}(D \ominus x), D \ominus x\big) - b_i\big(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D\big) \\ = b_i\big(\boldsymbol{\alpha}(D \boxminus x), \boldsymbol{\lambda}(D \boxminus x), D \boxminus x\big) - b_i\big(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D\big).\end{aligned}$$

The rest of this section is concerned with the two cases discussed in Section 3.2.4.1. Proposition 3.36 considers the case where α and λ are constant (Conditions C-α and C-λ) and Proposition 3.37 the case where α is as in Example 3.2 and λ as in Example 3.5. Proposition 3.36 (respectively, Proposition 3.37) is closely related to Proposition 3.25 (respectively, Proposition 3.31) by means of Proposition 3.35. The interpretation of the change in the GKB centrality is therefore the same as in the context of Propositions 3.25 and 3.31, in particular with regard to the connection strength as a gauge for the magnitude of the change.

Proposition 3.36 *Suppose Conditions* C-α *and* C-λ *are satisfied and* $\boldsymbol{\alpha}(D \ominus x) = [\boldsymbol{\alpha}(D \boxminus x)]_{-x}$ *and* $\boldsymbol{\lambda}(D \ominus x) = [\boldsymbol{\lambda}(D \boxminus x)]_{-x}$. *If* $1 \notin \sigma(\operatorname{diag}(\boldsymbol{\lambda}(D \ominus x))\dot{\boldsymbol{A}}(D \ominus x)) \cup \sigma(\operatorname{diag}(\boldsymbol{\lambda}(D))\dot{\boldsymbol{A}}(D))$, *then, for all* $i \in \mathcal{I}(n) \setminus \{x\}$,

$$\begin{aligned}b_{\hat{\imath}_x(i)}\big(\boldsymbol{\alpha}(D \ominus x), \boldsymbol{\lambda}(D \ominus x), D \ominus x\big) - b_i\big(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D\big) \\ = -\frac{b_i\big(\boldsymbol{e}_x, \boldsymbol{\lambda}(D), D\big)}{b_x\big(\boldsymbol{e}_x, \boldsymbol{\lambda}(D), D\big)}\, b_x\big(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D\big) \\ = -S_{D,i,x}\big(\boldsymbol{\lambda}(D)\big)\, b_x\big(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D\big).\end{aligned}$$

Proposition 3.37 *Suppose α is as in Example 3.2, that is, $\boldsymbol{\alpha}(D) = \dot{\boldsymbol{A}}(D)\mathbf{1}_n$ and $\boldsymbol{\alpha}(D \ominus x) = \dot{\boldsymbol{A}}(D \ominus x)\mathbf{1}_{n-1}$, λ is as in Example 3.5, that is, $\boldsymbol{\lambda}(D) = \bar{\lambda}\mathbf{1}_n$ and $\boldsymbol{\lambda}(D \ominus x) = \bar{\lambda}\mathbf{1}_{n-1}$ for some $\bar{\lambda} \in \mathbb{R}$, and $1 \notin \sigma(\bar{\lambda}\dot{\boldsymbol{A}}(D \ominus x)) \cup \sigma(\bar{\lambda}\dot{\boldsymbol{A}}(D))$. If $\bar{\lambda} = 0$, then, for all $i \in \mathcal{I}(n) \setminus \{x\}$,*

$$b_{\hat{\imath}_x(i)}\big(\boldsymbol{\alpha}(D \ominus x), \boldsymbol{\lambda}(D \ominus x), D \ominus x\big) - b_i\big(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D\big) = \deg^+_{D\ominus x}\big(\hat{\imath}_x(i)\big) - \deg^+_D(i),$$

and if $\bar{\lambda} \neq 0$, then, for all $i \in \mathcal{I}(n) \setminus \{x\}$,

$$\begin{aligned} &b_{\hat{\imath}_x(i)}\big(\boldsymbol{\alpha}(D \ominus x), \boldsymbol{\lambda}(D \ominus x), D \ominus x\big) - b_i\big(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D\big) \\ &\quad = -\frac{b_i(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)}{b_x(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)}\left(\frac{1}{\bar{\lambda}} + b_x\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D\big)\right) \\ &\quad = -S_{D,i,x}\big(\boldsymbol{\lambda}(D)\big)\left(\frac{1}{\bar{\lambda}} + b_x\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D\big)\right). \end{aligned}$$

3.2.5 Measures of the degree of idiosyncrasy

This section introduces different notions of the degree of idiosyncrasy that measure the extent to which the GKB centrality is a local or global characteristic.

Suppose the GKB centrality in D with profile of vertex idiosyncrasies $\boldsymbol{\alpha} >_c \mathbf{0}_n$ and profile of localness parameters $\boldsymbol{\lambda}$, $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$, exists and is unique and positive. Recall that, for all $i \in \mathcal{I}(n)$,

$$b_i\big(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D\big) = \alpha_i + \lambda_i \sum_{j \in \mathcal{N}^+_D(i)} b_j\big(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D\big).$$

Thus, the strength of the association between vertex i's centrality and the centralities of its out-neighbors increases with the magnitude of λ_i. The centralities of vertex i's out-neighbors depend on their own idiosyncracies and localness parameters and the centralities of their out-neighbors, and so forth, possibly ad infinitum. The recursive nature of this dependence mirrors vertex i's position within the digraph D, with the dependence being stronger, the larger the magnitude of λ_i. If λ_i is close to zero, then $b_i(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ is close to α_i. In this case, the degree to which $b_i(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ is determined by α_i is high. As a result, $b_i(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ represents a local characteristic of vertex i. In contrast, if λ_i is large in magnitude, then vertex i's centrality may be strongly affected by its position. If vertex i is isolated, then $b_i(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) = \alpha_i$. If $\lambda_i > 0$ and vertex i's out-degree is high, then only a small portion of $b_i(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ may be accounted for by α_i, so that the degree to which $b_i(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ is determined by α_i is low. As a result, $b_i(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ represents a global (in the sense that the position within the digraph is taken into account) characteristic of vertex i.

The following definition introduces three different notions of *degree of idiosyncrasy*: a measure of the extent to which the GKB centrality of a single vertex is a local or global characteristic and two aggregate measures.

Definition DI The *degree of idiosyncrasy of vertex i (in D)* is defined by

$$\mathrm{VDI}(i, \boldsymbol{\alpha}, \boldsymbol{\lambda}, D) := \frac{\alpha_i}{b_i(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)}.$$

The *average degree of idiosyncrasy (in D)* is defined by

$$\mathrm{ADI}(\boldsymbol{\alpha},\boldsymbol{\lambda},D) := \frac{1}{n}\sum_{i=1}^{n}\mathrm{VDI}(i,\boldsymbol{\alpha},\boldsymbol{\lambda},D).$$

The *total degree of idiosyncrasy (in D)* is defined by

$$\mathrm{TDI}(\boldsymbol{\alpha},\boldsymbol{\lambda},D) := \frac{\|\boldsymbol{\alpha}\|}{\|\boldsymbol{b}(\boldsymbol{\alpha},\boldsymbol{\lambda},D)\|}.$$

where $\|\cdot\|$ is any norm on $\mathbb{R}^n$.[31]

As intended, a vertex's degree of idiosyncrasy is high if its centrality is largely determined by its idiosyncrasy. Other interesting properties of all three degrees of idiosyncrasy are listed in the following result.

Proposition 3.38 *Suppose $\|\cdot\|$ is monotonic in the positive orthant.*[32]

(*3.38.1*) *If $\boldsymbol{\lambda} \geq_c \mathbf{0}_n$, then all three degrees of idiosyncrasy lie in the interval $(0,1]$. If $\boldsymbol{\lambda} \leq_c \mathbf{0}_n$, then the reciprocal of all three degrees of idiosyncrasy lie in the interval $(0,1]$.*

(*3.38.2*) *Suppose Condition I-α-λ is satisfied and the localness parameters vary continuously. Let $(i,j) \in \mathcal{I}(n)^2$ with $i \neq j$. If $[\boldsymbol{\lambda}]_{-i} \geq_c \mathbf{0}_{n-1}$ and $\mathcal{N}_D^+(i) \neq \emptyset$, then $\lambda_i \mapsto \mathrm{VDI}(i,\boldsymbol{\alpha},\boldsymbol{\lambda},D)$ and $\lambda_i \mapsto \mathrm{ADI}(\boldsymbol{\alpha},\boldsymbol{\lambda},D)$ are strictly decreasing and $\lambda_i \mapsto \mathrm{TDI}(\boldsymbol{\alpha},\boldsymbol{\lambda},D)$ is decreasing on $\Gamma_i(\boldsymbol{\lambda},D)$. If $[\boldsymbol{\lambda}]_{-j} \geq_c \mathbf{0}_{n-1}$, $\mathcal{N}_D^+(j) \neq \emptyset$, and there exists a walk $(i_0,\ldots,i_p)$ in D of length p from i to j such that $(\lambda_{i_0},\ldots,\lambda_{i_{p-1}}) >_c \mathbf{0}_p$, then $\lambda_j \mapsto \mathrm{VDI}(i,\boldsymbol{\alpha},\boldsymbol{\lambda},D)$ is strictly decreasing on $\Gamma_j(\boldsymbol{\lambda},D)$.*

Under the conditions stated in Result 3.38.2, if $\|\cdot\|$ is the p-norm ($1 \leq p < \infty$), then $\lambda_i \mapsto \mathrm{TDI}(\boldsymbol{\alpha},\boldsymbol{\lambda},D)$ is strictly decreasing on $\Gamma_i(\boldsymbol{\lambda},D)$, and if $\|\cdot\|$ is the maximum norm, then $\lambda_i \mapsto \mathrm{TDI}(\boldsymbol{\alpha},\boldsymbol{\lambda},D)$ is decreasing (but not necessarily strictly decreasing) on $\Gamma_i(\boldsymbol{\lambda},D)$.

3.3 Key player analysis

The key player analysis is divided into two parts. Section 3.3.1 describes the network game for which two key player problems are defined; it is a static, noncooperative game of complete information, where the players take their decisions simultaneously and independently of one another. Section 3.3.2 defines the two key player problems and discusses their solutions for two specific cases.

31. Note that $\|\boldsymbol{b}(\boldsymbol{\alpha},\boldsymbol{\lambda},D)\| > 0$ because $\boldsymbol{b}(\boldsymbol{\alpha},\boldsymbol{\lambda},D) >_c \mathbf{0}_n$.

32. A norm $\|\cdot\|$ on $\mathbb{R}^n$ is called *monotonic* if, for all $\boldsymbol{v},\boldsymbol{w} \in \mathbb{R}^n$, $|\boldsymbol{v}| \leq_c |\boldsymbol{w}|$ implies that $\|\boldsymbol{v}\| \leq \|\boldsymbol{w}\|$ (Bauer, Stoer, and Witzgall 1961, p. 257), where $|\boldsymbol{v}|$ denotes the vector the components of which are the absolute values of the components of $\boldsymbol{v}$. A norm $\|\cdot\|$ on $\mathbb{R}^n$ is called *monotonic in the positive orthant (in $\mathbb{R}^n$)* if, for all $\boldsymbol{v},\boldsymbol{w} \in \mathbb{R}^n_+$, $\boldsymbol{v} \leq_c \boldsymbol{w}$ implies that $\|\boldsymbol{v}\| \leq \|\boldsymbol{w}\|$ (p. 259). The p-norm ($1 \leq p < \infty$) and the maximum norm are both monotonic and monotonic in the positive orthant.

3.3.1 The network game

There are $n > 1$ players. The set of all players is identified with the set $\mathcal{I}(n) = \{1, \dots, n\}$, which is abbreviated as $\mathcal{I}$. The players have a common action space $\mathbb{R}_+$. The action of player $i \in \mathcal{I}$ is denoted by y_i. The set of all possible action profiles (column vectors) $(y_1, \dots, y_n)$ is equal to $\mathbb{R}^n_+$. A player's utility (or payoff) depends not only on his own action but may also depend on the actions of other players. This dependence is made explicit by a nonempty digraph D on $\mathcal{I}$, which is common knowledge. The digraph D encodes information about the identities of the players who directly affect a player's utility through their actions. For a particular player i, the set of all players who directly affect his utility is given by his out-neighborhood (in D), $\mathcal{N}^+_D(i)$. Note that, for all $i \in \mathcal{I}$, $i \notin \mathcal{N}^+_D(i)$, because D has no self-loops, which is a consequence of the definition of a digraph. The assumption of a digraph implies that a player is not necessarily an out-neighbor of his out-neighbors, that is, for all $i \in \mathcal{I}$, $j \in \mathcal{N}^+_D(i)$ does not necessarily imply that $i \in \mathcal{N}^+_D(j)$. The dependence of a player's utility on his out-neighbors' actions is therefore potentially unidirectional. The players' utility (or payoff) functions $\{u_i \colon \mathbb{R}^n_+ \to \mathbb{R}\}_{i \in \mathcal{I}}$ are given by

$$u_i(y_1, \dots, y_n) := \alpha_i y_i - \frac{1}{2} y_i^2 + \lambda_i y_i \sum_{j \in \mathcal{N}^+_D(i)} y_j = \Bigl(\alpha_i + \lambda_i \sum_{j \in \mathcal{I}} \dot{a}_{i,j}(D) y_j\Bigr) y_i - \frac{1}{2} y_i^2,$$

where $\alpha_i \geq 0$ and λ_i can be of any sign.[33] As in Section 3.2, α_i and λ_i are the short forms of $\alpha_i(D)$ and $\lambda_i(D)$, which are defined by $\alpha_i(D) := \alpha(D)(i)$ and $\lambda_i(D) := \lambda(D)(i)$, where α and λ are two mappings that assign to every digraph E (of any order) the functions $\alpha(E) \colon \mathcal{V}(E) \to \mathbb{R}_+$ and $\lambda(E) \colon \mathcal{V}(E) \to \mathbb{R}$. The utility that player i ascribes to the action profile $(y_1, \dots, y_n)$ consists of two components: the *private component* and the *social component*. The private component is given by the sum of the private benefit, $\alpha_i y_i$, and the private cost, $-(1/2)y_i^2$, with the marginal private benefit equal to α_i. The social component is given by $\lambda_i y_i \sum_{j \in \mathcal{N}^+_D(i)} y_j$ and represents player i's social benefit (if $\lambda_i > 0$) or social cost (if $\lambda_i < 0$) from playing action y_i. Player i's utility function is symmetric in his out-neighbors' actions.[34] It exhibits local strategic complements (respectively, substitutes) if $\lambda_i > 0$ (respectively, $\lambda_i < 0$) because

$$\frac{\partial^2 u_i(y_1, \dots, y_n)}{\partial y_i \partial y_j} = \begin{cases} \lambda_i & \text{if } j \in \mathcal{N}^+_D(i), \\ 0 & \text{if } j \notin \mathcal{N}^+_D(i); \end{cases}$$

it exhibits positive (respectively, negative) local externalities if $\lambda_i > 0$ (respectively, $\lambda_i < 0$).[35] The profiles (column vectors) $\boldsymbol{\alpha}$ and $\boldsymbol{\lambda}$ are defined as in Section 3.2, and, like their components, they may depend on D. If this dependence is material to a

33. Recall that $\dot{a}_{i,j}(D)$ denotes the component in row i and column j of the adjacency matrix of D (with respect to the identity mapping on $\mathcal{I}$), $\dot{\boldsymbol{A}}(D)$.

34. That is, for all permutations π of $\mathcal{I}$ with fixed points $\mathcal{I} \setminus \mathcal{N}^+_D(i)$, $u_i(y_{\pi(1)}, \dots, y_{\pi(n)}) = u_i(y_1, \dots, y_n)$.

35. In accordance with the terminology introduced by Galeotti et al. (2010, pp. 226–27), player i's utility function is said to exhibit positive (respectively, negative) local externalities if for all $(y_1, \dots, y_n) \in \mathbb{R}^n_+$ and for all $(\tilde{y}_1, \dots, \tilde{y}_n) \in \mathbb{R}^n_+$ with $\tilde{y}_i = y_i$ and $\{j \in \mathcal{N}^+_D(i) \mid \tilde{y}_j \geq y_j\} = \mathcal{N}^+_D(i)$, $u_i(\tilde{y}_1, \dots, \tilde{y}_n) \geq u_i(y_1, \dots, y_n)$ (respectively, $u_i(\tilde{y}_1, \dots, \tilde{y}_n) \leq u_i(y_1, \dots, y_n)$).

result, the two profiles are written as $\boldsymbol{\alpha}(D)$ and $\boldsymbol{\lambda}(D)$. The present network game is denoted by $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$. The players of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ are called *ex ante homogeneous with respect to* α, or α*-homogeneous* for short, if $\boldsymbol{\alpha} = \bar{\alpha}\mathbf{1}_n$ for some $\bar{\alpha} \in \mathbb{R}_+$; they are *called ex ante homogeneous with respect to* λ, or λ*-homogeneous* for short, if $\boldsymbol{\lambda} = \bar{\lambda}\mathbf{1}_n$ for some $\bar{\lambda} \in \mathbb{R}$. The network game of Ballester, Calvó-Armengol, and Zenou (2006) corresponds to $\Gamma(\boldsymbol{\alpha}, \bar{\lambda}\mathbf{1}_n, D)$ with $\boldsymbol{\alpha} >_c \mathbf{0}_n$ and $\bar{\lambda} \in \mathbb{R}_{++}$.

Sufficient conditions for the existence of a unique and interior Nash equilibrium, or NE for short, of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ are stated in Propositions 3.39 and 3.40, where the former result considers the case $\boldsymbol{\lambda}^- = \mathbf{0}_n$, that is, $\boldsymbol{\lambda} \geq_c \mathbf{0}_n$, and the latter the case $\boldsymbol{\lambda}^- \neq \mathbf{0}_n$.[36]

Proposition 3.39 *The game* $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ *has a unique and interior NE* $\boldsymbol{y}^\star \in \mathbb{R}^n_{++}$*, which is given by*

$$\boldsymbol{y}^\star = \boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D), \tag{3.19}$$

if three conditions are satisfied: (3.39.1) $\boldsymbol{\lambda}^- = \mathbf{0}_n$*, (3.39.2)* $\rho(\mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$*, and (3.39.3)* $\mathbf{0}_n <_c \boldsymbol{\alpha}$*.*

Proposition 3.40 *The game* $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ *has a unique and interior NE* $\boldsymbol{y}^\star \in \mathbb{R}^n_{++}$*, which is given by (3.19), if four conditions are satisfied: (3.40.1)* $\boldsymbol{\lambda}^- \neq \mathbf{0}_n$*; (3.40.2) Condition C-*ρ*; (3.40.3)* $\mathrm{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)(\boldsymbol{I}_n - \mathrm{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D))^{-1}\boldsymbol{\alpha} <_c \boldsymbol{\alpha}$*; and (3.40.4) for all nonempty and proper subsets* $\mathcal{J}$ *of* $\mathcal{I}$*,* $1 \notin \sigma([\mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)]_{\mathcal{J},\mathcal{J}})$ *and* $[\boldsymbol{\alpha}]_{\mathcal{I}\setminus\mathcal{J}} + [\mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)]_{\mathcal{I}\setminus\mathcal{J},\mathcal{J}}(\boldsymbol{I}_{|\mathcal{J}|} - [\mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)]_{\mathcal{J},\mathcal{J}})^{-1}[\boldsymbol{\alpha}]_{\mathcal{J}} >_c \mathbf{0}_{|\mathcal{I}\setminus\mathcal{J}|}$*.*

The unique and interior NE of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$, provided it exists, is given by the GKB centrality in D with profile of vertex idiosyncrasies $\boldsymbol{\alpha}$ and profile of localness parameters $\boldsymbol{\lambda}$, $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$. This characterization of the NE is of the same kind as that of Ballester, Calvó-Armengol, and Zenou (2006). It goes without saying that all comparative statics results of Section 3.2.4 apply to the unique and interior NE of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$. Note that *equilibrium welfare* in $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$, defined as the sum of the players' utilities at the NE $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$, is equal to $(1/2)\|\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)\|_2^2$.

I briefly discuss Proposition 3.40. The game $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ has a unique interior NE if Conditions 3.40.1, 3.40.2, and 3.40.3 are satisfied.[37] Conditions 3.40.2 and 3.40.3 imply that $\boldsymbol{\alpha} >_c \mathbf{0}_n$ (see the discussion of Result 3.14.2 in Section 3.2.3) and rule out that the profile $\mathbf{0}_n$ is a boundary NE of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$.[38] These two conditions entail a lower bound for the negative components of $\boldsymbol{\lambda}$ (see again the discussion of Result 3.14.2 in Section 3.2.3). Condition 3.40.4 rules out that a situation where some players play zero and the remaining players play a positive action is a boundary NE of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$, provided that Conditions 3.40.1, 3.40.2, and 3.40.3 are satisfied. For the purpose of discussing Condition 3.40.4, suppose $\boldsymbol{\lambda} = \bar{\lambda}\mathbf{1}_n$ for some $\bar{\lambda} < 0$, which implies that $\boldsymbol{\lambda}^- \neq \mathbf{0}_n$ (Condition 3.40.1), Conditions 3.40.2, 3.40.3, and 3.40.4

36. Recall that $\boldsymbol{\lambda}^-$ denotes the componentwise negative part of $\boldsymbol{\lambda}$.
37. Note the distinction between a *unique interior* NE and a *unique and interior* NE of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$.
38. A boundary NE of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ is a NE of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ that involves actions at the topological boundary of $\mathbb{R}_+$, $\{0\}$.

are satisfied, and $\mathcal{J}$ is a nonempty and proper subset of $\mathcal{I}$. Condition 3.40.3 reduces to $(\boldsymbol{I}_n - |\bar{\lambda}|\dot{\boldsymbol{A}}(D))\boldsymbol{\alpha} >_c \boldsymbol{0}_n$, which implies that

$$\left(\boldsymbol{I}_{|\mathcal{J}|} - |\bar{\lambda}|[\dot{\boldsymbol{A}}(D)]_{\mathcal{J},\mathcal{J}}\right)[\boldsymbol{\alpha}]_{\mathcal{J}} >_c |\bar{\lambda}|[\dot{\boldsymbol{A}}(D)]_{\mathcal{J},\mathcal{I}\setminus\mathcal{J}}[\boldsymbol{\alpha}]_{\mathcal{I}\setminus\mathcal{J}}. \tag{3.20}$$

Condition 3.40.4 implies that

$$|\bar{\lambda}|[\dot{\boldsymbol{A}}(D)]_{\mathcal{I}\setminus\mathcal{J},\mathcal{J}}\left(\boldsymbol{I}_{|\mathcal{J}|} - \bar{\lambda}[\dot{\boldsymbol{A}}(D)]_{\mathcal{J},\mathcal{J}}\right)^{-1}[\boldsymbol{\alpha}]_{\mathcal{J}} <_c [\boldsymbol{\alpha}]_{\mathcal{I}\setminus\mathcal{J}}, \tag{3.21}$$

where $(\boldsymbol{I}_{|\mathcal{J}|} - \bar{\lambda}[\dot{\boldsymbol{A}}(D)]_{\mathcal{J},\mathcal{J}})^{-1}[\boldsymbol{\alpha}]_{\mathcal{J}} >_c \boldsymbol{0}_{|\mathcal{J}|}$ according to (3.20) and Conditions 3.40.2 and 3.40.3.[39] Note that the vector on the left-hand side of inequality (3.21) is strictly increasing in the magnitude of $\bar{\lambda}$ on some interval of the negative real line for which Conditions 3.40.2 and 3.40.3 are satisfied. It follows that Condition 3.40.4 entails a lower bound for $\bar{\lambda}$ (in terms of $\dot{\boldsymbol{A}}(D)$ and $\boldsymbol{\alpha}$).

The remainder of this section discusses briefly three special cases of the network game $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$, thereby highlighting its nexus with existing games or models in the literature on the economics of social interactions. The first case (Example 3.41) shows that the so-called *local-aggregate game* (Calvó-Armengol and Zenou 2004; Ballester, Calvó-Armengol, and Zenou 2006) is a special case of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$. The second case (Example 3.42) establishes a relation between the so-called *local-average game* (Patacchini and Zenou 2012) and $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$. The third case (Example 3.43) is motivated by the work of Mas and Moretti (2009), who study the productivity of cashiers in a national supermarket chain. They find strong evidence of productivity spillovers and substantial *heterogeneity* in how workers respond to peers; specifically, the spillover effect is positive and large for some workers, and it is small or even *negative* for others (see, in particular, Figure 3).

Example 3.41 (Local-aggregate game) Suppose $\boldsymbol{\alpha} >_c \boldsymbol{0}_n$ and $\boldsymbol{\lambda} = \bar{\lambda}\boldsymbol{1}_n$ for some $\bar{\lambda} \in \mathbb{R}_{++}$ (Example 3.5), that is, the localness parameters are positive and homogeneous across players. It follows from (3.19) that the system of best reply functions at the unique and interior NE $(y_1^\star, \ldots, y_n^\star)$ of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ is given by

$$\forall i \in \mathcal{I} \quad y_i^\star = \alpha_i + \bar{\lambda} \sum_{j \in \mathcal{N}_D^+(i)} y_j^\star. \tag{3.22}$$

39. The proof of $(\boldsymbol{I}_{|\mathcal{J}|} - \bar{\lambda}[\dot{\boldsymbol{A}}(D)]_{\mathcal{J},\mathcal{J}})^{-1}[\boldsymbol{\alpha}]_{\mathcal{J}} >_c \boldsymbol{0}_{|\mathcal{J}|}$ is as follows. First, note that $\rho(|\bar{\lambda}|\dot{\boldsymbol{A}}(D)) < 1$ (Condition 3.40.2). Second, note that $\rho(|\bar{\lambda}|[\dot{\boldsymbol{A}}(D)]_{\mathcal{J},\mathcal{J}}) \leq \rho(|\bar{\lambda}|\dot{\boldsymbol{A}}(D))$ because $|\bar{\lambda}|\dot{\boldsymbol{A}}(D)$ is nonnegative and $|\bar{\lambda}|[\dot{\boldsymbol{A}}(D)]_{\mathcal{J},\mathcal{J}}$ is a principal submatrix of $|\bar{\lambda}|\dot{\boldsymbol{A}}(D)$ (see, for example, Berman and Plemmons 1994, Corollary 1.6 on p. 28). It follows that $\rho(|\bar{\lambda}|[\dot{\boldsymbol{A}}(D)]_{\mathcal{J},\mathcal{J}}) < 1$. We find

$$\begin{aligned}
\left(\boldsymbol{I}_{|\mathcal{J}|} - \bar{\lambda}[\dot{\boldsymbol{A}}(D)]_{\mathcal{J},\mathcal{J}}\right)^{-1}[\boldsymbol{\alpha}]_{\mathcal{J}} &= \left(\boldsymbol{I}_{|\mathcal{J}|} - \bar{\lambda}^2[\dot{\boldsymbol{A}}(D)]^2_{\mathcal{J},\mathcal{J}}\right)^{-1}\left(\boldsymbol{I}_{|\mathcal{J}|} - |\bar{\lambda}|[\dot{\boldsymbol{A}}(D)]_{\mathcal{J},\mathcal{J}}\right)[\boldsymbol{\alpha}]_{\mathcal{J}} \\
&\geq_c \left(\boldsymbol{I}_{|\mathcal{J}|} - |\bar{\lambda}|[\dot{\boldsymbol{A}}(D)]_{\mathcal{J},\mathcal{J}}\right)[\boldsymbol{\alpha}]_{\mathcal{J}} \\
&>_c |\bar{\lambda}|[\dot{\boldsymbol{A}}(D)]_{\mathcal{J},\mathcal{I}\setminus\mathcal{J}}[\boldsymbol{\alpha}]_{\mathcal{I}\setminus\mathcal{J}} \\
&>_c \boldsymbol{0}_{|\mathcal{J}|},
\end{aligned}$$

where the equality follows from Proposition 3.12, the first inequality from $\boldsymbol{I}_{|\mathcal{J}|} - \bar{\lambda}^2[\dot{\boldsymbol{A}}(D)]^2_{\mathcal{J},\mathcal{J}}$ being a nonsingular M-matrix (Proposition 3.12) whose inverse is bounded below by $\boldsymbol{I}_n$ (Lemma B.6), the second inequality from (3.20), and the last inequality form $[\boldsymbol{\alpha}]_{\mathcal{I}\setminus\mathcal{J}} >_c \boldsymbol{0}_{|\mathcal{J}|}$ (Conditions 3.40.2 and 3.40.3).

The system (3.22) is representative of the local-aggregate game of Calvó-Armengol and Zenou (2004) and Ballester, Calvó-Armengol, and Zenou (2006), which is a game with local strategic complements because $\bar{\lambda} > 0$. Statistical models based on (3.22) have been used to estimate network effects in crime (Liu et al. 2012; Lindquist and Zenou 2014). ◇

Example 3.42 (Local-average game) Suppose, for all $i \in \mathcal{I}$, $\deg_D^+(i) > 0$, $\alpha_i > 0$, and $\lambda_i = \bar{\phi}/\deg_D^+(i)$, where $\bar{\phi} \in \mathbb{R}_{++}$ (Example 3.6). It follows from (3.19) that the system of best reply functions at the unique and interior NE $(y_1^\star, \ldots, y_n^\star)$ of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ is given by

$$\forall i \in \mathcal{I} \quad y_i^\star = \alpha_i + \bar{\phi} \frac{\sum_{j \in \mathcal{N}_D^+(i)} y_j^\star}{\deg_D^+(i)}. \tag{3.23}$$

The system (3.23) has the same structure as the system of best reply functions at the unique and interior NE $(y_1^\star, \ldots, y_n^\star)$ of the local-average game of Patacchini and Zenou (2012), in which the players' utility functions are given by

$$u_i(y_1, \ldots, y_n) := \alpha_i y_i - \frac{1}{2} y_i^2 - \bar{\phi} \left(y_i - \frac{\sum_{j \in \mathcal{N}_D^+(i)} y_j}{\deg_D^+(i)} \right)^2$$

and the system of best reply functions at the NE by

$$\forall i \in \mathcal{I} \quad y_i^\star = \frac{\alpha_i}{1 + \bar{\phi}} + \frac{\bar{\phi}}{1 + \bar{\phi}} \frac{\sum_{j \in \mathcal{N}_D^+(i)} y_j^\star}{\deg_D^+(i)}. \tag{3.24}$$

Provided that the functional dependence of the intercept and slope parameters, that is, $\alpha_i/(1 + \bar{\phi})$ and $\bar{\phi}/(1 + \bar{\phi})$, is ignored in the translation of (3.24) to a statistical model (as, for example, in Patacchini and Zenou 2012), the resulting statistical model is observationally equivalent to the statistical model that corresponds to (3.23). ◇

Example 3.43 (Mas and Moretti 2009) Suppose, for all $i \in \mathcal{I}$, $\deg_D^+(i) > 0$, $\alpha_i > 0$, and $\lambda_i = \phi_i/\deg_D^+(i)$, where $\phi_i \in \mathbb{R}$ (Example 3.6). It follows from (3.19) that the system of best reply functions at the unique and interior NE $(y_1^\star, \ldots, y_n^\star)$ of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ is given by

$$\forall i \in \mathcal{I} \quad y_i^\star = \alpha_i + \phi_i \frac{\sum_{j \in \mathcal{N}_D^+(i)} y_j^\star}{\deg_D^+(i)}.$$

This system is the network analogue of the alternative model suggested by Mas and Moretti (2009) to analyse coworker productivity spillovers, where a worker's productivity is assumed to depend on the contemporaneous rather than the permanent productivity of coworkers (pp. 120–21). The negative spillover effects found by Mas and Moretti (2009) correspond to negative values of ϕ_i. The network game proposed in this section constitutes therefore a microeconomic foundation of the network analogue of Mas and Moretti's (2009) alternative model. Their empirical findings speak in particular for a network game where the localness parameters are potentially heterogeneous across player in terms of both magnitude and sign. ◇

3.3.2 Two key player problems

Suppose the game $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ has a unique NE that is given by $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ in what follows. A key player problem is defined relative to a central planner's objective function. In the context of the present game, there are two alternatives: aggregate equilibrium action or equilibrium welfare. The following analysis is confined to a variant of the former alternative, namely, *weighted aggregate equilibrium action*,

$$\Upsilon(\boldsymbol{\omega}, D) := \big\langle \boldsymbol{\omega}, \boldsymbol{b}\big(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D\big) \big\rangle = \sum_{i=1}^{n} \omega_i \, b_i\big(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D\big),$$

where $\boldsymbol{\omega} := (\omega_1, \ldots, \omega_n) \in [0,1]^n$ is a (column) vector of weights, referred to as *profile of weights*. Note that $\boldsymbol{\omega}$ need not necessarily satisfy $\langle \mathbf{1}_n, \boldsymbol{\omega} \rangle = 1$. Unweighted aggregate equilibrium action corresponds to the profile of weights $\mathbf{1}_n$. The focus is on the following two key player problems:

KPP-I Find the player whose *isolation* from all other players of the game $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ (by cutting all his or her arcs to all other players) results in the maximal decrease in weighted aggregate equilibrium action.

KPP-R Find the player whose *removal* from the game $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ (and thereby from the digraph D) results in the maximal decrease in weighted aggregate equilibrium action.

A key player problem with a profile of weights for which all components are positive is called a *global key player problem* (in $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$), and a key player problem with a profile of weights for which at least one component is zero is called a *local key player problem* (in $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$). The notion of a local key player problem embodies the idea that the planner attaches importance only to the players residing in a particular subdigraph K of D. In this case, only the components of the profile of weights with indices in the vertex set of K, $\mathcal{V}(K)$, receive positive weights, for example, weight 1 in case of equal weights, and all other components receive weight 0.

The distinction between the two types of key player problems is subtle but important in two respects: First, KPP-I and KPP-R are generally different in that they do not necessarily identify the same player(s) as the key player(s).[40] Second, the two key player problems involve different changes in the topology of the network. KPP-I entails the isolation of a player, whereas KPP-R entails the removal of a player. This difference is relevant in practice because only one type of change in the topology of the network may be feasible or appropriate for a given phenomenon. Consider, for example, a network of criminals where action represents an illegal act or some measure thereof. Suppose a convicted offender is sentenced to several years in prison and thereby physically removed from his network. Often, the imprisonment aims at cutting off all communication channels between the offender and the members of his (former) network. It goes without saying that this constitutes a major challenge for correctional institutions. If it is impossible for the convicted offender to commit

40. See Example 3.54 in Section 3.3.2.3 for an illustration.

the illegal act behind bars, which is, for example, the case for armed robbery, then the notion of the key player defined in the formulation of KPP-R is more appropriate than the notion in the formulation of KPP-I. Suppose the sentence or probation conditions of a convicted offender involve association, location, or even residence restrictions. The offender is thereby isolated from the rest of his network, but he may still be able to commit the illegal act. In this case, the notion of the key player of KPP-I is more appropriate than that of KPP-R.

A key assumption for the subsequent analysis is that the players adjust only their actions in response to the planner's alteration of the digraph by which they are connected but do not form new arcs or sever existing arcs themselves. This is a reasonable assumption in the short term. In the medium to long term, the players may adjust their connections, and they may also anticipate the planner's key player policy and take this into account in their decisions. The players' utility parameters, $\{(\alpha_i(D), \lambda_i(D)\}_{i\in\mathcal{I}}$, may change according to the mappings α and λ.

3.3.2.1 The KPP-I

Suppose, for all $x \in \mathcal{I}$, $\Gamma(\boldsymbol{\alpha}(D \boxminus x), \boldsymbol{\lambda}(D \boxminus x), D \boxminus x)$ has a unique NE that is given by $\boldsymbol{b}(\boldsymbol{\alpha}(D \boxminus x), \boldsymbol{\lambda}(D \boxminus x), D \boxminus x)$. The planner's objective is to maximally reduce weighted aggregate equilibrium action by isolating exactly one player from all other players of the game $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$. Formally, the planner solves

$$\text{KP-I}(\boldsymbol{\omega}, \alpha, \lambda, D) := \underset{x\in\{y\in\mathcal{I}\mid \Delta Y^I(y,\boldsymbol{\omega},D)<0\}}{\arg\min} \left\{x \mapsto \Delta Y^I(x, \boldsymbol{\omega}, D)\right\},$$

where $\Delta Y^I(x, \boldsymbol{\omega}, D)$ denotes the change in weighted aggregate equilibrium action caused by isolating player x:

$$\Delta Y^I(x, \boldsymbol{\omega}, D) := Y(\boldsymbol{\omega}, D \boxminus x) - Y(\boldsymbol{\omega}, D).$$

A player in KP-I$(\boldsymbol{\omega}, \alpha, \lambda, D)$ is called a *key player* of KPP-I in $\Gamma(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D)$. Proposition 3.24, equation (3.7) in particular, forms the basis for solving a specific KPP-I. Compact, interpretable expressions of $\Delta Y^I(x, \boldsymbol{\omega}, D)$ are available under certain assumptions. The focus of the remainder of this section is on the two cases discussed in Section 3.2.4.1. The first case assumes that α and λ are constant (Conditions C-α and C-λ). The corresponding result is stated as Proposition 3.44 and is based on Proposition 3.25. The second case assumes that α is as in Example 3.2 and λ as in Example 3.5, that is, the players of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ are λ-homogeneous. The corresponding result is stated as Proposition 3.48 and is based on Proposition 3.31.

Proposition 3.44 *If Conditions C-α and C-λ are satisfied and $\rho(\operatorname{diag}(|\boldsymbol{\lambda}|)\dot{\boldsymbol{A}}(D)) < 1$, then, for all $x \in \mathcal{I}$, $b_x(\boldsymbol{e}_x, \boldsymbol{\lambda}, D) \neq 0$ and*

$$\Delta Y^I(x, \boldsymbol{\omega}, D) = \omega_x \alpha_x - \frac{b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)}{b_x(\boldsymbol{e}_x, \boldsymbol{\lambda}, D)} \langle \boldsymbol{\omega}, \boldsymbol{b}(\boldsymbol{e}_x, \boldsymbol{\lambda}, D)\rangle.$$

By isolating player x, his action changes from $b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ to α_x, his marginal private benefit.[41] The magnitude of the difference between $\Delta Y^I(x, \boldsymbol{\omega}, D)$ and $\alpha_x - b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ is a measure of the extent to which a social multiplier is at work (via the social components of the players' utility functions). If $\boldsymbol{\lambda}^- \neq \mathbf{0}_n$, then $\Delta Y^I(x, \boldsymbol{\omega}, D)$ may be negative, zero, or positive (see Example 3.52 in Section 3.3.2.3). If $\boldsymbol{\lambda}^- = \mathbf{0}_n$, that is, $\boldsymbol{\lambda} \geq_c \mathbf{0}_n$, then $\Delta Y^I(x, \boldsymbol{\omega}, D)$ is nonpositive. This and other properties of $\Delta Y^I(x, \boldsymbol{\omega}, D)$ are stated in Proposition 3.45.

Proposition 3.45 *Suppose Conditions C-α and C-λ are satisfied, $\boldsymbol{\alpha} >_c \mathbf{0}_n$, $\boldsymbol{\lambda} \geq_c \mathbf{0}_n$, and $\rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$. For all $x \in \mathcal{I}$,*

$$\Delta Y^I(x, \boldsymbol{\omega}, D) = -\omega_x\big(b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) - \alpha_x\big) - b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) \sum_{j \in \mathcal{I}\setminus\{x\}} \omega_j S_{D,j,x}(\boldsymbol{\lambda}) \leq 0, \tag{3.25}$$

where $S_{D,j,x}(\boldsymbol{\lambda})$ is the connection strength of (j, x) in D at $\boldsymbol{\lambda}$ (Definition CS). Let $x \in \mathcal{I}$, and let $(*)$ denote the following condition: There exists a player $i \in \mathcal{I} \setminus \{x\}$ for whom $\omega_i > 0$ and there exists a walk $(i_0, \ldots, i_p)$ in D of length p from i to x such that $(\lambda_{i_0}, \ldots, \lambda_{i_{p-1}}) >_c \mathbf{0}_p$.

(3.45.1) If $\omega_x > 0$, $\lambda_x > 0$, and $\mathcal{N}_D^+(x) \neq \emptyset$, then $\Delta Y^I(x, \boldsymbol{\omega}, D) < 0$.

(3.45.2) If Condition $(*)$ is satisfied, then $\Delta Y^I(x, \boldsymbol{\omega}, D) < 0$.

(3.45.3) If Conditions I-α-λ and $(*)$ are satisfied, then $\Delta Y^I(x, \boldsymbol{\omega}, D)$ is strictly decreasing in α_x on $\mathbb{R}_{++}$.

(3.45.4) If Conditions I-α-λ and $(*)$ are satisfied and there exists a walk $(k_0, \ldots, k_q)$ in D of length q from x to i such that $(\lambda_{k_0}, \ldots, \lambda_{k_{q-1}}) >_c \mathbf{0}_q$, then $\Delta Y^I(x, \boldsymbol{\omega}, D)$ is strictly decreasing in α_i on $\mathbb{R}_{++}$.

(3.45.5) If Conditions I-α-λ and $(*)$ are satisfied and $\mathcal{N}_D^+(x) \neq \emptyset$, then $\Delta Y^I(x, \boldsymbol{\omega}, D)$ is strictly decreasing in λ_x on $\Gamma_x(\boldsymbol{\lambda}, D)$.

(3.45.6) If Conditions I-α-λ and $(*)$ are satisfied, then $\Delta Y^I(x, \boldsymbol{\omega}, D)$ is strictly decreasing in λ_i on $\Gamma_i(\boldsymbol{\lambda}, D)$.

If $\Delta Y^I(x, \boldsymbol{\omega}, D) < 0$ for some $x \in \mathcal{I}$, then KP-I$(\boldsymbol{\omega}, \alpha, \lambda, D) \neq \emptyset$ but not necessarily $|\text{KP-I}(\boldsymbol{\omega}, \alpha, \lambda, D)| = 1$. Sufficient conditions for $\Delta Y^I(x, \boldsymbol{\omega}, D) < 0$ are given in Proposition 3.45 (Results 3.45.1 and 3.45.2). As regards the case $\boldsymbol{\omega} = \mathbf{1}_n$, if Conditions C-$\alpha$ and C-λ are satisfied, $\boldsymbol{\alpha} >_c \mathbf{0}_n$, $\boldsymbol{\lambda} >_c \mathbf{0}_n$, and $\rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$, then there exists an $x \in \mathcal{I}$ with $\Delta Y^I(x, \mathbf{1}_n, D) < 0$ because D is not empty. Equation (3.25) shows that the stronger the players of $\Gamma(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D)$ are connected in D in terms of connection strength, the larger is the decrease in weighted aggregate equilibrium action.

41. If player i is isolated, then, for all action profiles $(y_1, \ldots, y_n)$, his utility satisfies $u_i(y_1, \ldots, y_n) = \alpha_i y_i - (1/2)y_i^2$, from which it follows that his best reply is given by his marginal private benefit, α_i.

A special case of Proposition 3.44, namely, where the players of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ are both α-homogeneous and λ-homogeneous, is stated in Corollary 3.46. To this end, note that if $\boldsymbol{\lambda} = \bar{\lambda}\mathbf{1}_n$ for some $\bar{\lambda} \in \mathbb{R}$ with $1 \notin \sigma(\bar{\lambda}\dot{\boldsymbol{A}}(D))$, then

$$\langle \boldsymbol{\omega}, \boldsymbol{b}(\boldsymbol{e}_x, \boldsymbol{\lambda}, D)\rangle = b_x(\boldsymbol{\omega}, \bar{\lambda}\mathbf{1}_n, D^\top), \tag{3.26}$$

where $D^\top$ denotes the transpose of D with $\dot{\boldsymbol{A}}(D^\top) = \dot{\boldsymbol{A}}(D)^\top$.

Corollary 3.46 *If α is as in Example 3.1 with $\bar{\alpha} \in \mathbb{R}_{++}$, λ is as in Example 3.5 with $\bar{\lambda} \in \mathbb{R}$, and $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$, then, for all $x \in \mathcal{I}$, $b_x(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D) \neq 0$ and*

$$\Delta Y^I(x, \boldsymbol{\omega}, D) = \bar{\alpha}\left(\omega_x - \frac{b_x(\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D)\, b_x(\boldsymbol{\omega}, \bar{\lambda}\mathbf{1}_n, D^\top)}{b_x(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)}\right).$$

As regards the second case where α is as in Example 3.2 and λ as in Example 3.5, it is important to note that $\boldsymbol{b}(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D)$ is the unique but not necessarily interior NE of the game $\Gamma(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D)$ if $\bar{\lambda} \geq 0$ and $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$ (Proposition 3.47). Under the foregoing conditions, if a player's out-degree is zero, then he plays action zero, and a positive action otherwise.

Proposition 3.47 *The game $\Gamma(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D)$ has a unique NE $\boldsymbol{y}^\star \in \mathbb{R}^n_+$, which is given by $\boldsymbol{y}^\star = \boldsymbol{b}(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D)$, if two conditions are satisfied: (3.47.1) $\bar{\lambda} \geq 0$ and (3.47.2) $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$. The action profile $\boldsymbol{b}(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D)$ satisfies, for all $i \in \mathcal{I}$, if $\deg^+_D(i) = 0$, then $b_i(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D) = 0$, and $b_i(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D) > 0$ otherwise.*

Note that Proposition 3.47 is true for any game $\Gamma(\dot{\boldsymbol{A}}(D \boxminus x)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D \boxminus x)$.

Proposition 3.48 *Suppose α is as in Example 3.2, λ is as in Example 3.5 with $\bar{\lambda} \geq 0$, and $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D))$. For all $x \in \mathcal{I}$, if $\bar{\lambda} = 0$, then*

$$\Delta Y^I(x, \boldsymbol{\omega}, D) = -\left(\omega_x \deg^+_D(x) + \sum_{i \in \mathcal{N}^-_D(x)} \omega_i\right),$$

and if $\bar{\lambda} > 0$, then

$$\Delta Y^I(x, \boldsymbol{\omega}, D) = \frac{1}{\bar{\lambda}}\left(\omega_x - \frac{b_x(\boldsymbol{\omega}, \bar{\lambda}\mathbf{1}_n, D^\top)}{b_x(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)}\right) - \frac{b_x(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D)\, b_x(\boldsymbol{\omega}, \bar{\lambda}\mathbf{1}_n, D^\top)}{b_x(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)}.$$

Within the framework of Proposition 3.48, if $\bar{\lambda} > 0$, then

$$\begin{aligned}\Delta Y^I(x, \boldsymbol{\omega}, D) = &-\omega_x\big(b_x(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D) - 0\big) \\ &- \left(\frac{1}{\bar{\lambda}} + b_x(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D)\right) \sum_{j \in \mathcal{I}\setminus\{x\}} \omega_j S_{D,j,x}(\bar{\lambda}\mathbf{1}_n),\end{aligned}$$

which is less than the corresponding expression given in Proposition 3.45 if $\omega_x > 0$ or there exists a player $i \in \mathcal{I} \setminus \{x\}$ for whom $\omega_i > 0$ and there exists a walk in D from i to x, provided that the values of the profiles $\boldsymbol{\alpha}$ and $\boldsymbol{\lambda}$ in Proposition 3.45 agree with $\dot{\boldsymbol{A}}(D)\mathbf{1}_n$ and $\bar{\lambda}\mathbf{1}_n$, respectively. The reason for this is that the marginal private benefit of the player being isolated remains unchanged within the framework of Proposition 3.45, whereas it falls to zero within the framework of Proposition 3.48, which results in a decrease of his action from $b_x(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D)$ to 0. This decrease can be decomposed into two components: a change from $b_x(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D)$ to $\deg_D^+(x)$ and a change from $\deg_D^+(x)$ to 0. Both changes cause other connected players to reduce their actions. The former change results in a decline of weighted aggregate equilibrium action by $b_x(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D) \sum_{j\in\mathcal{I}\setminus\{x\}} \omega_j S_{D,j,x}(\bar{\lambda}\mathbf{1}_n)$ and the latter by $(1/\bar{\lambda}) \sum_{j\in\mathcal{I}\setminus\{x\}} \omega_j S_{D,j,x}(\bar{\lambda}\mathbf{1}_n)$.

3.3.2.2 The KPP-R

Suppose, for all $x \in \mathcal{I}$, $\Gamma(\boldsymbol{\alpha}(D \ominus x), \boldsymbol{\lambda}(D \ominus x), D \ominus x)$ has a unique NE that is given by $\boldsymbol{b}(\boldsymbol{\alpha}(D \ominus x), \boldsymbol{\lambda}(D \ominus x), D \ominus x)$. The planner's objective is to maximally reduce weighted aggregate equilibrium action by removing exactly one player from the game $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$. Formally, the planner solves

$$\text{KP-R}(\boldsymbol{\omega}, \alpha, \lambda, D) := \operatorname*{arg\,min}_{x\in\{y\in\mathcal{I} \mid \Delta Y^R(y,\boldsymbol{\omega},D)<0\}} \{x \mapsto \Delta Y^R(x, \boldsymbol{\omega}, D)\},$$

where $\Delta Y^R(x, \boldsymbol{\omega}, D)$ denotes the change in weighted aggregate equilibrium action caused by removing player x:[42]

$$\Delta Y^R(x, \boldsymbol{\omega}, D) := Y([\boldsymbol{\omega}]_{-x}, D \ominus x) - Y(\boldsymbol{\omega}, D).$$

A player in $\text{KP-R}(\boldsymbol{\omega}, \alpha, \lambda, D)$ is called a *key player* of KPP-R in $\Gamma(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D)$. Propositions 3.24 and 3.35 form the basis for solving a specific KPP-R. Expressions of $\Delta Y^R(x, \boldsymbol{\omega}, D)$ for the two cases discussed in Section 3.3.2.1 are given in Propositions 3.49 and 3.51.

Proposition 3.49 *If Conditions C-α and C-λ are satisfied, for all $x \in \mathcal{I}$, $\boldsymbol{\alpha}(D \ominus x) = [\boldsymbol{\alpha}(D \boxminus x)]_{-x}$ and $\boldsymbol{\lambda}(D \ominus x) = [\boldsymbol{\lambda}(D \boxminus x)]_{-x}$, and $\rho(\mathrm{diag}(|\boldsymbol{\lambda}(D)|)\dot{\boldsymbol{A}}(D)) < 1$, then, for all $x \in \mathcal{I}$, $\Delta Y^R(x, \boldsymbol{\omega}, D) = \Delta Y^I(x, \boldsymbol{\omega}, D) - \omega_x \alpha_x$.*

By removing player x from the game, his action changes from $b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ to 0. The magnitude of the difference between $\Delta Y^R(x, \boldsymbol{\omega}, D)$ and $-\,b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ is a measure of the extent to which a social multiplier is at work. If $\boldsymbol{\lambda}^- \neq \mathbf{0}_n$, then $\Delta Y^R(x, \boldsymbol{\omega}, D)$ may be negative, zero, or positive (see Example 3.53 in Section 3.3.2.3). If $\boldsymbol{\lambda}^- = \mathbf{0}_n$, then $\Delta Y^R(x, \boldsymbol{\omega}, D)$ is nonpositive. Results 3.45.2 to 3.45.6 are true also for $\Delta Y^R(x, \boldsymbol{\omega}, D)$ because

$$\Delta Y^R(x, \boldsymbol{\omega}, D) = -\omega_x\, b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) - b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) \sum_{j\in\mathcal{I}\setminus\{x\}} \omega_j S_{D,j,x}(\boldsymbol{\lambda})$$

42. In the terminology of graph theory, the additive inverse of $\Delta Y^R(x, \boldsymbol{\omega}, D)$ is a *vitality index* (Koschützki et al. 2005, Definition 3.6.1).

if Conditions C-α and C-λ are satisfied, $\boldsymbol{\lambda} \geq_c \mathbf{0}_n$, for all $x \in \mathcal{I}$, $\boldsymbol{\alpha}(D \ominus x) = [\boldsymbol{\alpha}(D \boxminus x)]_{-x}$ and $\boldsymbol{\lambda}(D \ominus x) = [\boldsymbol{\lambda}(D \boxminus x)]_{-x}$, and $\rho(\mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$. The condition of Result 3.45.1 can be weakened to $\omega_x > 0$.

Proposition 3.49 shows that KPP-I and KPP-R are closely related under the stated conditions. The difference between $\Delta Y^R(x, \boldsymbol{\omega}, D)$ and $\Delta Y^I(x, \boldsymbol{\omega}, D)$ is due to the facts that player x's action decreases from $b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ to α_x if he is being isolated and from $b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ to 0 if he is being removed from the game and the effects on all other players are the same in both cases (Proposition 3.35). Moreover, Proposition 3.49 implies that the two key player problems have the same key player(s) if the players of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ are α-homogeneous but not necessarily λ-homogeneous. This result is stated in Corollary 3.50.

Corollary 3.50 *If the players of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ are α-homogeneous, Condition C-λ is satisfied, for all $x \in \mathcal{I}$, $\boldsymbol{\lambda}(D \ominus x) = [\boldsymbol{\lambda}(D \boxminus x)]_{-x}$, and $\rho(\mathrm{diag}(|\boldsymbol{\lambda}(D)|)\dot{\boldsymbol{A}}(D)) < 1$, then* KP-I$(\boldsymbol{\omega}, \alpha, \lambda, D) =$ KP-R$(\boldsymbol{\omega}, \alpha, \lambda, D)$.

Proposition 3.51 shows that α-homogeneity is not a necessary condition for KPP-I and KPP-R to have the same key player(s).

Proposition 3.51 *If α is as in Example 3.2, λ is as in Example 3.5 with $\bar{\lambda} \geq 0$, and $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$, then, for all $x \in \mathcal{I}$, $\Delta Y^R(x, \boldsymbol{\omega}, D) = \Delta Y^I(x, \boldsymbol{\omega}, D)$.*

3.3.2.3 Examples

Examples 3.52 and 3.53 show that $\Delta Y^I(x, \boldsymbol{\omega}, D)$ and $\Delta Y^R(x, \boldsymbol{\omega}, D)$ can be negative, zero, or positive for some player x if $\boldsymbol{\lambda}^- \neq \mathbf{0}_n$. Example 3.54 shows that KPP-I and KPP-R are generally different in that they do not necessarily identify the same player(s) as the key player(s). Example 3.55 shows that a key player of KPP-I is not necessarily the player with the largest difference between action and marginal private benefit and a key player of KPP-R is not necessarily the player with the highest action. Example 3.56 shows that a key player problem may identify several players as key players. Example 3.57 shows that a local key player problem can be different from the corresponding global key player problem in that the two problems identify different key players.

Example 3.52 Suppose $D = (\{1, 2, \ldots, 7\}, \{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2), (3,4), (4,3), (4,5), (4,6), (4,7), (5,4), (5,6), (6,4), (6,5), (7,4)\})$. See Figure 3.4 for an illustration of D. In addition, suppose $\boldsymbol{\alpha}(D) = \boldsymbol{\alpha}(D \boxminus 7) = \mathbf{1}_7$ and $\boldsymbol{\lambda}(D) = \boldsymbol{\lambda}(D \boxminus 7) = (1/3)(1, 1, 1, 1, 1, 1, -c)$, where $c \in \{8/28, 9/28, 10/28\}$. Straightforward but tedious calculations show that

$$\Delta Y^I(7, \mathbf{1}_7, D) = \frac{56c - 18}{2c + 9}.$$

If follows that $\Delta Y^I(7, \mathbf{1}_7, D) < 0$ if $c = 8/28$, $\Delta Y^I(7, \mathbf{1}_7, D) = 0$ if $c = 9/28$, and $\Delta Y^I(7, \mathbf{1}_7, D) > 0$ if $c = 10/28$. $\diamond$

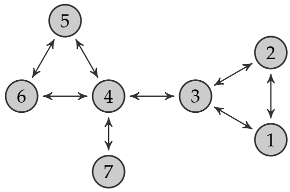


Figure 3.4. A symmetric digraph of order 7 and size 16 (Examples 3.52, 3.54, and 3.55)

Example 3.53 Consider the digraph D of Example 3.21. Suppose $\boldsymbol{\alpha}(D) = \mathbf{1}_4$, $\boldsymbol{\lambda}(D) = (1/6)(-2, -c, 6, -3)$, $\boldsymbol{\alpha}(D \ominus 3) = \mathbf{1}_3$, and $\boldsymbol{\lambda}(D \ominus 3) = (1/6)(-2, -c, -3)$, where $c \in \{2, 3, 4\}$. Straightforward but tedious calculations show that

$$\Delta Y^R(3, \mathbf{1}_4, D) = \frac{2c - 6}{15}.$$

If follows that $\Delta Y^R(3, \mathbf{1}_4, D) < 0$ if $c = 2$, $\Delta Y^R(3, \mathbf{1}_4, D) = 0$ if $c = 3$, and $\Delta Y^R(3, \mathbf{1}_4, D) > 0$ if $c = 4$. ◇

Example 3.54 Consider the digraph D of Example 3.52. Suppose Conditions C-α and C-λ are satisfied, $\boldsymbol{\alpha}(D) = (5, 5, 6, 5, 5, 5, 5)$ and $\boldsymbol{\lambda}(D) = (1/12)\mathbf{1}_7$, and for all $x \in \mathcal{I}$, $\boldsymbol{\alpha}(D \ominus x) = [\boldsymbol{\alpha}(D \boxminus x)]_{-x}$ and $\boldsymbol{\lambda}(D \ominus x) = [\boldsymbol{\lambda}(D \boxminus x)]_{-x}$. Straightforward calculations show that KP-I$(\mathbf{1}_7, \alpha, \lambda, D) = \{4\}$ and KP-R$(\mathbf{1}_7, \alpha, \lambda, D) = \{3\}$ (see Table 3.1), that is, the key player of the unweighted global KPP-I in $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ is player 4, and the key player of the unweighted global KPP-R in $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ is player 3. By isolating player 4, his action changes from $b_4(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) = 7.12$ to $\alpha_4 = 5$. This decrease by 2.12 is the largest among all players, which results in a decline in unweighted aggregate equilibrium action by 4.72. By removing player 3 from the game, his action changes from $b_3(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) = 7.62$ to 0. This decrease is the largest among all players and results in a decline in unweighted aggregate equilibrium action by 9.82. ◇

Example 3.55 Consider Example 3.54 with $\boldsymbol{\alpha}(D) = (2, 2, 3, 4, 2, 2, 1)$ and $\boldsymbol{\lambda}(D) = (1/10)(2, 2, 2, 1, 2, 2, 2)$. Straightforward calculations show that KP-I$(\mathbf{1}_7, \alpha, \lambda, D) =$ KP-R$(\mathbf{1}_7, \alpha, \lambda, D) = \{4\}$ (see Table 3.2), that is, player 4 is the key player of both unweighted global KPP-I and KPP-R in $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$. Note that player 4 is not the player with the largest difference between action and marginal private benefit or the player with the highest action, who, in both cases, is player 3. ◇

Example 3.56 Suppose $D = (\{1, 2, \ldots, 7\}, \{(1,2), (1,3), (2,1), (2,3), (3,4), (4,3), (5,4), (5,6), (6,4), (6,5), (7,4)\})$. See Figure 3.5 for an illustration of D. In addition, suppose Conditions C-α and C-λ are satisfied, $\boldsymbol{\alpha}(D) = (2, 2, 1, 1, 2, 2, 1)$, $\boldsymbol{\lambda}(D) = (1/3)\mathbf{1}_7$, and for all $x \in \mathcal{I}$, $\boldsymbol{\alpha}(D \ominus x) = [\boldsymbol{\alpha}(D \boxminus x)]_{-x}$ and $\boldsymbol{\lambda}(D \ominus x) = [\boldsymbol{\lambda}(D \boxminus x)]_{-x}$. Straightforward calculations yield KP-I$(\mathbf{1}_7, \alpha, \lambda, D) = \{4\}$ and KP-R$(\mathbf{1}_7, \alpha, \lambda, D) = \{1, 2, 5, 6\}$ (see Table 3.3), that is, the key player of the unweighted global KPP-I

Table 3.1. Unweighted global key player problems (Example 3.54)

x	α_x	$b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$	$b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) - \alpha_x$	$\Delta Y^I(x, \mathbf{1}_7, D)$	$\Delta Y^R(x, \mathbf{1}_7, D)$
1	5	6.15	1.15	−2.33	−7.33
2	5	6.15	1.15	−2.33	−7.33
3	6⋆	7.62⋆	1.62	−3.82	−9.82⋆
4	5	7.12	2.12⋆	−4.72⋆	−9.72
5	5	6.10	1.10	−2.32	−7.32
6	5	6.10	1.10	−2.32	−7.32
7	5	5.59	0.59	−1.20	−6.20

Notes: All real numbers in decimal notation are rounded to two decimal places. Numbers with a star indicate a column minimum or maximum.

Table 3.2. Unweighted global key player problems (Example 3.55)

x	α_x	$b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$	$b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) - \alpha_x$	$\Delta Y^I(x, \mathbf{1}_7, D)$	$\Delta Y^R(x, \mathbf{1}_7, D)$
1	2	3.92	1.92	−4.09	−6.09
2	2	3.92	1.92	−4.09	−6.09
3	3	5.68⋆	2.68⋆	−6.56	−9.56
4	4⋆	5.56	1.56	−7.30⋆	−11.30⋆
5	2	3.89	1.89	−3.53	−5.53
6	2	3.89	1.89	−3.53	−5.53
7	1	2.11	1.11	−1.53	−2.53

Notes: All real numbers in decimal notation are rounded to two decimal places. Numbers with a star indicate a column minimum or maximum.

Table 3.3. Unweighted global key player problems (Example 3.56)

x	α_x	$b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$	$b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) - \alpha_x$	$\Delta Y^I(x, \mathbf{1}_7, D)$	$\Delta Y^R(x, \mathbf{1}_7, D)$
1	2⋆	3.75⋆	1.75⋆	−3.00	−5.00⋆
2	2⋆	3.75⋆	1.75⋆	−3.00	−5.00⋆
3	1	1.50	0.50	−3.17	−4.17
4	1	1.50	0.50	−3.50⋆	−4.50
5	2⋆	3.75⋆	1.75⋆	−3.00	−5.00⋆
6	2⋆	3.75⋆	1.75⋆	−3.00	−5.00⋆
7	1	1.50	0.50	−0.50	−1.50

Notes: All real numbers in decimal notation are rounded to two decimal places. Numbers with a star indicate a column minimum or maximum.

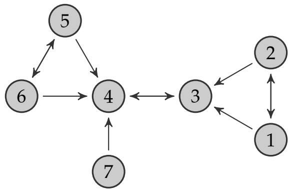


Figure 3.5. An asymmetric digraph of order 7 and size 11 (Examples 3.56)

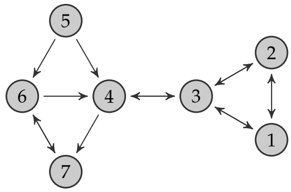


Figure 3.6. An asymmetric digraph of order 7 and size 14 (Example 3.57)

in $\mathit{\Gamma}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ is player 4, who is different from any of the key players of the unweighted global KPP-R in $\mathit{\Gamma}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$, players 1, 2, 5, and 6. ◇

Example 3.57 Suppose $D = (\{1, 2, \ldots, 7\}, \{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2), (3,4), (4,3), (4,7), (5,4), (5,6), (6,4), (6,7), (7,6)\})$. See Figure 3.6 for an illustration of D. Suppose $\boldsymbol{\omega} = (0,0,0,1,1,1,1)$. Let K denote the subdigraph of D induced by the subset of players $\mathcal{J} := \{i \in \mathcal{I} \mid \omega_i > 0\} = \{4,5,6,7\}$, that is, $K := D\langle \mathcal{J} \rangle$. Suppose Conditions C-$\alpha$ and C-λ are satisfied, $\boldsymbol{\alpha}(D) = (2,2,3,2,2,2,1)$, $\boldsymbol{\lambda}(D) = (1/3)\mathbf{1}_7$, for all $x \in \mathcal{I}$, $\boldsymbol{\alpha}(D \ominus x) = [\boldsymbol{\alpha}(D \boxminus x)]_{-x}$ and $\boldsymbol{\lambda}(D \ominus x) = [\boldsymbol{\lambda}(D \boxminus x)]_{-x}$, $\boldsymbol{\alpha}(K) = [\boldsymbol{\alpha}(D)]_{\mathcal{J}} = (2,2,2,1)$, $\boldsymbol{\lambda}(K) = [\boldsymbol{\lambda}(D)]_{\mathcal{J}} = (1/3)\mathbf{1}_4$, and for all $x \in \mathcal{I}(4)$, $\boldsymbol{\alpha}(K \ominus x) = [\boldsymbol{\alpha}(K \boxminus x)]_{-x}$ and $\boldsymbol{\lambda}(K \ominus x) = [\boldsymbol{\lambda}(K \boxminus x)]_{-x}$. Straightforward calculations show that $\text{KP-R}(\boldsymbol{\omega}, \alpha, \lambda, D) = \{4\}$ and $3 + \text{KP-R}(\mathbf{1}_4, \alpha, \lambda, K) = \{6\}$ (see Table 3.4), that is, the key player of the local KPP-R in $\mathit{\Gamma}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$, player 4, is different from the key player of the unweighted global KPP-R in $\mathit{\Gamma}([\boldsymbol{\alpha}]_{\mathcal{J}}, [\boldsymbol{\lambda}]_{\mathcal{J}}, K)$, player 6.[43] The key player of the unweighted global KPP-R in $\mathit{\Gamma}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ is player 3, who is also the player with the highest action in $\mathit{\Gamma}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ ($b_3(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) = 10.74$). If there were no arcs between players 3 and 4, then player 4 would play action 2.74 ($b_{4-3}([\boldsymbol{\alpha}]_{\mathcal{J}}, [\boldsymbol{\lambda}]_{\mathcal{J}}, K) = 2.74$). Because player 4's utility function exhibits strategic complements ($\lambda_4 > 0$), his action in $\mathit{\Gamma}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$, $b_4(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) = 6.47$, is higher than his action in $\mathit{\Gamma}([\boldsymbol{\alpha}]_{\mathcal{J}}, [\boldsymbol{\lambda}]_{\mathcal{J}}, K)$, $b_{4-3}([\boldsymbol{\alpha}]_{\mathcal{J}}, [\boldsymbol{\lambda}]_{\mathcal{J}}, K) = 2.74$. The difference $6.47 - 2.74 = 3.73$ represents the effect of the actions of players 1, 2, and 3 on player 4's action in $\mathit{\Gamma}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$. This effect is taken into account in the local KPP-R in $\mathit{\Gamma}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ but is completely ignored in the unweighted global KPP-R in $\mathit{\Gamma}([\boldsymbol{\alpha}]_{\mathcal{J}}, [\boldsymbol{\lambda}]_{\mathcal{J}}, K)$; as a result, the key player of the local KPP-R in $\mathit{\Gamma}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ is different from the key player of the unweighted global KPP-R in $\mathit{\Gamma}([\boldsymbol{\alpha}]_{\mathcal{J}}, [\boldsymbol{\lambda}]_{\mathcal{J}}, K)$. ◇

43. The set $3 + \text{KP-R}(\mathbf{1}_4, \alpha, \lambda, K)$ is equal to $\{3 + i \mid i \in \text{KP-R}(\mathbf{1}_4, \alpha, \lambda, K)\}$.

Table 3.4. Global and local key player problems (Example 3.57)

x	α_x	$b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$	$\Delta Y^I(x, \mathbf{1}_7, D)$	$\Delta Y^R(x, \mathbf{1}_7, D)$
Global key player problems in $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ with $\boldsymbol{\omega} = \mathbf{1}_7$				
1	2	8.37	−18.85	−20.85
2	2	8.37	−18.85	−20.85
3	3*	10.74*	−25.79*	−28.79*
4	2	6.47	−17.15	−19.15
5	2	5.84	−3.84	−5.84
6	2	5.05	−7.99	−9.99
7	1	2.68	−5.86	−6.86
Local key player problems in $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ with $\boldsymbol{\omega} = (0,0,0,1,1,1,1)$				
x	α_x	$b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$	$\Delta Y^I(x, \boldsymbol{\omega}, D)$	$\Delta Y^R(x, \boldsymbol{\omega}, D)$
4	2*	6.47*	−10.68*	−12.68*
5	2*	5.84	−3.84	−5.84
6	2*	5.05	−7.32	−9.32
7	1	2.68	−4.79	−5.79
Global key player problems in $\Gamma([\boldsymbol{\alpha}]_{\mathcal{J}}, [\boldsymbol{\lambda}]_{\mathcal{J}}, K)$ with $\boldsymbol{\omega} = \mathbf{1}_4$[a]				
x	α_x	$b_{x-3}([\boldsymbol{\alpha}]_{\mathcal{J}}, [\boldsymbol{\lambda}]_{\mathcal{J}}, K)$	$\Delta Y^I(x-3, \mathbf{1}_4, K)$	$\Delta Y^R(x-3, \mathbf{1}_4, K)$
4	2*	2.74	−3.36	−5.36
5	2*	4.13*	−2.13	−4.13
6	2*	3.65	−4.63*	−6.63*
7	1	2.22	−3.52	−4.52

Notes: All real numbers in decimal notation are rounded to two decimal places. Numbers with a star indicate a column minimum or maximum. [a] $K = D\langle \mathcal{J} \rangle$ with $\mathcal{J} = \{4, 5, 6, 7\}$.

3.4 Concluding remarks

The present work contributes to the literature on key player analysis by introducing the concept of the local key player in the context of a network game where the players are heterogeneous with respect to the marginal private benefit from taking an action and with respect to the network effects exerted on own behavior by the actions of connected players, of which the magnitude and sign can vary across players. The qualifier local refers thereby to the social planner's objective to reduce aggregate activity of only those players who reside in a certain local area or part of the network.

The proposed network game or, more precisely, its translation to a statistical model nests several prominent models used in the literature on the economics of social interactions to estimate peer or network effects, such as the local-aggregate model (Calvó-Armengol and Zenou 2004; Ballester, Calvó-Armengol, and Zenou 2006), the local-average model (Patacchini and Zenou 2012), and a network version of a model discussed in Mas and Moretti (2009) to estimate the extent to which the productivity of a worker depends on the productivity of co-workers in the same team. Recent research combined the local-aggregate and local-average models into a hybrid model to test which of the two is more adequate to describe a certain phenomenon (Liu, Patacchini, and Zenou 2014; Lindquist, Sauermann, and Zenou 2016). A translation of the proposed network game to an identifiable statistical model enables to test *at the individual level* to which extent network effects are driven by the aggregate or average action of an individual's neighbors. As a first step, future research needs to discuss suitable estimators of the parameters of such a statistical model.

Empirical work on key players in social networks is relatively scarce (see Zenou 2016 for a survey). The proposed refinement of the key player concept may contribute to the growth of this literature because it widens the field of application by adding a local flavor. An example is the design of compensation plans for employees according to their contribution to overall team productivity in a team organization where the teams form a universe of interconnected islands; specifically, the team member who would reduce team productivity most if he or she gave notice receives the highest salary. The local flavor may be an important feature when implementing key player based policies in the real world in the context of crime. It enables in particular the design of a key player based policing policy to fight crime locally in all police areas (or other geographical areas of political importance or interest) of a jurisdiction, thereby taking into account that criminal networks spread across area boundaries. Needless to say, the design of such a key player based policing policy is only possible if relevant data is available. Lindquist and Zenou (2014) demonstrate with their analysis of co-offending networks in Sweden that a policing policy based on the *global* key player concept is in fact implementable. They identify the key offender in each weakly connected component. They show that a key player based policing policy outperforms other reasonable policing policies such as targeting the most active criminals or targeting criminals who have the highest betweenness or eigenvector centrality in the network. Provided that their data can be complemented with spatial data on crime, for example, the average coordinate of the geographical locations of all crimes attributed to a criminal over a certain period of time, the implementation of a policing policy based on the *local* key player concept seems realistic.

Appendix A

Basic Concepts in Graph Theory

This appendix reviews basic concepts of the theory of digraphs (for a comprehensive introduction see, for example, Bang-Jensen and Gutin 2009).

Basic terminology

A *directed graph* (or *digraph* for short) D consists of a nonempty, finite set of elements called *vertices* and a finite set of ordered pairs of distinct vertices called *arcs*. The set of vertices of D is called the *vertex set* of D and is denoted by $\mathcal{V}(D)$. The set of arcs of D is called the *arc set* of D and is denoted by $\mathcal{A}(D)$. It follows that D is represented by the pair $(\mathcal{V}(D), \mathcal{A}(D))$.

The *order* of D is the cardinality of $\mathcal{V}(D)$. The *size* of D is the cardinality of $\mathcal{A}(D)$. The digraph D is called *empty* if $\mathcal{A}(D) = \emptyset$.

Suppose D is of order at least two and not empty. An arc (u, v) in D is directed from u to v, where u and v are called the *tail* and the *head* of (u, v), respectively. The definition of a digraph implies that D contains no multiple arcs, that is, pairs of arcs with the same head and the same tail, and no self-loops, that is, arcs whose head and tail are equal. Two vertices u and v in D are called *adjacent* in D if the arc (u, v) or the arc (v, u) is in D.

Symmetric and transitive digraph

The *transpose* of D, denoted by D^T, is defined as $(\mathcal{V}(D), \{(u, v) \mid (v, u) \in \mathcal{A}(D)\})$.

The digraph D is called *symmetric* if for all distinct vertices u and v in D, (u, v) is an arc in D if and only if (v, u) is an arc in D, that is, $D^\mathsf{T} = D$.

The digraph D is called *transitive* if for all pairs $((u, v), (v, w))$ of arcs in D with $u \neq w$, the arc (u, w) is also in D (see, for example, Bang-Jensen and Gutin 2009, Section 2.3).

Walk and path

Let x and y be two (not necessarily distinct) vertices in D, and let p be a positive integer. A *walk* in D of length p from x to y is a finite sequence $(v_0, v_1, \ldots, v_p)$ in $\mathcal{V}(D)$ of length $p + 1$ such that $v_0 = x$, $v_p = y$, and for all $k \in \{1, \ldots, p\}$,

$(v_{k-1}, v_k) \in \mathcal{A}(D)$. The *inverse* of the walk $(v_0, v_1, \ldots, v_p)$ in D is the finite sequence $(v_p, v_{p-1}, \ldots, v_0)$, which may or may not be a walk in D. A walk in D is called *path* if it contains no repeated vertices.

Weakly and strongly connected component

A digraph K is called *subdigraph* of D if $\mathcal{V}(K) \subset \mathcal{V}(D)$, $\mathcal{A}(K) \subset \mathcal{A}(D)$, and $\mathcal{A}(K) \subset \mathcal{V}(K)^2$. If K is a subdigraph of D, then D is called a *superdigraph* of K. A subdigraph K of D is called *complete* if for all pairs (u, v) of distinct vertices in K, both arcs (u, v) and (v, u) are in K. The *subdigraph of D induced by $L \subset \mathcal{V}(D)$*, denoted by $D\langle L\rangle$, is the unique subdigraph of D such that $\mathcal{V}(D\langle L\rangle) = L$ and every arc in D with both head and tail in L is in $D\langle L\rangle$. A subdigraph K of D is called *strongly connected* if $|\mathcal{V}(K)| = 1$ or else for all pairs (u, v) of distinct vertices in K, there exists a walk in K from u to v and a walk in K from v to u. A *strongly connected component* of D is a maximal induced subdigraph of D that is strongly connected. A *complete component* of D is a strongly connected component that is complete. The *complete biorientation* of D, denoted by $CB(D)$, is the unique digraph obtained from D by adding the arc (v, u) to D if (u, v) but not (v, u) is in D. A subdigraph K of D is called a *weakly connected component* of D if $CB(K)$ is a strongly connected component of $CB(D)$.

Distance and diameter

The *distance* between two vertices u and v in D, denoted by $\operatorname{dist}_D(u, v)$, is defined as follows: it is zero if $u = v$, else it is the length (that is, the number of arcs) of a shortest path in D from u to v if such a path exists and $+\infty$ otherwise. The *diameter* of D, denoted by $\operatorname{diam}(D)$, is defined as $\max\{\operatorname{dist}_D(u, v) \mid (u, v) \in \mathcal{V}(D)^2\}$.

Neighborhoods

Let u be a vertex in D, and let r be a positive integer. The *in-neighborhood* of u (in D) is the set $\mathcal{N}_D^-(u) := \{v \in \mathcal{V}(D) \mid (v, u) \in \mathcal{A}(D)\}$, and the *out-neighborhood* of u (in D) is the set $\mathcal{N}_D^+(u) := \{v \in \mathcal{V}(D) \mid (u, v) \in \mathcal{A}(D)\}$. The vertices of $\mathcal{N}_D^-(u)$ and $\mathcal{N}_D^+(u)$ are called *in-neighbors* and *out-neighbors* of u (in D), respectively. The *in-degree* of u (in D) is defined by $\deg_D^-(u) := |\mathcal{N}_D^-(u)|$ and the *out-degree* of u (in D) by $\deg_D^+(u) := |\mathcal{N}_D^+(u)|$. The vertex u is called *isolated* (in D) if $\mathcal{N}_D^-(u) = \mathcal{N}_D^+(u) = \emptyset$. The *out-neighborhood of order r* of u (in D), denoted by $\mathcal{N}_{D,r}^+(u)$, is defined recursively by

$$\begin{aligned} &\mathcal{N}_{D,1}^+(u) := \mathcal{N}_D^+(u), \\ \forall r > 1 \quad &\mathcal{N}_{D,r}^+(u) := \bigcup_{v \in \mathcal{N}_{D,r-1}^+(u)} \mathcal{N}_D^+(v). \end{aligned}$$

The vertices of the set $\bigcup_{r \in \mathbb{Z}_{++} \setminus \{1\}} \mathcal{N}_{D,r}^+(u)$ are called *higher-order out-neighbors* of u (in D). The *in-neighborhood of order r* of u (in D), which is denoted by $\mathcal{N}_{D,r}^-(u)$, and the *higher-order in-neighbors* of u (in D) are defined analogously.

Adjacency matrix

A digraph D of order $N > 1$ can be represented by a square matrix of order N. This can be seen as follows. Let $h\colon \mathcal{V}(D) \to \{1, \ldots, N\}$ be a bijection. By means of h, the vertex set $\mathcal{V}(D)$ can be identified with the set $\{1, \ldots, N\}$ and the arc set $\mathcal{A}(D)$ with a subset of $\{1, \ldots, N\}^2$, namely, $\mathcal{A}_h(D) := \{(h(u), h(v)) \mid (u, v) \in \mathcal{A}(D)\}$. It follows that h is a digraph isomorphism from $(\mathcal{V}(D), \mathcal{A}(D))$ to $(\{1, \ldots, N\}, \mathcal{A}_h(D))$, that is, h is an arc-preserving bijection. The *adjacency matrix* of D with respect to h, denoted by $\dot{\boldsymbol{A}}_h(D)$, is the square matrix of order N with the component in row i and column j equal to one if $(i, j) \in \mathcal{A}_h(D)$ and zero else. Note that $\dot{\boldsymbol{A}}_h(D)$ is different from $\boldsymbol{O}_N$, the zero matrix of order N, if D is not empty. Also note that all components on the main diagonal of $\dot{\boldsymbol{A}}_h(D)$ vanish because D has no self-loops. If $\mathcal{V}(D) \subset \mathbb{R}$ and h is the unique order isomorphism, then $\dot{\boldsymbol{A}}_h(D)$ is written as $\dot{\boldsymbol{A}}(D)$.[1]

1. An order isomorphism is an order-preserving bijection. For example, if $\mathcal{V}(D) = \{1, 3, 4, 7, 8\}$, then $h\colon \mathcal{V}(D) \to \{1, 2, 3, 4, 5\}$ is given by $h(1) = 1$, $h(3) = 2$, $h(4) = 3$, $h(7) = 4$, and $h(8) = 5$.

Appendix B

Basic Results in Matrix Analysis

This appendix contains a collection of basic results in matrix analysis referenced either in the text or in the proofs of the main results. Lemma B.2, Result B.3.1 of Lemma B.3, Lemma B.5, Result B.6.1 of Lemma B.6, and Lemmata B.7, B.8, and B.9 are well known in the literature. Let $n \in \mathbb{Z}_{++} \setminus \{1\}$.

Lemma B.1 *Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, and let $\boldsymbol{A} \in \mathcal{M}(n, \mathbb{R})$.*

(*B.1.1*) *If $\boldsymbol{x} \leq_c \boldsymbol{y}$ and $\boldsymbol{O}_n \leq_c \boldsymbol{A}$, then $\boldsymbol{Ax} \leq_c \boldsymbol{Ay}$.*

(*B.1.2*) *If $\boldsymbol{x} <_c \boldsymbol{y}$ and $\boldsymbol{I}_n \leq_c \boldsymbol{A}$, then $\boldsymbol{Ax} <_c \boldsymbol{Ay}$.*

Proof Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, and let $\boldsymbol{A} \in \mathcal{M}(n, \mathbb{R})$. Let $\boldsymbol{z} := \boldsymbol{x} - \boldsymbol{y}$.

Proof of Result B.1.1 Suppose $\boldsymbol{z} \leq_c \boldsymbol{0}_n$ and $\boldsymbol{O}_n \leq_c \boldsymbol{A}$. We find $\boldsymbol{Ax} = \boldsymbol{Ay} + \boldsymbol{Az} \leq_c \boldsymbol{Ay}$ because $\boldsymbol{z} \leq_c \boldsymbol{0}_n$ and $\boldsymbol{O}_n \leq_c \boldsymbol{A}$ imply that $\boldsymbol{Az} \leq_c \boldsymbol{0}_n$. ■

Proof of Result B.1.2 Suppose $\boldsymbol{z} <_c \boldsymbol{0}_n$ and $\boldsymbol{I}_n \leq_c \boldsymbol{A}$. We find $\boldsymbol{Ax} = \boldsymbol{Ay} + \boldsymbol{Az} <_c \boldsymbol{Ay}$ because $\boldsymbol{z} <_c \boldsymbol{0}_n$ and $\boldsymbol{I}_n \leq_c \boldsymbol{A}$ imply that $\boldsymbol{Az} <_c \boldsymbol{0}_n$. ■

Lemma B.2 *Let $\boldsymbol{A} \in \mathcal{M}(n, \mathbb{R})$ with $\boldsymbol{A} \neq \boldsymbol{O}_n$. The Neumann series $\sum_{k=0}^{\infty} \boldsymbol{A}^k$ converges (strongly) if and only if $\rho(\boldsymbol{A}) < 1$. If $\sum_{k=0}^{\infty} \boldsymbol{A}^k$ converges (strongly), then $\boldsymbol{I}_n - \boldsymbol{A}$ is nonsingular with inverse $\sum_{k=0}^{\infty} \boldsymbol{A}^k$.*

Proof See, for example, Meyer (2000, pp. 618–19) or Frommer (1990, Satz A.2.2). ■

Lemma B.3 *Let $c \in \mathbb{R} \setminus \{0\}$, and let $\boldsymbol{A} \in \mathcal{M}(n, \mathbb{R})$ with $\boldsymbol{A} \neq \boldsymbol{O}_n$.*

(*B.3.1*) *The matrix $\boldsymbol{I}_n - c\boldsymbol{A}$ is nonsingular if and only if $1 \notin \sigma(c\boldsymbol{A})$.*

(*B.3.2*) *If $|c|\rho(\boldsymbol{A}) < 1$, then, for all $p \in \mathbb{Z}_{++}$, the Neumann series $\sum_{k=0}^{\infty} c^{pk} \boldsymbol{A}^{pk}$ converges (strongly) and $\boldsymbol{I}_n - c^p \boldsymbol{A}^p$ is nonsingular with inverse $\sum_{k=0}^{\infty} c^{pk} \boldsymbol{A}^{pk}$.*

Proof Let $c \in \mathbb{R} \setminus \{0\}$, and let $\boldsymbol{A} \in \mathcal{M}(n, \mathbb{R})$ with $\boldsymbol{A} \neq \boldsymbol{O}_n$.

Proof of Result B.3.1 First, note that $\boldsymbol{I}_n - c\boldsymbol{A}$ is nonsingular if and only if $(1/c)\boldsymbol{I}_n - \boldsymbol{A}$ is nonsingular because $\boldsymbol{I}_n - c\boldsymbol{A} = c((1/c)\boldsymbol{I}_n - \boldsymbol{A})$. Second, note that, by the definition of $\sigma(\boldsymbol{A})$, $(1/c)\boldsymbol{I}_n - \boldsymbol{A}$ is nonsingular if and only if $(1/c) \notin \sigma(\boldsymbol{A})$. Finally, note that $(1/c) \notin \sigma(\boldsymbol{A})$ is equivalent to $1 \notin \sigma(c\boldsymbol{A})$. ■

Proof of Result B.3.2 Suppose $|c|\rho(\boldsymbol{A}) < 1$. Let $p \in \mathbb{Z}_{++}$. First, note that $|c|\rho(\boldsymbol{A}) < 1$ is equivalent to $(|c|\rho(\boldsymbol{A}))^p < 1$. Second, note that, for all $a \in \mathbb{R}$, $\rho(a\boldsymbol{A}) = |a|\rho(\boldsymbol{A})$ because $\sigma(a\boldsymbol{A}) = a\sigma(\boldsymbol{A})$. Third, note that $\rho(\boldsymbol{A}^p) = \rho(\boldsymbol{A})^p$, which is a consequence of the *spectral mapping theorem* (see, for example, Kubrusly 2011, Theorem 6.19 and Corollary 6.20). Using the preceding results, we find $\rho(c^p\boldsymbol{A}^p) = |c^p|\rho(\boldsymbol{A}^p) = |c|^p\rho(\boldsymbol{A})^p = (|c|\rho(\boldsymbol{A}))^p$, which implies that $(|c|\rho(\boldsymbol{A}))^p < 1$ is equivalent to $\rho(c^p\boldsymbol{A}^p) < 1$. Finally, note that, according to Lemma B.2, the Neumann series $\sum_{k=0}^{\infty} c^{pk}\boldsymbol{A}^{pk}$ converges (strongly) and $\boldsymbol{I}_n - c^p\boldsymbol{A}^p$ is nonsingular with inverse $\sum_{k=0}^{\infty} c^{pk}\boldsymbol{A}^{pk}$. ■

Lemma B.4 *Let $(c_1, c_2) \in \mathbb{R}^2$, and let $\boldsymbol{A} \in \mathcal{M}(n, \mathbb{R})$. Suppose $1 \notin \sigma(c_2\boldsymbol{A})$. The two matrices $\boldsymbol{I}_n - c_1\boldsymbol{A}$ and $(\boldsymbol{I}_n - c_2\boldsymbol{A})^{-1}$ commute.*

Proof Let $(c_1, c_2) \in \mathbb{R}^2$, and let $\boldsymbol{A} \in \mathcal{M}(n, \mathbb{R})$. Suppose $1 \notin \sigma(c_2\boldsymbol{A})$. The statement is trivial if $c_2 = 0$. Suppose $c_2 \neq 0$ in what follows. Premultiplying (respectively, postmultiplying) both sides of the identity $\boldsymbol{I}_n = (\boldsymbol{I}_n - c_2\boldsymbol{A}) + c_2\boldsymbol{A}$ by $(\boldsymbol{I}_n - c_2\boldsymbol{A})^{-1}$ gives $(\boldsymbol{I}_n - c_2\boldsymbol{A})^{-1} = \boldsymbol{I}_n + c_2(\boldsymbol{I}_n - c_2\boldsymbol{A})^{-1}\boldsymbol{A}$ (respectively, $(\boldsymbol{I}_n - c_2\boldsymbol{A})^{-1} = \boldsymbol{I}_n + c_2\boldsymbol{A}(\boldsymbol{I}_n - c_2\boldsymbol{A})^{-1}$), from which it follows that $(\boldsymbol{I}_n - c_2\boldsymbol{A})^{-1}\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{I}_n - c_2\boldsymbol{A})^{-1}$. Using this result, we find $(\boldsymbol{I}_n - c_1\boldsymbol{A})(\boldsymbol{I}_n - c_2\boldsymbol{A})^{-1} = (\boldsymbol{I}_n - c_2\boldsymbol{A})^{-1} - c_1\boldsymbol{A}(\boldsymbol{I}_n - c_2\boldsymbol{A})^{-1}$ $= (\boldsymbol{I}_n - c_2\boldsymbol{A})^{-1} - c_1(\boldsymbol{I}_n - c_2\boldsymbol{A})^{-1}\boldsymbol{A} = (\boldsymbol{I}_n - c_2\boldsymbol{A})^{-1}(\boldsymbol{I}_n - c_1\boldsymbol{A})$, that is, $\boldsymbol{I}_n - c_1\boldsymbol{A}$ and $(\boldsymbol{I}_n - c_2\boldsymbol{A})^{-1}$ commute. ■

Lemma B.5 (Perron 1907; Frobenius 1912) *Let $\boldsymbol{A} \in \mathcal{M}(n, \mathbb{R})$ be nonnegative.*

(B.5.1) The matrix $\boldsymbol{A}$ has a nonnegative real eigenvalue that is equal to its spectral radius, that is, $\rho(\boldsymbol{A}) \in \sigma(\boldsymbol{A})$.

(B.5.2) To the eigenvalue $\rho(\boldsymbol{A})$ of $\boldsymbol{A}$ there corresponds a nonnegative eigenvector, that is, there exists an $\boldsymbol{x} \in \mathbb{R}^n_+ \setminus \{\boldsymbol{0}_n\}$ with $\boldsymbol{A}\boldsymbol{x} = \rho(\boldsymbol{A})\boldsymbol{x}$.

Proof See, for example, Varga (2000, Theorem 2.20). ■

Lemma B.6 *Let $c \in \mathbb{R}_+$, and let $\boldsymbol{A} \in \mathcal{M}(n, \mathbb{R})$ be nonnegative.*

(B.6.1) The matrix $\boldsymbol{I}_n - c\boldsymbol{A}$ is nonsingular with a nonnegative inverse if and only if $c\rho(\boldsymbol{A}) < 1$.

(B.6.2) If $c\rho(\boldsymbol{A}) < 1$, then $\boldsymbol{I}_n \leq_c (\boldsymbol{I}_n - c\boldsymbol{A})^{-1}$.

(B.6.3) If $c > 0$, $c\rho(\boldsymbol{A}) < 1$, and $\boldsymbol{A}$ is irreducible, then $\boldsymbol{O}_n <_c (\boldsymbol{I}_n - c\boldsymbol{A})^{-1}$.

Proof Let $c \in \mathbb{R}_+$, and let $\boldsymbol{A} \in \mathcal{M}(n, \mathbb{R})$ be nonnegative. Results B.6.1 and B.6.2 are trivial if $c = 0$. Suppose $c > 0$ in what follows.

Proof of Result B.6.1 First, note that $\boldsymbol{I}_n - c\boldsymbol{A}$ is nonsingular with nonnegative inverse if and only if $(1/c)\boldsymbol{I}_n - \boldsymbol{A}$ is nonsingular with nonnegative inverse because $c > 0$ and $\boldsymbol{I}_n - c\boldsymbol{A} = c((1/c)\boldsymbol{I}_n - \boldsymbol{A})$. Second, note that $(1/c)\boldsymbol{I}_n - \boldsymbol{A}$ is an M-matrix if and only if $(1/c) \geq \rho(\boldsymbol{A})$ (for the definition of M-matrices see, for example, Berman and Plemmons 1994, Definition 1.2 on p. 133). Third, note that $(1/c)\boldsymbol{I}_n - \boldsymbol{A}$ is singular if $1/c = \rho(\boldsymbol{A})$. Indeed, if $1/c = \rho(\boldsymbol{A})$, then $(1/c) \in \sigma(\boldsymbol{A})$ (Lemma B.5). Thus, $(1/c)\boldsymbol{I}_n - \boldsymbol{A}$ is a nonsingular M-matrix if and only if $(1/c) > \rho(\boldsymbol{A})$. Fourth, note that $(1/c)\boldsymbol{I}_n - \boldsymbol{A}$ is nonsingular with nonnegative inverse if and only if $(1/c)\boldsymbol{I}_n - \boldsymbol{A}$ is a nonsingular M-matrix (see, for example, Theorem 2.3 on pp. 134–38, in particular Condition N_{38}). The foregoing results imply that $\boldsymbol{I}_n - c\boldsymbol{A}$ is nonsingular with nonnegative inverse if and only if $1/c > \rho(\boldsymbol{A})$ or, equivalently, $c\rho(\boldsymbol{A}) < 1$. ■

Proof of Result B.6.2 Suppose $c\rho(\boldsymbol{A}) < 1$. Note that $\boldsymbol{I}_n - c\boldsymbol{A}$ is nonsingular with nonnegative inverse (Result B.6.1). Postmultiplying both sides of $\boldsymbol{I}_n = (\boldsymbol{I}_n - c\boldsymbol{A}) + c\boldsymbol{A}$ by $(\boldsymbol{I}_n - c\boldsymbol{A})^{-1}$ gives $(\boldsymbol{I}_n - c\boldsymbol{A})^{-1} = \boldsymbol{I}_n + c\boldsymbol{A}(\boldsymbol{I}_n - c\boldsymbol{A})^{-1}$, which implies that $\boldsymbol{I}_n \leq_c (\boldsymbol{I}_n - c\boldsymbol{A})^{-1}$ because $c > 0$ and both $\boldsymbol{A}$ and $(\boldsymbol{I}_n - c\boldsymbol{A})^{-1}$ are nonnegative. ■

Proof of Result B.6.3 Suppose $c\rho(\boldsymbol{A}) < 1$ and $\boldsymbol{A}$ is irreducible. Note that $\boldsymbol{I}_n - c\boldsymbol{A}$ is nonsingular with nonnegative inverse (Result B.6.1). For all $(i,j) \in \{1,\ldots,n\}^2$, there exists an $m_{i,j} \in \mathbb{Z}_{++}$ such that $[\boldsymbol{A}^{m_{i,j}}]_{i,j} > 0$ because $\boldsymbol{A}$ is nonnegative and irreducible (see, for example, Berman and Plemmons 1994, Theorem 2.1 on p. 29). Let $m := \max\{m_{i,j} \mid (i,j) \in \{1,\ldots,n\}^2\}$. According to the proof of Result B.6.2, $(\boldsymbol{I}_n - c\boldsymbol{A})^{-1} = \boldsymbol{I}_n + c\boldsymbol{A}(\boldsymbol{I}_n - c\boldsymbol{A})^{-1}$, from which

$$(\boldsymbol{I}_n - c\boldsymbol{A})^{-1} = \sum_{k=0}^{m} c^k \boldsymbol{A}^k + c^{m+1}\boldsymbol{A}^{m+1}(\boldsymbol{I}_n - c\boldsymbol{A})^{-1}$$

follows by recursive substitution. We find, for all $(i,j) \in \{1,\ldots,n\}^2$,

$$[(\boldsymbol{I}_n - c\boldsymbol{A})^{-1}]_{i,j} = \sum_{k=0}^{m} c^k [\boldsymbol{A}^k]_{i,j} + c^{m+1}[\boldsymbol{A}^{m+1}(\boldsymbol{I}_n - c\boldsymbol{A})^{-1}]_{i,j} \geq c^{m_{i,j}}[\boldsymbol{A}^{m_{i,j}}]_{i,j} > 0$$

because $c > 0$ and both $\boldsymbol{A}$ and $(\boldsymbol{I}_n - c\boldsymbol{A})^{-1}$ are nonnegative. ■

Lemma B.7 *For any sub-multiplicative matrix norm* $\|\cdot\|$ *on* $\mathcal{M}(n,\mathbb{C})$ *and for any* $\boldsymbol{A} \in \mathcal{M}(n,\mathbb{C})$, $\rho(\boldsymbol{A}) \leq \|\boldsymbol{A}\|$.

Proof The proof follows the lines in Meyer (2000, Example 7.1.4). Let $\|\cdot\|$ be a sub-multiplicative matrix norm on $\mathcal{M}(n,\mathbb{C})$, and let $\boldsymbol{A} \in \mathcal{M}(n,\mathbb{C})$. I show that

$$\forall \lambda \in \sigma(\boldsymbol{A}) \quad |\lambda| \leq \|\boldsymbol{A}\|, \tag{B.1}$$

from which $\rho(\boldsymbol{A}) \leq \|\boldsymbol{A}\|$ follows. Let $\lambda \in \sigma(\boldsymbol{A})$ with associated eigenvector $\boldsymbol{v} \in \mathbb{C}^n$. By the definition of an eigenvector, $\boldsymbol{v} \neq \boldsymbol{0}_n$. Let $\boldsymbol{B} \in \mathcal{M}(n,\mathbb{C})$ with $[\boldsymbol{B}]_{\bullet,1} = \boldsymbol{v}$ and for all $j \in \{2,3,\ldots,n\}$, $[\boldsymbol{B}]_{\bullet,j} = \boldsymbol{0}_n$. We find $\lambda\boldsymbol{B} = \boldsymbol{A}\boldsymbol{B}$ and $|\lambda|\|\boldsymbol{B}\| = \|\lambda\boldsymbol{B}\| = \|\boldsymbol{A}\boldsymbol{B}\| \leq \|\boldsymbol{A}\|\|\boldsymbol{B}\|$, which is equivalent to $|\lambda| \leq \|\boldsymbol{A}\|$ because $\|\boldsymbol{B}\| > 0$ ($\boldsymbol{v} \neq \boldsymbol{0}_n$ implies that $\boldsymbol{B} \neq \boldsymbol{O}_n$). ■

Lemma B.8 *If $\boldsymbol{A} \in \mathcal{M}(n, \mathbb{R})$ is nonnegative and row-normalized, then $\rho(\boldsymbol{A}) = 1$.*

Proof Let $\boldsymbol{A} \in \mathcal{M}(n, \mathbb{R})$ be nonnegative and row-normalized, where $\boldsymbol{A} \in \mathcal{M}(n, \mathbb{C})$ by the canonical embedding $\mathbb{R} \hookrightarrow \mathbb{C}$. Recall that the maximum absolute row sum norm $\|\cdot\|_\infty$ on $\mathcal{M}(n, \mathbb{C})$ is sub-multiplicative. First, I show that $\rho(\boldsymbol{A}) \leq 1$. We find $\|\boldsymbol{A}\|_\infty = \max\{\sum_{j=1}^{n} |[\boldsymbol{A}]_{ij}| \mid i \in \{1, \ldots, n\}\} = 1$ because $\boldsymbol{A}$ is nonnegative and row-normalized. Lemma B.7 implies that $\rho(\boldsymbol{A}) \leq 1$. Second, I show that $\rho(\boldsymbol{A}) \geq 1$. We find $\boldsymbol{A}\boldsymbol{1}_n = 1\boldsymbol{1}_n$ because $\boldsymbol{A}$ is row-normalized, that is, 1 is an eigenvalue of $\boldsymbol{A}$ with corresponding eigenvector $\boldsymbol{1}_n$. It follows that $\rho(\boldsymbol{A}) \geq 1$. ■

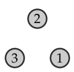
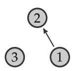
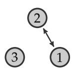
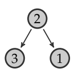
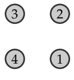
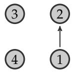
Lemma B.9 *For all $\boldsymbol{A} \in \mathcal{M}(n, \mathbb{R})$, $\rho(-\boldsymbol{A}) = \rho(\boldsymbol{A})$.*

Proof Let $\boldsymbol{A} \in \mathcal{M}(n, \mathbb{R})$. According to the Gelfand–Beurling formula (see, for example, Kubrusly 2011, Proposition 6.21), for any consistent matrix norm $\|\cdot\|$ on $\mathcal{M}(n, \mathbb{R})$, $\rho(\boldsymbol{A}) = \lim_{k\to\infty} \|\boldsymbol{A}^k\|^{1/k}$, from which it follows that $\rho(-\boldsymbol{A}) = \rho(\boldsymbol{A})$. ■

Appendix C

Tables

Table C.1. Non-isomorphic digraphs D of order $n \in \{3, 4\}$ with geometric multiplicity g.m.$(1, \bar{\boldsymbol{A}}(D)) > 1$

Digraph D	$\lvert\mathcal{A}(D)\rvert^{a}$ $\lvert\mathcal{I}_0(D)\rvert^{e}$	$\lvert[D]\rvert^{b}$	$\mathrm{Eig}(1, \bar{\boldsymbol{A}}(D))^{c}$	g.m.$(1, \bar{\boldsymbol{A}}(D))^{d}$ g.m.$(1, \bar{\boldsymbol{A}}(D)) - \lvert\mathcal{I}_0(D)\rvert$
$n = 3$				
	0 3	1	$\boldsymbol{e}_1, \boldsymbol{e}_2, \mathbf{1}_3$	3 0
	1 1	6	$\boldsymbol{e}_3, \mathbf{1}_3$	2 1
	2 1	3	$\boldsymbol{e}_3, \mathbf{1}_3$	2 1
	2 0	3	$\boldsymbol{e}_1 - \boldsymbol{e}_3, \mathbf{1}_3$	2 2
$n = 4$				
	0 4	1	$\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \mathbf{1}_4$	4 0
	1 2	12	$\boldsymbol{e}_3, \boldsymbol{e}_4, \mathbf{1}_4$	3 1

[a] The size of D, that is, the number of arcs of D. [b] The cardinality of the isomorphism class $[D]$ of D. [c] A basis for $\mathrm{Eig}(1, \bar{\boldsymbol{A}}(D))$, the eigenspace of $\bar{\boldsymbol{A}}(D)$ associated with the eigenvalue 1. The standard basis for $\mathbb{R}^n$ is given by $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$. The common "dimension" of the standard basis vectors is entirely omitted in this notion and must be inferred from the context. [d] The geometric multiplicity of the eigenvalue 1 of $\bar{\boldsymbol{A}}(D)$, that is, the dimension of $\mathrm{Eig}(1, \bar{\boldsymbol{A}}(D))$. [e] The number of isolated vertices (players) of D.

Table C.1. Non-isomorphic digraphs D of order $n \in \{3,4\}$ with geometric multiplicity g.m.$(1, \bar{\boldsymbol{A}}(D)) > 1$ (continued)

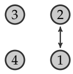
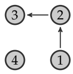
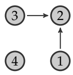
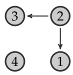
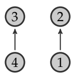
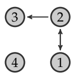
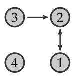
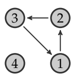
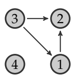
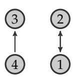
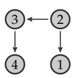
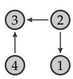
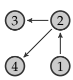
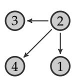
Digraph D	$\lvert\mathcal{A}(D)\rvert$ $\lvert\mathcal{I}_0(D)\rvert$	$\lvert[D]\rvert$	$\mathrm{Eig}(1, \bar{\boldsymbol{A}}(D))$	g.m.$(1, \bar{\boldsymbol{A}}(D))$ g.m.$(1, \bar{\boldsymbol{A}}(D)) - \lvert\mathcal{I}_0(D)\rvert$
	2 2	6	$\boldsymbol{e}_3, \boldsymbol{e}_4, \mathbf{1}_4$	3 1
	2 1	24	$\boldsymbol{e}_4, \mathbf{1}_4$	2 1
	2 1	12	$\boldsymbol{e}_4, \mathbf{1}_4$	2 1
	2 1	12	$\boldsymbol{e}_1 - \boldsymbol{e}_3, \boldsymbol{e}_4, \mathbf{1}_4$	3 2
	2 0	24	$\boldsymbol{e}_1 + \boldsymbol{e}_2, \mathbf{1}_4$	2 2
	3 1	24	$\boldsymbol{e}_4, \mathbf{1}_4$	2 1
	3 1	24	$\boldsymbol{e}_4, \mathbf{1}_4$	2 1
	3 1	8	$\boldsymbol{e}_4, \mathbf{1}_4$	2 1
	3 1	24	$\boldsymbol{e}_4, \mathbf{1}_4$	2 1
	3 0	12	$\boldsymbol{e}_1 + \boldsymbol{e}_2, \mathbf{1}_4$	2 2
	3 0	24	$2\boldsymbol{e}_1 + \boldsymbol{e}_2, \mathbf{1}_4$	2 2
	3 0	24	$2\boldsymbol{e}_1 + \boldsymbol{e}_2, \mathbf{1}_4$	2 2
	3 0	12	$\boldsymbol{e}_3 - \boldsymbol{e}_4, \mathbf{1}_4$	2 2
	3 0	4	$\boldsymbol{e}_1 - \boldsymbol{e}_3, \boldsymbol{e}_3 - \boldsymbol{e}_4, \mathbf{1}_4$	3 3

Table C.1. Non-isomorphic digraphs D of order $n \in \{3,4\}$ with geometric multiplicity g.m.$(1, \bar{\boldsymbol{A}}(D)) > 1$ (continued)

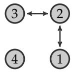
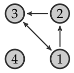
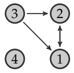
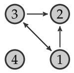
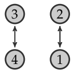
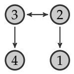
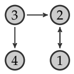
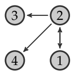
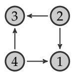
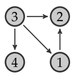
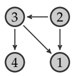
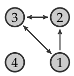
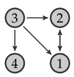
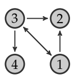
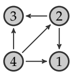
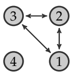
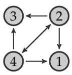
Digraph D	$\lvert\mathcal{A}(D)\rvert$ $\lvert\mathcal{I}_0(D)\rvert$	$\lvert[D]\rvert$	Eig$(1, \bar{\boldsymbol{A}}(D))$	g.m.$(1, \bar{\boldsymbol{A}}(D))$ g.m.$(1, \bar{\boldsymbol{A}}(D)) - \lvert\mathcal{I}_0(D)\rvert$
	4 1	12	$\boldsymbol{e}_4, \boldsymbol{1}_4$	2 1
	4 1	24	$\boldsymbol{e}_4, \boldsymbol{1}_4$	2 1
	4 1	12	$\boldsymbol{e}_4, \boldsymbol{1}_4$	2 1
	4 1	12	$\boldsymbol{e}_4, \boldsymbol{1}_4$	2 1
	4 0	3	$\boldsymbol{e}_1 + \boldsymbol{e}_2, \boldsymbol{1}_4$	2 2
	4 0	12	$3\boldsymbol{e}_1 + 2\boldsymbol{e}_2 + \boldsymbol{e}_3, \boldsymbol{1}_4$	2 2
	4 0	24	$2\boldsymbol{e}_1 + 2\boldsymbol{e}_2 + \boldsymbol{e}_3, \boldsymbol{1}_4$	2 2
	4 0	12	$\boldsymbol{e}_3 - \boldsymbol{e}_4, \boldsymbol{1}_4$	2 2
	4 0	6	$\boldsymbol{e}_1 - \boldsymbol{e}_3, \boldsymbol{1}_4$	2 2
	4 0	16	$\boldsymbol{e}_1 + \boldsymbol{e}_2 - 2\boldsymbol{e}_4, \boldsymbol{1}_4$	2 2
	4 0	24	$2\boldsymbol{e}_1 + \boldsymbol{e}_2 - 2\boldsymbol{e}_4, \boldsymbol{1}_4$	2 2
	5 1	24	$\boldsymbol{e}_4, \boldsymbol{1}_4$	2 1
	5 0	12	$\boldsymbol{e}_1 + \boldsymbol{e}_2 - 2\boldsymbol{e}_4, \boldsymbol{1}_4$	2 2
	5 0	24	$\boldsymbol{e}_1 + 2\boldsymbol{e}_2 - 3\boldsymbol{e}_4, \boldsymbol{1}_4$	2 2

Table C.1. Non-isomorphic digraphs D of order $n \in \{3,4\}$ with geometric multiplicity $\mathrm{g.m.}(1, \bar{\boldsymbol{A}}(D)) > 1$ (continued)

Digraph D	$\lvert\mathcal{A}(D)\rvert$ $\lvert\mathcal{I}_0(D)\rvert$	$\lvert[D]\rvert$	$\mathrm{Eig}(1, \bar{\boldsymbol{A}}(D))$	$\mathrm{g.m.}(1, \bar{\boldsymbol{A}}(D))$ $\mathrm{g.m.}(1, \bar{\boldsymbol{A}}(D)) - \lvert\mathcal{I}_0(D)\rvert$
	5 0	12	$\boldsymbol{e}_1 - \boldsymbol{e}_3, \mathbf{1}_4$	2 2
	6 1	4	$\boldsymbol{e}_4, \mathbf{1}_4$	2 1
	6 0	6	$\boldsymbol{e}_1 - \boldsymbol{e}_3, \mathbf{1}_4$	2 2

Appendix D

Proofs of Chapter 1 Results

Proof of Lemma 1.1

See, for example, Amann and Escher (2005, Theorem 5.7 on p. 274, Corollary 1.2 on p. 302, and Theorem 1.8 on p. 306). ■

Example 1.3

Let the function $d\colon \mathbb{R}^2_{++} \to \mathbb{R}$ be defined by $d(x,y) := (\log(x/y))^2$. It is straightforward to show that d is a semimetric on $\mathbb{R}_{++}$.

Let $(x,y) \in \mathbb{R}^2_{++}$. We find

$$\frac{\partial d(x,y)}{\partial x} = \frac{2}{x}\log\left(\frac{x}{y}\right),$$

which implies that

$$\frac{\partial d(x,y)}{\partial x}\begin{cases} < 0 & \text{if and only if } x < y, \\ = 0 & \text{if and only if } x = y, \\ > 0 & \text{if and only if } x > y. \end{cases}$$

It follows that the function $x \mapsto d(x,y)$ is strictly decreasing on $(0,y)$ and strictly increasing on $(y,+\infty)$.

Let $(x,\Delta x) \in \mathbb{R}^2_{++}$ with $\Delta x < x$, and let $x_0 := \sqrt{(x-\Delta x)(x+\Delta x)}$. Note that $\log((x-\Delta x)/(x+\Delta x)) < 0$ because $0 < (x-\Delta x)/(x+\Delta x) < 1$. We find

$$d(x-\Delta x,y) - d(x+\Delta x,y) = 2\log\left(\frac{x_0}{y}\right)\log\left(\frac{x-\Delta x}{x+\Delta x}\right),$$

which implies that

$$d(x-\Delta x,y) - d(x+\Delta x,y)\begin{cases} < 0 & \text{if and only if } y < x_0, \\ = 0 & \text{if and only if } y = x_0, \\ > 0 & \text{if and only if } y > x_0. \end{cases}$$

It follows that the function $y \mapsto d(x - \Delta x, y) - d(x + \Delta x, y)$ is negative on $(0, x_0)$ and positive on $(x_0, +\infty)$. ■

Proof of Corollary 1.7

Let $\Gamma := (\mathcal{I}, G, \mathcal{Y}, \{(\alpha_i, \beta, \gamma)\}_{i \in \mathcal{I}}, f)$ be a generic NALA game, and let $\boldsymbol{\alpha}$ be defined as in Section 1.3.2. Suppose G is complete.

By definition, if the players of Γ are strongly ex ante homogeneous, then they are weakly ex ante homogeneous.

Suppose the players of Γ are weakly ex ante homogeneous, that is, $\boldsymbol{\alpha} = \bar{\boldsymbol{A}}(G)\boldsymbol{\alpha}$. Since G is complete, we have $\bar{\boldsymbol{A}}(G) = (1/(n-1))(\mathbf{1}_n \mathbf{1}_n^\top - \boldsymbol{I}_n)$. We find

$$\begin{aligned}
\boldsymbol{\alpha} = \bar{\boldsymbol{A}}(G)\boldsymbol{\alpha} \quad &\Leftrightarrow \quad \boldsymbol{\alpha} = \frac{\mathbf{1}_n \mathbf{1}_n^\top - \boldsymbol{I}_n}{n-1}\boldsymbol{\alpha} \\
&\Leftrightarrow \quad (n-1)\boldsymbol{\alpha} = \langle \mathbf{1}_n, \boldsymbol{\alpha} \rangle \mathbf{1}_n - \boldsymbol{\alpha} \\
&\Leftrightarrow \quad \boldsymbol{\alpha} = \frac{\langle \mathbf{1}_n, \boldsymbol{\alpha} \rangle}{n} \mathbf{1}_n.
\end{aligned}$$

It follows that the players of Γ are strongly ex ante homogeneous. ■

Proof of Proposition 1.8

Let $\Gamma := (\mathcal{I}, G, \mathcal{Y}, \{(\alpha_i, \beta, \gamma)\}_{i \in \mathcal{I}}, f)$ be a generic NALA game, and let $\boldsymbol{\alpha}$ be defined as in Section 1.3.2. Note that $\mathbf{1}_n \in \mathrm{Eig}(1, \bar{\boldsymbol{A}}(G))$, so that $\mathrm{g.m.}(1, \bar{\boldsymbol{A}}(G)) \geq 1$.

Proof of Result 1.8.1 Suppose the players of Γ are weakly and not strongly ex ante homogeneous. It follows that $\boldsymbol{\alpha} = \bar{\boldsymbol{A}}(G)\boldsymbol{\alpha}$ and $\boldsymbol{\alpha} \notin \mathrm{span}\{\mathbf{1}_n\}$. We conclude that $\mathrm{span}\{\mathbf{1}_n\} \subsetneq \mathrm{span}\{\mathbf{1}_n, \boldsymbol{\alpha}\} \subset \mathrm{Eig}(1, \bar{\boldsymbol{A}}(G))$ and $\mathrm{g.m.}(1, \bar{\boldsymbol{A}}(G)) > 1$. ■

Proof of Result 1.8.2 Suppose G is strongly connected. It follows that $\mathrm{sl}(G)$ is strongly connected. Since $\mathrm{sl}(G)$ is strongly connected, $\bar{\boldsymbol{A}}(G)$ is irreducible (see, for example, Berman and Plemmons 1994, Theorem 2.7 on p. 30). Since $\bar{\boldsymbol{A}}(G)$ is nonnegative and irreducible, $\rho(\bar{\boldsymbol{A}}(G))$ is a simple eigenvalue of $\bar{\boldsymbol{A}}(G)$ (see, for example, Varga 2000, Theorem 2.7), that is, the eigenvalue $\rho(\bar{\boldsymbol{A}}(G))$ has algebraic multiplicity one. Since $\bar{\boldsymbol{A}}(G)$ is nonnegative and row-normalized, $\rho(\bar{\boldsymbol{A}}(G)) = 1$ (Lemma B.8). The geometric multiplicity of an eigenvalue does not exceed its algebraic multiplicity (see, for example, Horn and Johnson 2012, Theorem 1.4.10). It follows that $\rho(\bar{\boldsymbol{A}}(G))$ must have geometric multiplicity one, that is, the dimension of the eigenspace of $\bar{\boldsymbol{A}}(G)$ associated with the eigenvalue 1 is equal to 1. Finally, we find $\mathrm{Eig}(1, \bar{\boldsymbol{A}}(G)) = \mathrm{span}\{\mathbf{1}_n\}$ because $\mathbf{1}_n \in \mathrm{Eig}(1, \bar{\boldsymbol{A}}(G))$ and $\mathrm{g.m.}(1, \bar{\boldsymbol{A}}(G)) = 1$. ■

Proof of Result 1.8.3 Suppose player $x \in \mathcal{I}$ is isolated in G. We have $\bar{\boldsymbol{A}}(G)\boldsymbol{e}_x = \boldsymbol{e}_x$, from which $\mathrm{span}\{\mathbf{1}_n, \boldsymbol{e}_x\} \subset \mathrm{Eig}(1, \bar{\boldsymbol{A}}(G))$ and $\mathrm{g.m.}(1, \bar{\boldsymbol{A}}(G)) > 1$ follow. ■

Proof of Corollary 1.9

The statement follows from Result 1.8.2. ∎

Example 1.12

Consider the setup of Example 1.12. Let $\boldsymbol{P}_\pi$ be the permutation matrix of π, that is, $\boldsymbol{P}_\pi = (\boldsymbol{e}_2, \boldsymbol{e}_3, \ldots, \boldsymbol{e}_{n-1}, \boldsymbol{e}_n, \boldsymbol{e}_1)^\top = (\boldsymbol{e}_n, \boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{n-2}, \boldsymbol{e}_{n-1})$. We have $\bar{\boldsymbol{A}}(G) = \boldsymbol{P}_\pi$.

First, I show that Γ cannot have a NE that lies in the interior of $\mathbb{R}^n_+$. Suppose, for the sake of contradiction, Γ has an interior NE $\boldsymbol{y}^\star := (y_1^\star, \ldots, y_n^\star) \in \mathbb{R}^n_{++}$. It follows that the players' best reply functions at $\boldsymbol{y}^\star$ satisfy

$$\forall i \in \mathcal{I} \quad y_i^\star = \frac{1}{\gamma}\alpha_i + y^\star_{\pi(i)} \qquad \Leftrightarrow \qquad \boldsymbol{y}^\star = \frac{1}{\gamma}\boldsymbol{\alpha} + \boldsymbol{P}_\pi \boldsymbol{y}^\star.$$

The NE $\boldsymbol{y}^\star$ is therefore given by $(\boldsymbol{I}_n - \boldsymbol{P}_\pi)\boldsymbol{y}^\star = (1/\gamma)\boldsymbol{\alpha}$, where $\boldsymbol{I}_n - \boldsymbol{P}_\pi$ is singular because $\boldsymbol{1}_n \in \ker(\boldsymbol{I}_n - \boldsymbol{P}_\pi)$. Consequently, $\boldsymbol{\alpha}$ must lie in the column space of $\boldsymbol{I}_n - \boldsymbol{P}_\pi$, that is, there exists a $\boldsymbol{c} := (c_1, \ldots, c_n) \in \mathbb{R}^n$ such that $(\boldsymbol{I}_n - \boldsymbol{P}_\pi)\boldsymbol{c} = \boldsymbol{\alpha}$. We find $0 < \sum_{i\in\mathcal{I}} \alpha_i = \langle \boldsymbol{1}_n, \boldsymbol{\alpha}\rangle = \langle \boldsymbol{1}_n, (\boldsymbol{I}_n - \boldsymbol{P}_\pi)\boldsymbol{c}\rangle = \langle \boldsymbol{1}_n, \boldsymbol{c}\rangle - \langle \boldsymbol{1}_n, \boldsymbol{P}_\pi \boldsymbol{c}\rangle = \sum_{i\in\mathcal{I}} c_i - \sum_{i\in\mathcal{I}} c_i = 0$, a contradiction.

Second, I show that Γ cannot have a NE that does not lie in the interior of $\mathbb{R}^n_+$. Suppose, for the sake of contradiction, Γ has a NE $\boldsymbol{y}^\star := (y_1^\star, \ldots, y_n^\star) \in \mathbb{R}^n_+$ with $\{i \in \mathcal{I} \mid y_i^\star = 0\} \neq \emptyset$. Let $k \in \mathcal{I}$ with $y_k^\star = 0$. Note that

$$\frac{\partial u_k(\boldsymbol{y}^\star)}{\partial y_k} = \alpha_k + \gamma y^\star_{\pi(k)}.$$

Since $y_k^\star = 0$, we must have $\partial u_k(\boldsymbol{y}^\star)/\partial y_k \leq 0$ or, equivalently, $y^\star_{\pi(k)} \leq -\alpha_k/\gamma$. It follows that $y^\star_{\pi(k)} < 0$ because $\alpha_k > 0$ and $\gamma > 0$, a contradiction to $y^\star_{\pi(k)} \in \mathbb{R}_+$. ∎

Proof of Proposition 1.13

Let $\Gamma := (\mathcal{I}, G, \mathbb{R}, \{(\alpha_i, \beta, \gamma)\}_{i\in\mathcal{I}}, f)$ be a NALA game, and let $\boldsymbol{\alpha}$ be defined as in Section 1.3.2. Suppose Conditions 1.13.1 to 1.13.4 are satisfied. In what follows, the component in row i and column j of $\bar{\boldsymbol{A}}(G)$, $\bar{a}_{i,j}(G)$, is abbreviated to $\bar{a}_{i,j}$.

First, note that $f(\mathbb{R}) = \mathbb{R}$ because $f(\mathbb{R})$ is an interval (Lemma 1.1) and f is strictly increasing (Assumption F) and not bounded below and above (Condition 1.13.4). Second, note that Condition 1.13.3 is equivalent to $1 \notin \sigma(\gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G))$.

Let $\mathcal{Y}^\star \subset \mathbb{R}^n$ denote the set of all NEs in pure strategies of Γ. I show that $|\mathcal{Y}^\star| = 1$. To this end, let $\boldsymbol{y}^\star := (y_1^\star, \ldots, y_n^\star) \in \mathbb{R}^n$. Note that $\boldsymbol{y}^\star \in \mathcal{Y}^\star$ if and only if

$$\forall i \in \mathcal{I} \quad \frac{\partial u_i(\boldsymbol{y}^\star)}{\partial y_i} = 0 \tag{D.1}$$

and

$$\forall i \in \mathcal{I} \quad \frac{\partial^2 u_i(\boldsymbol{y}^\star)}{\partial {y_i}^2} < 0. \tag{D.2}$$

Let $i \in \mathcal{I}$. For all $\boldsymbol{y} := (y_1, \ldots, y_n) \in \mathbb{R}^n$, we have

$$\frac{\partial u_i(\boldsymbol{y})}{\partial y_i} = \left(\alpha_i - \beta f(y_i) - \gamma\left(f(y_i) - \sum_{j\in\mathcal{I}} \bar{a}_{i,j} f(y_j)\right)\right)\partial f(y_i)$$

and

$$\begin{aligned}\frac{\partial^2 u_i(\boldsymbol{y})}{\partial {y_i}^2} &= \left(\alpha_i - \beta f(y_i) - \gamma\left(f(y_i) - \sum_{j\in\mathcal{I}} \bar{a}_{i,j} f(y_j)\right)\right)\partial^2 f(y_i)\\ &\quad - (\beta + \gamma \mathbb{1}_{\mathcal{I}\setminus\mathcal{I}_0^+(G)}(i))\partial f(y_i)^2.\end{aligned}$$

Since $\partial f > 0$ (Assumption F), the preceding two results imply that

$$\begin{aligned} & 0 = \frac{\partial u_i(\boldsymbol{y}^\star)}{\partial y_i}\\ \Leftrightarrow\quad & 0 = \alpha_i - \beta f(y_i^\star) - \gamma\left(f(y_i^\star) - \sum_{j\in\mathcal{I}} \bar{a}_{i,j} f(y_j^\star)\right) \qquad \text{(D.3)}\\ \Leftrightarrow\quad & f(y_i^\star) = \frac{\alpha_i}{\beta+\gamma} + \frac{\gamma}{\beta+\gamma}\sum_{j\in\mathcal{I}} \bar{a}_{i,j} f(y_j^\star) \qquad \text{(D.4)}\end{aligned}$$

and

$$\begin{aligned} & 0 > \frac{\partial^2 u_i(\boldsymbol{y}^\star)}{\partial {y_i}^2}\\ \Leftrightarrow\quad & 0 > \underbrace{\left(\alpha_i - \beta f(y_i^\star) - \gamma\left(f(y_i^\star) - \sum_{j\in\mathcal{I}} \bar{a}_{i,j} f(y_j^\star)\right)\right)}_{=0 \text{ by (D.3)}}\partial^2 f(y_i^\star)\\ &\quad - (\beta + \gamma \mathbb{1}_{\mathcal{I}\setminus\mathcal{I}_0^+(G)}(i))\partial f(y_i^\star)^2\\ \Leftrightarrow\quad & 0 < \beta + \gamma \mathbb{1}_{\mathcal{I}\setminus\mathcal{I}_0^+(G)}(i).\end{aligned}$$

Using (D.4), we find

$$\begin{aligned} \text{(D.1)} \quad \Leftrightarrow \quad & \boldsymbol{f}(\boldsymbol{y}^\star) = \frac{1}{\beta+\gamma}\boldsymbol{\alpha} + \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\\ \Leftrightarrow \quad & \left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)\boldsymbol{f}(\boldsymbol{y}^\star) = \frac{1}{\beta+\gamma}\boldsymbol{\alpha}. \qquad \text{(D.5)}\end{aligned}$$

Condition 1.13.3 implies that the system of linear equations

$$\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)\boldsymbol{z}^\star = \frac{1}{\beta+\gamma}\boldsymbol{\alpha} \tag{D.6}$$

has a unique solution $\boldsymbol{z}^\star := (z_1^\star, \dots, z_n^\star) \in \mathbb{R}^n$ (Lemma B.3), which is given by

$$\boldsymbol{z}^\star = \frac{1}{\beta+\gamma}\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1}\boldsymbol{\alpha}. \tag{D.7}$$

It is clear from a comparison of the system of equations (D.5) and the system of linear equations (D.6) that $\boldsymbol{y}^\star$ is a solution to the system of equations $\boldsymbol{z}^\star = \boldsymbol{f}(\boldsymbol{y}^\star)$. This system has a unique solution $\boldsymbol{y}^\star \in \mathbb{R}^n$ because f is bijective (Lemma 1.1) and $f(\mathbb{R})^n = \mathbb{R}^n$. In summary, the system of equations (D.1) has a unique solution $\boldsymbol{y}^\star$, which is given by (1.7). Finally, note that, for all $i \in \mathcal{I}$, $\beta + \gamma \mathbb{1}_{\mathcal{I}\setminus\mathcal{I}_0^+(G)}(i) > 0$, because $\beta > 0$ (Condition 1.13.1) and $\beta + \gamma > 0$ (Condition 1.13.2). The preceding arguments demonstrate that there exists a unique $\boldsymbol{y}^\star \in \mathbb{R}^n$ that satisfies both (D.1) and (D.2). To sum up, I have shown that $\boldsymbol{y}^\star$ as given by (1.7) lies in $\mathcal{Y}^\star$ and that $|\mathcal{Y}^\star| = 1$. ■

Proof of Proposition 1.14

Let $\Gamma := (\mathcal{I}, G, [0,\bar{v}], \{(\alpha_i, \beta, \gamma)\}_{i\in\mathcal{I}}, f)$ be a NALA game, let $\boldsymbol{\alpha}$ be defined as in Section 1.3.2, and let $\alpha_{\min}$ and $\alpha_{\max}$ be defined as in Section 1.3.3. Suppose Conditions 1.14.1 to 1.14.4 are satisfied. In what follows, the component in row i and column j of $\bar{\boldsymbol{A}}(G)$, $\bar{a}_{i,j}(G)$, is abbreviated to $\bar{a}_{i,j}$.

First, note that $f([0,\bar{v}]) = [f(0), f(\bar{v})]$ and $\operatorname{int}(f([0,\bar{v}])^n) = (f(0), f(\bar{v}))^n$ because $f([0,\bar{v}])$ is an interval (Lemma 1.1) and f is strictly increasing (Assumption F). Second, note that $\rho(\bar{\boldsymbol{A}}(G)) = 1$ (Lemma B.8). Third, note that Conditions 1.14.1 and 1.14.2 imply that $\beta + \gamma > 0$ and $0 \leq \gamma/(\beta+\gamma) < 1$. Fourth, note that Condition 1.14.3 is equivalent to $\beta f(0)\mathbf{1}_n <_c \boldsymbol{\alpha}$ and Condition 1.14.4 is equivalent to $\boldsymbol{\alpha} <_c \beta f(\bar{v})\mathbf{1}_n$.

Let $\mathcal{Y}^\star \subset (0,\bar{v})^n$ denote the set of all interior NEs in pure strategies of Γ. I show that $|\mathcal{Y}^\star| = 1$. To this end, let $\boldsymbol{y}^\star := (y_1^\star, \dots, y_n^\star) \in [0,\bar{v}]^n$. Note that $\boldsymbol{y}^\star \in \mathcal{Y}^\star$ if and only if

$$\boldsymbol{y}^\star \in (0,\bar{v})^n, \tag{D.8}$$

$$\forall i \in \mathcal{I} \quad \frac{\partial u_i(\boldsymbol{y}^\star)}{\partial y_i} = 0, \tag{D.9}$$

and

$$\forall i \in \mathcal{I} \quad \frac{\partial^2 u_i(\boldsymbol{y}^\star)}{\partial {y_i}^2} < 0. \tag{D.10}$$

Analogous to the proof of Proposition 1.13, we find

$$\text{(D.9)} \quad \Leftrightarrow \quad \left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)\boldsymbol{f}(\boldsymbol{y}^\star) = \frac{1}{\beta+\gamma}\boldsymbol{\alpha} \tag{D.11}$$

and

$$\text{(D.10)} \quad \Leftrightarrow \quad \forall i \in \mathcal{I} \quad \beta + \gamma \mathbb{1}_{\mathcal{I}\setminus\mathcal{I}_0^+(G)}(i) > 0.$$

The matrix $\boldsymbol{I}_n - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G)$ is nonsingular with a nonnegative inverse that is bounded below by $\boldsymbol{I}_n$ because $0 \le \gamma/(\beta+\gamma) < 1$ and $\rho(\bar{\boldsymbol{A}}(G)) = 1$ (Lemma B.6). It follows that the system (D.6) has a unique solution $\boldsymbol{z}^\star := (z_1^\star, \ldots, z_n^\star) \in \mathbb{R}^n$, which is given by (D.7). The two inequalities $\boldsymbol{I}_n \le_c (\boldsymbol{I}_n - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G))^{-1}$ and $\beta f(0)\mathbf{1}_n <_c \boldsymbol{\alpha}$ imply that $f(0)\mathbf{1}_n <_c \boldsymbol{z}^\star$. Indeed,

$$\frac{1}{\beta+\gamma}\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1}\boldsymbol{\alpha} >_c \frac{\beta f(0)}{\beta+\gamma}\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1}\mathbf{1}_n = f(0)\mathbf{1}_n,$$

where the inequality is according to Lemma B.1 and the equality follows from the fact that

$$\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1}\mathbf{1}_n = \frac{\beta+\gamma}{\beta}\mathbf{1}_n. \tag{D.12}$$

Similarly, $\boldsymbol{I}_n \le_c (\boldsymbol{I}_n - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G))^{-1}$ and $\boldsymbol{\alpha} <_c \beta f(\bar{v})\mathbf{1}_n$ imply that $\boldsymbol{z}^\star <_c f(\bar{v})\mathbf{1}_n$. It follows from $f(0)\mathbf{1}_n <_c \boldsymbol{z}^\star$ and $\boldsymbol{z}^\star <_c f(\bar{v})\mathbf{1}_n$ that $\boldsymbol{z}^\star \in (f(0), f(\bar{v}))^n$. It is clear from a comparison of the right system of equations of (D.11) and the system of linear equations (D.6) that $\boldsymbol{y}^\star$ is a solution to the system of equations $\boldsymbol{z}^\star = \boldsymbol{f}(\boldsymbol{y}^\star)$. This system has a unique solution $\boldsymbol{y}^\star \in (0, \bar{v})^n$ because f is bijective (Lemma 1.1) and $\boldsymbol{z}^\star \in (f(0), f(\bar{v}))^n$. In summary, the system of equations (D.9) has a unique solution $\boldsymbol{y}^\star$, which is given by (1.7). Finally, note that, for all $i \in \mathcal{I}$, $\beta + \gamma \mathbb{1}_{\mathcal{I}\setminus\mathcal{I}_0^+(G)}(i) > 0$, because $\beta > 0$ (Condition 1.14.1) and $\beta+\gamma > 0$ (Conditions 1.14.1 and 1.14.2). The preceding arguments demonstrate that there exists a unique $\boldsymbol{y}^\star \in [0, \bar{v}]^n$ that satisfies (D.8), (D.9), and (D.10). To sum up, I have shown that $\boldsymbol{y}^\star$ as given by (1.7) lies in $\mathcal{Y}^\star$ and that $|\mathcal{Y}^\star| = 1$.

In the remainder of the proof, I show that Γ has no boundary NEs. Table D.1 gives an overview of all possible types of boundary NEs of Γ. For example, a boundary NE of type B-3 is characterized by the fact that some players play zero and the remaining players play an action in the interior of the action space. Let $\tilde{\boldsymbol{y}}^\star := (\tilde{y}_1^\star, \ldots, \tilde{y}_n^\star) \in [0, \bar{v}]^n$.

Table D.1. Possible types of boundary Nash equilibria of Γ (Propositions 1.14 and 1.15)

	Actions and set of actions		
Type	0	$(0, \bar{v})$	$\bar{v}$
B-1	$\times$		
B-2			$\times$
B-3	$\times$	$\times$	
B-4	$\times$		$\times$
B-5		$\times$	$\times$
B-6	$\times$	$\times$	$\times$

First, $\mathbf{0}_n$ (this is referred to as a boundary NE of type B-1 in Table D.1) cannot be a boundary NE of Γ. Suppose, for the sake of contradiction, $\mathbf{0}_n$ is a boundary

NE of Γ. Then we must have

$$\forall i \in \mathcal{I} \quad \frac{\partial u_i(\mathbf{0}_n)}{\partial y_i} \leq 0. \tag{D.13}$$

Let $i \in \mathcal{I}$. We find

$$\begin{aligned}\frac{\partial u_i(\mathbf{0}_n)}{\partial y_i} &= \Big(\alpha_i - \beta f(0) - \gamma\Big(f(0) - \sum_{j\in\mathcal{I}} \bar{a}_{i,j} f(0)\Big)\Big)\partial f(0)\\ &= \big(\alpha_i - \beta f(0)\big)\partial f(0)\\ &> 0\end{aligned}$$

because $\sum_{j\in\mathcal{I}} \bar{a}_{i,j} = 1$, $\partial f(0) > 0$ (Assumption F), and $\alpha_i > \beta f(0)$ (Condition 1.14.3). The preceding inequality contradicts (D.13). Consequently, $\mathbf{0}_n$ cannot be a boundary NE of Γ.

Second, I show that $\bar{v}\mathbf{1}_n$ (this is referred to as a boundary NE of type B-2 in Table D.1) cannot be a boundary NE of Γ. Suppose, for the sake of contradiction, $\bar{v}\mathbf{1}_n$ is a boundary NE of Γ. Then we must have

$$\forall i \in \mathcal{I} \quad \frac{\partial u_i(\bar{v}\mathbf{1}_n)}{\partial y_i} \geq 0. \tag{D.14}$$

Let $i \in \mathcal{I}$. We find

$$\begin{aligned}\frac{\partial u_i(\bar{v}\mathbf{1}_n)}{\partial y_i} &= \Big(\alpha_i - \beta f(\bar{v}) - \gamma\Big(f(\bar{v}) - \sum_{j\in\mathcal{I}} \bar{a}_{i,j} f(\bar{v})\Big)\Big)\partial f(\bar{v})\\ &= \big(\alpha_i - \beta f(\bar{v})\big)\partial f(\bar{v})\\ &< 0\end{aligned}$$

because $\sum_{j\in\mathcal{I}} \bar{a}_{i,j} = 1$, $\partial f(\bar{v}) > 0$ (Assumption F), and $\alpha_i < \beta f(\bar{v})$ (Condition 1.14.4). The preceding inequality contradicts (D.14). Consequently, $\bar{v}\mathbf{1}_n$ cannot be a boundary NE of Γ.

Third, I show that a situation where some players play zero and the remaining players play an action in the interior of the action space (this is referred to as a boundary NE of type B-3 in Table D.1) cannot be a boundary NE of Γ. Suppose, for the sake of contradiction, $\tilde{\boldsymbol{y}}^\star$ is a boundary NE of Γ of type B-3. Specifically, suppose there exists a $\mathcal{J} \subset \mathcal{I}$ with $0 < |\mathcal{J}| < n$ such that (i) for all $i \in \mathcal{I} \setminus \mathcal{J}$, $\tilde{y}_i^\star = 0$, and (ii) for all $i \in \mathcal{J}$, $\tilde{y}_i^\star \in (0, \bar{v})$. Then we must have

$$\forall i \in \mathcal{I} \setminus \mathcal{J} \quad \frac{\partial u_i(\tilde{\boldsymbol{y}}^\star)}{\partial y_i} \leq 0. \tag{D.15}$$

Let $i \in \mathcal{I} \setminus \mathcal{J}$. Since $\gamma \geq 0$ (Condition 1.14.2), $\bar{\boldsymbol{A}}(G)$ is nonnegative, and f is strictly increasing (Assumption F), we have

$$\gamma \sum_{j\in\mathcal{J}} \bar{a}_{i,j}\big(f(0) - f(\tilde{y}_j^\star)\big) \leq 0. \tag{D.16}$$

Using (D.16), we find

$$\begin{aligned}
\frac{\partial u_i(\tilde{\boldsymbol{y}}^\star)}{\partial y_i} &= \Big(\alpha_i - \beta f(0) - \gamma\Big(f(0) - \sum_{j\in\mathcal{I}\setminus\mathcal{J}} \bar{a}_{i,j} f(0) - \sum_{j\in\mathcal{J}} \bar{a}_{i,j} f(\tilde{y}_j^\star)\Big)\Big)\partial f(0)\\
&= \begin{cases} \big(\alpha_i - \beta f(0)\big)\partial f(0) & \text{if } \deg_G^+(i) = 0,\\ \big(\alpha_i - \beta f(0) - \gamma \sum_{j\in\mathcal{J}} \bar{a}_{i,j}\big(f(0) - f(\tilde{y}_j^\star)\big)\big)\partial f(0) & \text{if } \deg_G^+(i) > 0,\end{cases}\\
&\geq \big(\alpha_i - \beta f(0)\big)\partial f(0)\\
&> 0
\end{aligned}$$

because $\sum_{j\in\mathcal{I}\setminus\mathcal{J}} \bar{a}_{i,j} + \sum_{j\in\mathcal{J}} \bar{a}_{i,j} = 1$, $\partial f(0) > 0$ (Assumption F), and $\alpha_i > \beta f(0)$ (Condition 1.14.3). The preceding inequality contradicts (D.15). Consequently, $\tilde{\boldsymbol{y}}^\star$ cannot be a boundary NE of Γ of type B-3.

Fourth, I show that a situation where some players play zero and the remaining players play the maximum possible action (this is referred to as a boundary NE of type B-4 in Table D.1) cannot be a boundary NE of Γ. Suppose, for the sake of contradiction, $\tilde{\boldsymbol{y}}^\star$ is a boundary NE of Γ of type B-4. Specifically, suppose there exists a $\mathcal{J} \subset \mathcal{I}$ with $0 < |\mathcal{J}| < n$ such that (i) for all $i \in \mathcal{I} \setminus \mathcal{J}$, $\tilde{y}_i^\star = 0$, and (ii) for all $i \in \mathcal{J}$, $\tilde{y}_i^\star = \bar{v}$. Then we must have

$$\forall i \in \mathcal{I} \setminus \mathcal{J} \quad \frac{\partial u_i(\tilde{\boldsymbol{y}}^\star)}{\partial y_i} \leq 0 \tag{D.17}$$

and

$$\forall i \in \mathcal{J} \quad \frac{\partial u_i(\tilde{\boldsymbol{y}}^\star)}{\partial y_i} \geq 0. \tag{D.18}$$

Let $i \in \mathcal{I} \setminus \mathcal{J}$ and $k \in \mathcal{J}$. Since $\gamma \geq 0$ (Condition 1.14.2), $\bar{\boldsymbol{A}}(G)$ is nonnegative, and f is strictly increasing (Assumption F), we have

$$\gamma \sum_{j\in\mathcal{J}} \bar{a}_{i,j}\big(f(0) - f(\bar{v})\big) \leq 0 \tag{D.19}$$

and

$$\gamma \sum_{j\in\mathcal{I}\setminus\mathcal{J}} \bar{a}_{k,j}\big(f(\bar{v}) - f(0)\big) \geq 0. \tag{D.20}$$

Using (D.19), we find

$$\begin{aligned}
\frac{\partial u_i(\tilde{\boldsymbol{y}}^\star)}{\partial y_i} &= \Big(\alpha_i - \beta f(0) - \gamma\Big(f(0) - \sum_{j\in\mathcal{I}\setminus\mathcal{J}} \bar{a}_{i,j} f(0) - \sum_{j\in\mathcal{J}} \bar{a}_{i,j} f(\bar{v})\Big)\Big)\partial f(0)\\
&= \begin{cases} \big(\alpha_i - \beta f(0)\big)\partial f(0) & \text{if } \deg_G^+(i) = 0,\\ \big(\alpha_i - \beta f(0) - \gamma \sum_{j\in\mathcal{J}} \bar{a}_{i,j}\big(f(0) - f(\bar{v})\big)\big)\partial f(0) & \text{if } \deg_G^+(i) > 0,\end{cases}\\
&\geq \big(\alpha_i - \beta f(0)\big)\partial f(0)
\end{aligned}$$

$$> 0$$

because $\sum_{j\in\mathcal{I}\setminus\mathcal{J}} \bar{a}_{i,j} + \sum_{j\in\mathcal{J}} \bar{a}_{i,j} = 1$, $\partial f(0) > 0$ (Assumption F), and $\alpha_i > \beta f(0)$ (Condition 1.14.3). The preceding inequality contradicts (D.17). Using (D.20), we find

$$\begin{aligned}
\frac{\partial u_k(\tilde{\boldsymbol{y}}^\star)}{\partial y_k} &= \Bigg(\alpha_k - \beta f(\bar{v}) - \gamma\Bigg(f(\bar{v}) - \sum_{j\in\mathcal{I}\setminus\mathcal{J}} \bar{a}_{k,j} f(0) - \sum_{j\in\mathcal{J}} \bar{a}_{k,j} f(\bar{v})\Bigg)\Bigg)\partial f(\bar{v}) \\
&= \begin{cases} \big(\alpha_k - \beta f(\bar{v})\big)\partial f(\bar{v}) & \text{if } \deg_G^+(k) = 0, \\ \big(\alpha_k - \beta f(\bar{v}) - \gamma \sum_{j\in\mathcal{I}\setminus\mathcal{J}} \bar{a}_{k,j}\big(f(\bar{v}) - f(0)\big)\big)\partial f(\bar{v}) & \text{if } \deg_G^+(k) > 0, \end{cases} \\
&\le \big(\alpha_k - \beta f(\bar{v})\big)\partial f(\bar{v}) \\
&< 0
\end{aligned}$$

because $\sum_{j\in\mathcal{I}\setminus\mathcal{J}} \bar{a}_{i,j} + \sum_{j\in\mathcal{J}} \bar{a}_{i,j} = 1$, $\partial f(\bar{v}) > 0$ (Assumption F), and $\alpha_k < \beta f(\bar{v})$ (Condition 1.14.4). The preceding inequality contradicts (D.18). Consequently, $\tilde{\boldsymbol{y}}^\star$ cannot be a boundary NE of Γ of type B-4.

Fifth, I show that a situation where some players play an action in the interior of the action space and the remaining players play the maximum possible action (this is referred to as a boundary NE of type B-5 in Table D.1) cannot be a boundary NE of Γ. Suppose, for the sake of contradiction, $\tilde{\boldsymbol{y}}^\star$ is a boundary NE of Γ of type B-5. Specifically, suppose there exists a $\mathcal{J} \subset \mathcal{I}$ with $0 < |\mathcal{J}| < n$ such that (i) for all $i \in \mathcal{I} \setminus \mathcal{J}$, $\tilde{y}_i^\star \in (0, \bar{v})$, and (ii) for all $i \in \mathcal{J}$, $\tilde{y}_i^\star = \bar{v}$. Then we must have

$$\forall i \in \mathcal{J} \quad \frac{\partial u_i(\tilde{\boldsymbol{y}}^\star)}{\partial y_i} \ge 0. \tag{D.21}$$

Let $i \in \mathcal{J}$. Since $\gamma \ge 0$ (Condition 1.14.2), $\bar{\boldsymbol{A}}(G)$ is nonnegative, and f is strictly increasing (Assumption F), we have

$$\gamma \sum_{j\in\mathcal{I}\setminus\mathcal{J}} \bar{a}_{i,j}\big(f(\bar{v}) - f(\tilde{y}_j^\star)\big) \ge 0. \tag{D.22}$$

Using (D.22), we find

$$\begin{aligned}
\frac{\partial u_i(\tilde{\boldsymbol{y}}^\star)}{\partial y_i} &= \Bigg(\alpha_i - \beta f(\bar{v}) - \gamma\Bigg(f(\bar{v}) - \sum_{j\in\mathcal{I}\setminus\mathcal{J}} \bar{a}_{i,j} f(\tilde{y}_j^\star) - \sum_{j\in\mathcal{J}} \bar{a}_{i,j} f(\bar{v})\Bigg)\Bigg)\partial f(\bar{v}) \\
&= \begin{cases} \big(\alpha_i - \beta f(\bar{v})\big)\partial f(\bar{v}) & \text{if } \deg_G^+(i) = 0, \\ \big(\alpha_i - \beta f(\bar{v}) - \gamma \sum_{j\in\mathcal{I}\setminus\mathcal{J}} \bar{a}_{i,j}\big(f(\bar{v}) - f(\tilde{y}_j^\star)\big)\big)\partial f(\bar{v}) & \text{if } \deg_G^+(i) > 0, \end{cases} \\
&\le \big(\alpha_i - \beta f(\bar{v})\big)\partial f(\bar{v}) \\
&< 0
\end{aligned}$$

because $\sum_{j\in\mathcal{I}\setminus\mathcal{J}} \bar{a}_{i,j} + \sum_{j\in\mathcal{J}} \bar{a}_{i,j} = 1$, $\partial f(\bar{v}) > 0$ (Assumption F), and $\alpha_i < \beta f(\bar{v})$ (Condition 1.14.4). The preceding inequality contradicts (D.21). Consequently, $\tilde{\boldsymbol{y}}^\star$ cannot be a boundary NE of Γ of type B-5.

Sixth, I show that a situation where some players play zero, some players play an action in the interior of the action space, and the remaining players play the maximum possible action (this is referred to as a boundary NE of type B-6 in Table D.1) cannot be a boundary NE of Γ. Suppose, for the sake of contradiction, $\tilde{\boldsymbol{y}}^\star$ is a boundary NE of Γ of type B-6. Specifically, suppose there exist $\mathcal{J} \subset \mathcal{I}$ and $\mathcal{K} \subset \mathcal{I}$ with $0 < |\mathcal{J}| < n$, $0 < |\mathcal{K}| < n$, and $\mathcal{K} \cap \mathcal{I} = \emptyset$ such that (i) for all $i \in \mathcal{I} \setminus (\mathcal{J} \cup \mathcal{K})$, $\tilde{y}_i^\star = 0$, (ii) for all $i \in \mathcal{J}$, $\tilde{y}_i^\star \in (0, \bar{v})$, and (iii) for all $i \in \mathcal{K}$, $\tilde{y}_i^\star = \bar{v}$. Then we must have

$$\forall i \in \mathcal{I} \setminus (\mathcal{J} \cup \mathcal{K}) \quad \frac{\partial u_i(\tilde{\boldsymbol{y}}^\star)}{\partial y_i} \leq 0 \tag{D.23}$$

and

$$\forall i \in \mathcal{K} \quad \frac{\partial u_i(\tilde{\boldsymbol{y}}^\star)}{\partial y_i} \geq 0. \tag{D.24}$$

Let $i \in \mathcal{I} \setminus (\mathcal{J} \cup \mathcal{K})$ and $k \in \mathcal{K}$. Since $\gamma \geq 0$ (Condition 1.14.2), $\bar{\boldsymbol{A}}(G)$ is nonnegative, and f is strictly increasing (Assumption F), we have

$$\gamma \sum_{j \in \mathcal{J}} \bar{a}_{i,j}\big(f(0) - f(\tilde{y}_j^\star)\big) + \gamma \sum_{j \in \mathcal{K}} \bar{a}_{i,j}\big(f(0) - f(\bar{v})\big) \leq 0 \tag{D.25}$$

and

$$\gamma \sum_{j \in \mathcal{I} \setminus (\mathcal{J} \cup \mathcal{K})} \bar{a}_{k,j}\big(f(\bar{v}) - f(0)\big) + \gamma \sum_{j \in \mathcal{J}} \bar{a}_{k,j}\big(f(\bar{v}) - f(\tilde{y}_j^\star)\big) \geq 0. \tag{D.26}$$

Using (D.25), we find

$$\begin{aligned}
\frac{\partial u_i(\tilde{\boldsymbol{y}}^\star)}{\partial y_i} &= \Bigg(\alpha_i - \beta f(0) - \gamma\Bigg(f(0) - \sum_{j \in \mathcal{I} \setminus (\mathcal{J} \cup \mathcal{K})} \bar{a}_{i,j} f(0) - \sum_{j \in \mathcal{J}} \bar{a}_{i,j} f(\tilde{y}_j^\star) \\
&\qquad - \sum_{j \in \mathcal{K}} \bar{a}_{i,j} f(\bar{v})\Bigg)\Bigg)\partial f(0) \\
&= \begin{cases} \big(\alpha_i - \beta f(0)\big)\partial f(0) & \text{if } \deg_G^+(i) = 0, \\ \big(\alpha_i - \beta f(0) - \gamma \sum_{j \in \mathcal{J}} \bar{a}_{i,j}\big(f(0) - f(\tilde{y}_j^\star)\big) & \\ \quad - \gamma \sum_{j \in \mathcal{K}} \bar{a}_{i,j}\big(f(0) - f(\bar{v})\big)\big)\partial f(0) & \text{if } \deg_G^+(i) > 0, \end{cases} \\
&\geq \big(\alpha_i - \beta f(0)\big)\partial f(0) \\
&> 0
\end{aligned}$$

because $\sum_{j \in \mathcal{I} \setminus (\mathcal{J} \cup \mathcal{K})} \bar{a}_{i,j} + \sum_{j \in \mathcal{J}} \bar{a}_{i,j} + \sum_{j \in \mathcal{K}} \bar{a}_{i,j} = 1$, $\partial f(0) > 0$ (Assumption F), and $\alpha_i > \beta f(0)$ (Condition 1.14.3). The preceding inequality contradicts (D.23). Using (D.26), we find

$$\frac{\partial u_k(\tilde{\boldsymbol{y}}^\star)}{\partial y_k} = \Bigg(\alpha_k - \beta f(\bar{v}) - \gamma\Bigg(f(\bar{v}) - \sum_{j \in \mathcal{I} \setminus (\mathcal{J} \cup \mathcal{K})} \bar{a}_{k,j} f(0) - \sum_{j \in \mathcal{J}} \bar{a}_{k,j} f(\tilde{y}_j^\star)$$

$$
\begin{aligned}
&\qquad - \sum_{j\in\mathcal{K}} \bar{a}_{k,j} f(\bar{v})\Big)\Big)\partial f(\bar{v}) \\
&= \begin{cases} \big(\alpha_k - \beta f(\bar{v})\big)\partial f(\bar{v}) & \text{if } \deg_G^+(k) = 0, \\ \big(\alpha_k - \beta f(\bar{v}) - \gamma \sum_{j\in\mathcal{I}\setminus(\mathcal{J}\cup\mathcal{K})} \bar{a}_{k,j}\big(f(\bar{v}) - f(0)\big) & \\ \quad - \gamma \sum_{j\in\mathcal{J}} \bar{a}_{k,j}\big(f(\bar{v}) - f(\tilde{y}_j^\star)\big)\big)\partial f(\bar{v}) & \text{if } \deg_G^+(k) > 0, \end{cases} \\
&\le \big(\alpha_k - \beta f(\bar{v})\big)\partial f(\bar{v}) \\
&< 0
\end{aligned}
$$

because $\sum_{j\in\mathcal{I}\setminus(\mathcal{J}\cup\mathcal{K})} \bar{a}_{i,j} + \sum_{j\in\mathcal{J}} \bar{a}_{i,j} + \sum_{j\in\mathcal{K}} \bar{a}_{i,j} = 1$, $\partial f(\bar{v}) > 0$ (Assumption F), and $\alpha_k < \beta f(\bar{v})$ (Condition 1.14.4). The preceding inequality contradicts (D.24). Consequently, $\tilde{\boldsymbol{y}}^\star$ cannot be a boundary NE of Γ of type B-6. ■

Proof of Proposition 1.15

Let $\Gamma := (\mathcal{I}, G, [0,\bar{v}], \{(\alpha_i, \beta, \gamma)\}_{i\in\mathcal{I}}, f)$ be a NALA game, let $\boldsymbol{\alpha}$ be defined as in Section 1.3.2, and let $\alpha_{\min}$ and $\alpha_{\max}$ be defined as in Section 1.3.3. Suppose Conditions 1.15.1 to 1.15.4 are satisfied. In what follows, the component in row i and column j of $\bar{\boldsymbol{A}}(G)$, $\bar{a}_{i,j}(G)$, is abbreviated to $\bar{a}_{i,j}$.

First, note that $f([0,\bar{v}]) = [f(0), f(\bar{v})]$ and $\operatorname{int}(f([0,\bar{v}])^n) = (f(0), f(\bar{v}))^n$ because $f([0,\bar{v}])$ is an interval (Lemma 1.1) and f is strictly increasing (Assumption F). Second, note that $\rho(\bar{\boldsymbol{A}}(G)) = 1$ (Lemma B.8) and $\rho(\bar{\boldsymbol{A}}(G)^2) = \rho(\bar{\boldsymbol{A}}(G))^2$, which is a consequence of the *spectral mapping theorem* (see, for example, Kubrusly 2011, Theorem 6.19 and Corollary 6.20). Third, note that Conditions 1.15.1 and 1.15.2 imply that $\beta + \gamma > 0$ and $-1 < \gamma/(\beta+\gamma) < 0$. Fourth, note that Condition 1.15.3 is equivalent to $\beta f(0)\mathbf{1}_n + \gamma(f(0) - f(\bar{v}))\mathbf{1}_n <_c \boldsymbol{\alpha}$, where $\beta f(0)\mathbf{1}_n <_c \beta f(0)\mathbf{1}_n + \gamma(f(0) - f(\bar{v}))$ because $\gamma < 0$ (Condition 1.15.2) and f is strictly increasing (Assumption F), from which it follows that Condition 1.15.3 is stronger than Condition 1.14.3, so that, $\beta f(0)\mathbf{1}_n <_c \boldsymbol{\alpha}$. Fifth, note that Condition 1.15.4 is equivalent to $\boldsymbol{\alpha} <_c \beta f(\bar{v})\mathbf{1}_n + \gamma(f(\bar{v}) - f(0))\mathbf{1}_n$, where $\beta f(\bar{v})\mathbf{1}_n + \gamma(f(\bar{v}) - f(0))\mathbf{1}_n <_c \beta f(\bar{v})\mathbf{1}_n$ because $\gamma < 0$ (Condition 1.15.2) and f is strictly increasing (Assumption F), from which it follows that Condition 1.15.4 is stronger than Condition 1.14.4, so that, $\boldsymbol{\alpha} <_c \beta f(\bar{v})$. Sixth, note that Conditions 1.15.1 to 1.15.4 imply that (Lemma 1.19)

$$\frac{\beta(\beta+2\gamma)}{\beta+\gamma} f(0)\mathbf{1}_n <_c \left(\boldsymbol{I}_n + \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)\boldsymbol{\alpha} <_c \frac{\beta(\beta+2\gamma)}{\beta+\gamma} f(\bar{v})\mathbf{1}_n. \tag{D.27}$$

Let $\mathcal{Y}^\star \subset (0,\bar{v})^n$ denote the set of all interior NEs in pure strategies of Γ. I show that $|\mathcal{Y}^\star| = 1$. To this end, let $\boldsymbol{y}^\star := (y_1^\star, \ldots, y_n^\star) \in [0,\bar{v}]^n$. Note that $\boldsymbol{y}^\star \in \mathcal{Y}^\star$ if and only if

$$\boldsymbol{y}^\star \in (0,\bar{v})^n, \tag{D.28}$$

$$\forall i \in \mathcal{I} \quad \frac{\partial u_i(\boldsymbol{y}^\star)}{\partial y_i} = 0, \tag{D.29}$$

and

$$\forall i \in \mathcal{I} \quad \frac{\partial^2 u_i(\boldsymbol{y}^\star)}{\partial {y_i}^2} < 0. \tag{D.30}$$

Analogous to the proof of Proposition 1.13, we find

$$\text{(D.29)} \quad \Leftrightarrow \quad \left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)\boldsymbol{f}(\boldsymbol{y}^\star) = \frac{1}{\beta+\gamma}\boldsymbol{\alpha} \tag{D.31}$$

and

$$\text{(D.30)} \quad \Leftrightarrow \quad \forall i \in \mathcal{I} \quad \beta + \gamma\,\mathbb{1}_{\mathcal{I}\setminus\mathcal{I}_0^+(G)}(i) > 0.$$

The matrix $\boldsymbol{I}_n - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G)$ is nonsingular because $|\gamma/(\beta+\gamma)| < 1$ and $\rho(\bar{\boldsymbol{A}}(G)) = 1$ (Lemma B.3). It follows that the system of linear equations (D.6) has a unique solution $\boldsymbol{z}^\star := (z_1^\star, \ldots, z_n^\star) \in \mathbb{R}^n$, which is given by (D.7). Next, I show that $\boldsymbol{z}^\star \in (f(0), f(\bar{v}))^n$. To this end, I derive an alternative representation of $\boldsymbol{z}^\star$. Note that

$$\left(\boldsymbol{I}_n + \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right) = \boldsymbol{I}_n - \frac{\gamma^2}{(\beta+\gamma)^2}\bar{\boldsymbol{A}}(G)^2, \tag{D.32}$$

where $\boldsymbol{I}_n - \gamma^2/(\beta+\gamma)^2\bar{\boldsymbol{A}}(G)^2$ is nonsingular with a nonnegative inverse that is bounded below by $\boldsymbol{I}_n$ because $0 < \gamma^2/(\beta+\gamma)^2\rho(\bar{\boldsymbol{A}}(G)^2) = \gamma^2/(\beta+\gamma)^2 < 1$ (Lemma B.6). It follows that

$$\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1} = \left(\boldsymbol{I}_n - \frac{\gamma^2}{(\beta+\gamma)^2}\bar{\boldsymbol{A}}(G)^2\right)^{-1}\left(\boldsymbol{I}_n + \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right), \tag{D.33}$$

which leads to the following alternative representation of $\boldsymbol{z}^\star$:

$$\boldsymbol{z}^\star = \frac{1}{\beta+\gamma}\left(\boldsymbol{I}_n - \frac{\gamma^2}{(\beta+\gamma)^2}\bar{\boldsymbol{A}}(G)^2\right)^{-1}\left(\boldsymbol{I}_n + \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)\boldsymbol{\alpha}. \tag{D.34}$$

The inequality $\boldsymbol{I}_n \leq_c (\boldsymbol{I}_n - \gamma^2/(\beta+\gamma)^2\bar{\boldsymbol{A}}(G)^2)^{-1}$ and the left inequality of (D.27) imply that $\boldsymbol{z}^\star \in (f(0), +\infty)^n$. Indeed,

$$\begin{aligned}
&\frac{1}{\beta+\gamma}\left(\boldsymbol{I}_n - \frac{\gamma^2}{(\beta+\gamma)^2}\bar{\boldsymbol{A}}(G)^2\right)^{-1}\left(\boldsymbol{I}_n + \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)\boldsymbol{\alpha} \\
&\qquad >_c \frac{1}{\beta+\gamma}\left(\boldsymbol{I}_n - \frac{\gamma^2}{(\beta+\gamma)^2}\bar{\boldsymbol{A}}(G)^2\right)^{-1}\frac{\beta(\beta+2\gamma)}{\beta+\gamma}f(0)\mathbf{1}_n = f(0)\mathbf{1}_n,
\end{aligned}$$

where the inequality is according to Lemma B.1 and the equality follows from the fact that

$$\left(\boldsymbol{I}_n - \frac{\gamma^2}{(\beta+\gamma)^2}\bar{\boldsymbol{A}}(G)^2\right)^{-1}\mathbf{1}_n = \frac{(\beta+\gamma)^2}{\beta(\beta+2\gamma)}\mathbf{1}_n. \tag{D.35}$$

Similarly, the inequality $\boldsymbol{I}_n \leq_c (\boldsymbol{I}_n - \gamma^2/(\beta+\gamma)^2\bar{\boldsymbol{A}}(G)^2)^{-1}$ and the right inequality of (D.27) imply that $\boldsymbol{z}^\star \in (-\infty, f(\bar{v}))^n$. Combining $\boldsymbol{z}^\star \in (-\infty, f(\bar{v}))^n$ and $\boldsymbol{z}^\star \in (f(0), +\infty)^n$ gives $\boldsymbol{z}^\star \in (f(0), f(\bar{v}))^n$. It is clear from a comparison of the right system of equations of (D.31) and the system of linear equations (D.6) that $\boldsymbol{y}^\star$ is a solution to the system of equations $\boldsymbol{z}^\star = \boldsymbol{f}(\boldsymbol{y}^\star)$. This system has a unique solution $\boldsymbol{y}^\star \in (0, \bar{v})^n$ because f is bijective (Lemma 1.1) and $\boldsymbol{z}^\star \in (f(0), f(\bar{v}))^n$. In summary, the system of equations (D.29) has a unique solution $\boldsymbol{y}^\star$, which is given by (1.7). Finally, note that, for all $i \in \mathcal{I}$, $\beta + \gamma \mathbb{1}_{\mathcal{I} \setminus \mathcal{I}_0^+(G)}(i) > 0$, because $\beta > 0$ (Condition 1.15.1) and $\beta + \gamma > 0$ (Conditions 1.15.1 and 1.15.2). The preceding arguments demonstrate that there exists a unique $\boldsymbol{y}^\star \in [0, \bar{v}]^n$ that satisfies (D.28), (D.29), and (D.30). To sum up, I have shown that $\boldsymbol{y}^\star$ as given by (1.7) lies in $\mathcal{Y}^\star$ and that $|\mathcal{Y}^\star| = 1$.

In the remainder of the proof, I show that Γ has no boundary NEs. Table D.1 gives an overview of all possible types of boundary NEs of Γ. Let $\tilde{\boldsymbol{y}}^\star := (\tilde{y}_1^\star, \ldots, \tilde{y}_n^\star) \in [0, \bar{v}]^n$.

First, $\boldsymbol{0}_n$ (this is referred to as a boundary NE of type B-1 in Table D.1) cannot be a boundary NE of Γ. See the proof of Proposition 1.14 for the precise argument.

Second, $\bar{v}\boldsymbol{1}_n$ (this is referred to as a boundary NE of type B-2 in Table D.1) cannot be a boundary NE of Γ. See the proof of Proposition 1.14 for the precise argument.

Third, I show that a situation where some players play zero and the remaining players play an action in the interior of the action space (this is referred to as a boundary NE of type B-3 in Table D.1) cannot be a boundary NE of Γ. Suppose, for the sake of contradiction, $\tilde{\boldsymbol{y}}^\star$ is a boundary NE of Γ of type B-3. Specifically, suppose there exists a $\mathcal{J} \subset \mathcal{I}$ with $0 < |\mathcal{J}| < n$ such that (i) for all $i \in \mathcal{I} \setminus \mathcal{J}$, $\tilde{y}_i^\star = 0$, and (ii) for all $i \in \mathcal{J}$, $\tilde{y}_i^\star \in (0, \bar{v})$. Then we must have

$$\forall i \in \mathcal{I} \setminus \mathcal{J} \quad \frac{\partial u_i(\tilde{\boldsymbol{y}}^\star)}{\partial y_i} \leq 0. \tag{D.36}$$

Let $i \in \mathcal{I} \setminus \mathcal{J}$. Since $\gamma < 0$ (Condition 1.15.2), $\bar{\boldsymbol{A}}(G)$ is nonnegative, and f is strictly increasing (Assumption F), we have

$$\gamma \sum_{j \in \mathcal{J}} \bar{a}_{i,j}\big(f(0) - f(\tilde{y}_j^\star)\big) \leq \gamma\big(f(0) - f(\bar{v})\big). \tag{D.37}$$

Using (D.37), we find

$$\begin{aligned}
\frac{\partial u_i(\tilde{\boldsymbol{y}}^\star)}{\partial y_i} &= \left(\alpha_i - \beta f(0) - \gamma\left(f(0) - \sum_{j \in \mathcal{I} \setminus \mathcal{J}} \bar{a}_{i,j} f(0) - \sum_{j \in \mathcal{J}} \bar{a}_{i,j} f(\tilde{y}_j^\star)\right)\right)\partial f(0) \\
&= \begin{cases} \big(\alpha_i - \beta f(0)\big)\partial f(0) & \text{if } \deg_G^+(i) = 0, \\ \big(\alpha_i - \beta f(0) - \gamma \sum_{j \in \mathcal{J}} \bar{a}_{i,j}\big(f(0) - f(\tilde{y}_j^\star)\big)\big)\partial f(0) & \text{if } \deg_G^+(i) > 0, \end{cases} \\
&\geq \begin{cases} \big(\alpha_i - \beta f(0)\big)\partial f(0) & \text{if } \deg_G^+(i) = 0, \\ \big(\alpha_i - \beta f(0) - \gamma\big(f(0) - f(\bar{v})\big)\big)\partial f(0) & \text{if } \deg_G^+(i) > 0, \end{cases} \\
&> 0
\end{aligned}$$

because $\sum_{j\in\mathcal{I}\setminus\mathcal{J}} \bar{a}_{i,j} + \sum_{j\in\mathcal{J}} \bar{a}_{i,j} = 1$, $\partial f(0) > 0$ (Assumption F), $\alpha_i > \beta f(0)$ (Assumption F and Conditions 1.15.2 and 1.15.3), and $\alpha_i > \beta f(0) + \gamma(f(0) - f(\bar{v}))$ (Condition 1.15.3). The preceding inequality contradicts (D.36). Consequently, $\tilde{\boldsymbol{y}}^\star$ cannot be a boundary NE of Γ of type B-3.

Fourth, I show that a situation where some players play zero and the remaining players play the maximum possible action (this is referred to as a boundary NE of type B-4 in Table D.1) cannot be a boundary NE of Γ. Suppose, for the sake of contradiction, $\tilde{\boldsymbol{y}}^\star$ is a boundary NE of Γ of type B-4. Specifically, suppose there exists a $\mathcal{J} \subset \mathcal{I}$ with $0 < |\mathcal{J}| < n$ such that (i) for all $i \in \mathcal{I} \setminus \mathcal{J}$, $\tilde{y}_i^\star = 0$, and (ii) for all $i \in \mathcal{J}$, $\tilde{y}_i^\star = \bar{v}$. Then we must have

$$\forall i \in \mathcal{I} \setminus \mathcal{J} \quad \frac{\partial u_i(\tilde{\boldsymbol{y}}^\star)}{\partial y_i} \le 0 \tag{D.38}$$

and

$$\forall i \in \mathcal{J} \quad \frac{\partial u_i(\tilde{\boldsymbol{y}}^\star)}{\partial y_i} \ge 0. \tag{D.39}$$

Let $i \in \mathcal{I} \setminus \mathcal{J}$ and $k \in \mathcal{J}$. Since $\gamma < 0$ (Condition 1.15.2), $\bar{\boldsymbol{A}}(G)$ is nonnegative, and f is strictly increasing (Assumption F), we have

$$\gamma \sum_{j\in\mathcal{J}} \bar{a}_{i,j}\big(f(0) - f(\bar{v})\big) \le \gamma\big(f(0) - f(\bar{v})\big) \tag{D.40}$$

and

$$\gamma \sum_{j\in\mathcal{I}\setminus\mathcal{J}} \bar{a}_{k,j}\big(f(\bar{v}) - f(0)\big) \ge \gamma\big(f(\bar{v}) - f(0)\big). \tag{D.41}$$

Using (D.40), we find

$$\begin{aligned}
\frac{\partial u_i(\tilde{\boldsymbol{y}}^\star)}{\partial y_i} &= \left(\alpha_i - \beta f(0) - \gamma\left(f(0) - \sum_{j\in\mathcal{I}\setminus\mathcal{J}} \bar{a}_{i,j} f(0) - \sum_{j\in\mathcal{J}} \bar{a}_{i,j} f(\bar{v})\right)\right)\partial f(0) \\
&= \begin{cases} \big(\alpha_i - \beta f(0)\big)\partial f(0) & \text{if } \deg_G^+(i) = 0, \\ \big(\alpha_i - \beta f(0) - \gamma \sum_{j\in\mathcal{J}} \bar{a}_{i,j}\big(f(0) - f(\bar{v})\big)\big)\partial f(0) & \text{if } \deg_G^+(i) > 0, \end{cases} \\
&\ge \begin{cases} \big(\alpha_i - \beta f(0)\big)\partial f(0) & \text{if } \deg_G^+(i) = 0, \\ \big(\alpha_i - \beta f(0) - \gamma\big(f(0) - f(\bar{v})\big)\big)\partial f(0) & \text{if } \deg_G^+(i) > 0, \end{cases} \\
&> 0
\end{aligned}$$

because $\sum_{j\in\mathcal{I}\setminus\mathcal{J}} \bar{a}_{i,j} + \sum_{j\in\mathcal{J}} \bar{a}_{i,j} = 1$, $\partial f(0) > 0$ (Assumption F), $\alpha_i > \beta f(0)$ (Assumption F and Conditions 1.15.2 and 1.15.3), and $\alpha_i > \beta f(0) + \gamma(f(0) - f(\bar{v}))$ (Condition 1.15.3). The preceding inequality contradicts (D.38). Using (D.41), we find

$$\frac{\partial u_k(\tilde{\boldsymbol{y}}^\star)}{\partial y_k} = \left(\alpha_k - \beta f(\bar{v}) - \gamma\left(f(\bar{v}) - \sum_{j\in\mathcal{I}\setminus\mathcal{J}} \bar{a}_{k,j} f(0) - \sum_{j\in\mathcal{J}} \bar{a}_{k,j} f(\bar{v})\right)\right)\partial f(\bar{v})$$

$$
\begin{aligned}
&= \begin{cases} \big(\alpha_k - \beta f(\bar{v})\big)\partial f(\bar{v}) & \text{if } \deg_G^+(k) = 0, \\ \big(\alpha_k - \beta f(\bar{v}) - \gamma \sum_{j\in\mathcal{I}\setminus\mathcal{J}} \bar{a}_{k,j}\big(f(\bar{v}) - f(0)\big)\big)\partial f(\bar{v}) & \text{if } \deg_G^+(k) > 0, \end{cases} \\
&\le \begin{cases} \big(\alpha_k - \beta f(\bar{v})\big)\partial f(\bar{v}) & \text{if } \deg_G^+(k) = 0, \\ \big(\alpha_i - \beta f(\bar{v}) - \gamma\big(f(\bar{v}) - f(0)\big)\big)\partial f(\bar{v}) & \text{if } \deg_G^+(k) > 0, \end{cases} \\
&< 0
\end{aligned}
$$

because $\sum_{j\in\mathcal{I}\setminus\mathcal{J}} \bar{a}_{i,j} + \sum_{j\in\mathcal{J}} \bar{a}_{i,j} = 1$, $\partial f(\bar{v}) > 0$ (Assumption F), $\alpha_k < \beta f(\bar{v})$ (Assumption F and Conditions 1.15.2 and 1.15.4), and $\alpha_k < \beta f(\bar{v}) + \gamma(f(\bar{v}) - f(0))$ (Condition 1.15.4). The preceding inequality contradicts (D.39). Consequently, $\tilde{\boldsymbol{y}}^\star$ cannot be a boundary NE of Γ of type B-4.

Fifth, I show that a situation where some players play an action in the interior of the action space and the remaining players play the maximum possible action (this is referred to as a boundary NE of type B-5 in Table D.1) cannot be a boundary NE of Γ. Suppose, for the sake of contradiction, $\tilde{\boldsymbol{y}}^\star$ is a boundary NE of Γ of type B-5. Specifically, suppose there exists a $\mathcal{J} \subset \mathcal{I}$ with $0 < |\mathcal{J}| < n$ such that (i) for all $i \in \mathcal{I} \setminus \mathcal{J}$, $\tilde{y}_i^\star \in (0, \bar{v})$, and (ii) for all $i \in \mathcal{J}$, $\tilde{y}_i^\star = \bar{v}$. Then we must have

$$
\forall i \in \mathcal{J} \quad \frac{\partial u_i(\tilde{\boldsymbol{y}}^\star)}{\partial y_i} \ge 0. \tag{D.42}
$$

Let $i \in \mathcal{J}$. Since $\gamma < 0$ (Condition 1.15.2), $\bar{\boldsymbol{A}}(G)$ is nonnegative, and f is strictly increasing (Assumption F), we have

$$
\gamma \sum_{j\in\mathcal{I}\setminus\mathcal{J}} \bar{a}_{i,j}\big(f(\bar{v}) - f(\tilde{y}_j^\star)\big) \ge \gamma\big(f(\bar{v}) - f(0)\big). \tag{D.43}
$$

Using (D.43), we find

$$
\begin{aligned}
\frac{\partial u_i(\tilde{\boldsymbol{y}}^\star)}{\partial y_i} &= \left(\alpha_i - \beta f(\bar{v}) - \gamma\left(f(\bar{v}) - \sum_{j\in\mathcal{I}\setminus\mathcal{J}} \bar{a}_{i,j} f(\tilde{y}_j^\star) - \sum_{j\in\mathcal{J}} \bar{a}_{i,j} f(\bar{v})\right)\right)\partial f(\bar{v}) \\
&= \begin{cases} \big(\alpha_i - \beta f(\bar{v})\big)\partial f(\bar{v}) & \text{if } \deg_G^+(i) = 0, \\ \big(\alpha_i - \beta f(\bar{v}) - \gamma \sum_{j\in\mathcal{I}\setminus\mathcal{J}} \bar{a}_{i,j}\big(f(\bar{v}) - f(\tilde{y}_j^\star)\big)\big)\partial f(\bar{v}) & \text{if } \deg_G^+(i) > 0, \end{cases} \\
&\le \begin{cases} \big(\alpha_i - \beta f(\bar{v})\big)\partial f(\bar{v}) & \text{if } \deg_G^+(i) = 0, \\ \big(\alpha_i - \beta f(\bar{v}) - \gamma\big(f(\bar{v}) - f(0)\big)\big)\partial f(\bar{v}) & \text{if } \deg_G^+(i) > 0, \end{cases} \\
&< 0
\end{aligned}
$$

because $\sum_{j\in\mathcal{I}\setminus\mathcal{J}} \bar{a}_{i,j} + \sum_{j\in\mathcal{J}} \bar{a}_{i,j} = 1$, $\partial f(\bar{v}) > 0$ (Assumption F), $\alpha_i < \beta f(\bar{v})$ (Assumption F and Conditions 1.15.2 and 1.15.4), and $\alpha_i < \beta f(\bar{v}) + \gamma(f(\bar{v}) - f(0))$ (Condition 1.15.4). The preceding inequality contradicts (D.42). Consequently, $\tilde{\boldsymbol{y}}^\star$ cannot be a boundary NE of Γ of type B-5.

Sixth, I show that a situation where some players play zero, some players play an action in the interior of the action space, and the remaining players play the maximum possible action (this is referred to as a boundary NE of type B-6 in

Table D.1) cannot be a boundary NE of Γ. Suppose, for the sake of contradiction, $\tilde{\boldsymbol{y}}^\star$ is a boundary NE of Γ of type B-6. Specifically, suppose there exist $\mathcal{J} \subset \mathcal{I}$ and $\mathcal{K} \subset \mathcal{I}$ with $0 < |\mathcal{J}| < n$, $0 < |\mathcal{K}| < n$, and $\mathcal{K} \cap \mathcal{I} = \emptyset$ such that (i) for all $i \in \mathcal{I} \setminus (\mathcal{J} \cup \mathcal{K})$, $\tilde{y}_i^\star = 0$, (ii) for all $i \in \mathcal{J}$, $\tilde{y}_i^\star \in (0, \bar{v})$, and (iii) for all $i \in \mathcal{K}$, $\tilde{y}_i^\star = \bar{v}$. Then we must have

$$\forall i \in \mathcal{I} \setminus (\mathcal{J} \cup \mathcal{K}) \quad \frac{\partial u_i(\tilde{\boldsymbol{y}}^\star)}{\partial y_i} \leq 0 \tag{D.44}$$

and

$$\forall i \in \mathcal{K} \quad \frac{\partial u_i(\tilde{\boldsymbol{y}}^\star)}{\partial y_i} \geq 0. \tag{D.45}$$

Let $i \in \mathcal{I} \setminus (\mathcal{J} \cup \mathcal{K})$ and $k \in \mathcal{K}$. Since $\gamma < 0$ (Condition 1.15.2), $\bar{\boldsymbol{A}}(G)$ is nonnegative, and f is strictly increasing (Assumption F), we have

$$\gamma \sum_{j \in \mathcal{J}} \bar{a}_{i,j}\big(f(0) - f(\tilde{y}_j^\star)\big) + \gamma \sum_{j \in \mathcal{K}} \bar{a}_{i,j}\big(f(0) - f(\bar{v})\big) \leq \gamma\big(f(0) - f(\bar{v})\big) \tag{D.46}$$

and

$$\gamma \sum_{j \in \mathcal{I} \setminus (\mathcal{J} \cup \mathcal{K})} \bar{a}_{k,j}\big(f(\bar{v}) - f(0)\big) + \gamma \sum_{j \in \mathcal{J}} \bar{a}_{k,j}\big(f(\bar{v}) - f(\tilde{y}_j^\star)\big) \geq \gamma\big(f(\bar{v}) - f(0)\big). \tag{D.47}$$

Using (D.46), we find

$$\begin{aligned}
\frac{\partial u_i(\tilde{\boldsymbol{y}}^\star)}{\partial y_i} &= \Bigg(\alpha_i - \beta f(0) - \gamma\Bigg(f(0) - \sum_{j \in \mathcal{I} \setminus (\mathcal{J} \cup \mathcal{K})} \bar{a}_{i,j} f(0) - \sum_{j \in \mathcal{J}} \bar{a}_{i,j} f(\tilde{y}_j^\star) \\
&\qquad - \sum_{j \in \mathcal{K}} \bar{a}_{i,j} f(\bar{v})\Bigg)\Bigg)\partial f(0) \\
&= \begin{cases} (\alpha_i - \beta f(0))\partial f(0) & \text{if } \deg_G^+(i) = 0, \\ \big(\alpha_i - \beta f(0) - \gamma \sum_{j \in \mathcal{J}} \bar{a}_{i,j}\big(f(0) - f(\tilde{y}_j^\star)\big) & \\ \quad - \gamma \sum_{j \in \mathcal{K}} \bar{a}_{i,j}\big(f(0) - f(\bar{v})\big)\big)\partial f(0) & \text{if } \deg_G^+(i) > 0, \end{cases} \\
&\geq \begin{cases} (\alpha_i - \beta f(0))\partial f(0) & \text{if } \deg_G^+(i) = 0, \\ \big(\alpha_i - \beta f(0) - \gamma\big(f(0) - f(\bar{v})\big)\big)\partial f(0) & \text{if } \deg_G^+(i) > 0, \end{cases} \\
&> 0
\end{aligned}$$

because $\sum_{j \in \mathcal{I} \setminus (\mathcal{J} \cup \mathcal{K})} \bar{a}_{i,j} + \sum_{j \in \mathcal{J}} \bar{a}_{i,j} + \sum_{j \in \mathcal{K}} \bar{a}_{i,j} = 1$, $\partial f(0) > 0$ (Assumption F), $\alpha_i > \beta f(0)$ (Assumption F and Conditions 1.15.2 and 1.15.3), and $\alpha_i > \beta f(0) + \gamma(f(0) - f(\bar{v}))$ (Condition 1.15.3). The preceding inequality contradicts (D.44). Using (D.47), we find

$$\frac{\partial u_k(\tilde{\boldsymbol{y}}^\star)}{\partial y_k} = \Bigg(\alpha_k - \beta f(\bar{v}) - \gamma\Bigg(f(\bar{v}) - \sum_{j \in \mathcal{I} \setminus (\mathcal{J} \cup \mathcal{K})} \bar{a}_{k,j} f(0) - \sum_{j \in \mathcal{J}} \bar{a}_{k,j} f(\tilde{y}_j^\star)$$

$$
\begin{aligned}
&\qquad - \sum_{j \in \mathcal{K}} \bar{a}_{k,j} f(\bar{v}) \Big) \Big) \partial f(\bar{v}) \\
&= \begin{cases} \big(\alpha_k - \beta f(\bar{v})\big) \partial f(\bar{v}) & \text{if } \deg_G^+(k) = 0, \\ \big(\alpha_k - \beta f(\bar{v}) - \gamma \sum_{j \in \mathcal{I} \setminus (\mathcal{J} \cup \mathcal{K})} \bar{a}_{k,j} \big(f(\bar{v}) - f(0)\big) & \\ - \gamma \sum_{j \in \mathcal{J}} \bar{a}_{k,j} \big(f(\bar{v}) - f(\tilde{y}_j^\star)\big)\big) \partial f(\bar{v}) & \text{if } \deg_G^+(k) > 0, \end{cases} \\
&\leq \begin{cases} \big(\alpha_k - \beta f(\bar{v})\big) \partial f(\bar{v}) & \text{if } \deg_G^+(k) = 0, \\ \big(\alpha_k - \beta f(\bar{v}) - \gamma \big(f(\bar{v}) - f(0)\big)\big) \partial f(\bar{v}) & \text{if } \deg_G^+(k) > 0, \end{cases} \\
&< 0
\end{aligned}
$$

because $\sum_{j \in \mathcal{I} \setminus (\mathcal{J} \cup \mathcal{K})} \bar{a}_{i,j} + \sum_{j \in \mathcal{J}} \bar{a}_{i,j} + \sum_{j \in \mathcal{K}} \bar{a}_{i,j} = 1$, $\partial f(\bar{v}) > 0$ (Assumption F), $\alpha_k < \beta f(\bar{v})$ (Assumption F and Conditions 1.15.2 and 1.15.4), and $\alpha_k < \beta f(\bar{v}) + \gamma(f(\bar{v}) - f(0))$ (Condition 1.15.4). The preceding inequality contradicts (D.45). Consequently, $\tilde{\boldsymbol{y}}^\star$ cannot be a boundary NE of Γ of type B-6. ∎

Proof of Proposition 1.16

Let $\Gamma := (\mathcal{I}, G, \mathbb{R}_+, \{(\alpha_i, \beta, \gamma)\}_{i \in \mathcal{I}}, f)$ be a NALA game, let $\boldsymbol{\alpha}$ be defined as in Section 1.3.2, and let $\alpha_{\min}$ be defined as in Section 1.3.3. Suppose Conditions 1.16.1 to 1.16.4 are satisfied. In what follows, the component in row i and column j of $\bar{\boldsymbol{A}}(G)$, $\bar{a}_{i,j}(G)$, is abbreviated to $\bar{a}_{i,j}$.

First, note that $f(\mathbb{R}_+) = [f(0), +\infty)$ and $\operatorname{int}(f(\mathbb{R}_+)^n) = (f(0), +\infty)^n$ because $f(\mathbb{R}_+)$ is an interval (Lemma 1.1) and f is strictly increasing (Assumption F) and not bounded above (Condition 1.16.4). Second, note that $\rho(\bar{\boldsymbol{A}}(G)) = 1$ (Lemma B.8). Third, note that Conditions 1.16.1 and 1.16.2 imply that $\beta + \gamma > 0$ and $0 \leq \gamma/(\beta + \gamma) < 1$. Fourth, note that Condition 1.16.3 is equivalent to $\beta f(0) \mathbf{1}_n <_c \boldsymbol{\alpha}$.

Let $\mathcal{Y}^\star \subset \mathbb{R}^n_{++}$ denote the set of all interior NEs in pure strategies of Γ. I show that $|\mathcal{Y}^\star| = 1$. To this end, let $\boldsymbol{y}^\star := (y_1^\star, \dots, y_n^\star) \in \mathbb{R}^n_+$. Note that $\boldsymbol{y}^\star \in \mathcal{Y}^\star$ if and only if

$$\boldsymbol{y}^\star \in \mathbb{R}^n_{++}, \tag{D.48}$$

$$\forall i \in \mathcal{I} \quad \frac{\partial u_i(\boldsymbol{y}^\star)}{\partial y_i} = 0, \tag{D.49}$$

and

$$\forall i \in \mathcal{I} \quad \frac{\partial^2 u_i(\boldsymbol{y}^\star)}{\partial {y_i}^2} < 0. \tag{D.50}$$

Analogous to the proof of Proposition 1.13, we find

$$\text{(D.49)} \quad \Leftrightarrow \quad \Big(\boldsymbol{I}_n - \frac{\gamma}{\beta + \gamma} \bar{\boldsymbol{A}}(G)\Big) \boldsymbol{f}(\boldsymbol{y}^\star) = \frac{1}{\beta + \gamma} \boldsymbol{\alpha} \tag{D.51}$$

and

$$\text{(D.50)} \quad \Leftrightarrow \quad \forall i \in \mathcal{I} \quad \beta + \gamma \mathbb{1}_{\mathcal{I} \setminus \mathcal{I}_0^+(G)}(i) > 0.$$

The matrix $\boldsymbol{I}_n - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G)$ is nonsingular with a nonnegative inverse that is bounded below by $\boldsymbol{I}_n$ because $0 \leq \gamma/(\beta+\gamma) < 1$ and $\rho(\bar{\boldsymbol{A}}(G)) = 1$ (Lemma B.6). It follows that the system of linear equations (D.6) has a unique solution $\boldsymbol{z}^\star \in \mathbb{R}^n$, which is given by (D.7). Analogous to the proof of Proposition 1.14, the two inequalities $\boldsymbol{I}_n \leq_c (\boldsymbol{I}_n - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G))^{-1}$ and $\beta f(0)\boldsymbol{1}_n <_c \boldsymbol{\alpha}$ imply that $\boldsymbol{z}^\star \in (f(0), +\infty)^n$. It is clear from a comparison of the right system of equations of (D.51) and the system of linear equations (D.6) that $\boldsymbol{y}^\star$ is a solution to the system of equations $\boldsymbol{z}^\star = \boldsymbol{f}(\boldsymbol{y}^\star)$. This system has a unique solution $\boldsymbol{y}^\star \in \mathbb{R}^n_{++}$ because f is bijective (Lemma 1.1) and $\boldsymbol{z}^\star \in (f(0), +\infty)^n$. In summary, the system of equations (D.49) has a unique solution $\boldsymbol{y}^\star$, which is given by (1.7). Finally, note that, for all $i \in \mathcal{I}$, $\beta + \gamma \mathbb{1}_{\mathcal{I}\setminus\mathcal{I}_0^+(G)}(i) > 0$, because $\beta > 0$ (Condition 1.16.1) and $\beta + \gamma > 0$ (Conditions 1.16.1 and 1.16.2). The preceding arguments demonstrate that there exists a unique $\boldsymbol{y}^\star \in \mathbb{R}^n_+$ that satisfies (D.48), (D.49), and (D.50). To sum up, I have shown that $\boldsymbol{y}^\star$ as given by (1.7) lies in $\mathcal{Y}^\star$ and that $|\mathcal{Y}^\star| = 1$.

In the remainder of the proof, I show that Γ has no boundary NEs. Table D.2 gives an overview of all possible types of boundary NEs of Γ. For example, a boundary NE of type B-2 is characterized by the fact that some players play zero and the remaining players play a positive action.

Table D.2. Possible types of boundary Nash equilibria of Γ (Propositions 1.16 and 1.17)

	Action and set of actions	
Type	0	$\mathbb{R}_{++}$
B-1	$\times$	
B-2	$\times$	$\times$

First, $\boldsymbol{0}_n$ (this is referred to as a boundary NE of type B-1 in Table D.2) cannot be a boundary NE of Γ. See the proof of Proposition 1.14 for the precise argument.

Second, I show that a situation where some players play zero and the remaining players play a positive action (this is referred to as a boundary NE of type B-2 in Table D.2) cannot be a boundary NE of Γ. Suppose, for the sake of contradiction, $\tilde{\boldsymbol{y}}^\star := (\tilde{y}_1^\star, \ldots, \tilde{y}_n^\star) \in \mathbb{R}^n_+$ is a boundary NE of Γ of type B-2. Specifically, suppose there exists a $\mathcal{J} \subset \mathcal{I}$ with $0 < |\mathcal{J}| < n$ such that (i) for all $i \in \mathcal{I} \setminus \mathcal{J}$, $\tilde{y}_i^\star = 0$, and (ii) for all $i \in \mathcal{J}$, $\tilde{y}_i^\star \in \mathbb{R}_{++}$. Then we must have

$$\forall i \in \mathcal{I} \setminus \mathcal{J} \quad \frac{\partial u_i(\tilde{\boldsymbol{y}}^\star)}{\partial y_i} \leq 0. \tag{D.52}$$

Let $i \in \mathcal{I} \setminus \mathcal{J}$. Since $\gamma \geq 0$ (Condition 1.16.2), $\bar{\boldsymbol{A}}(G)$ is nonnegative, and f is strictly increasing (Assumption F), we have

$$\gamma \sum_{j \in \mathcal{J}} \bar{a}_{i,j}\big(f(0) - f(\tilde{y}_j^\star)\big) \leq 0. \tag{D.53}$$

Using (D.53), we find

$$
\begin{aligned}
\frac{\partial u_i(\tilde{\boldsymbol{y}}^\star)}{\partial y_i} &= \Big(\alpha_i - \beta f(0) - \gamma\Big(f(0) - \sum_{j\in\mathcal{I}\setminus\mathcal{J}} \bar{a}_{i,j} f(0) - \sum_{j\in\mathcal{J}} \bar{a}_{i,j} f(\tilde{y}_j^\star)\Big)\Big)\partial f(0) \\
&= \begin{cases} (\alpha_i - \beta f(0))\partial f(0) & \text{if } \deg_G^+(i) = 0, \\ (\alpha_i - \beta f(0) - \gamma \sum_{j\in\mathcal{J}} \bar{a}_{i,j}(f(0) - f(\tilde{y}_j^\star)))\partial f(0) & \text{if } \deg_G^+(i) > 0, \end{cases} \\
&\geq (\alpha_i - \beta f(0))\partial f(0) \\
&> 0
\end{aligned}
$$

because $\sum_{j\in\mathcal{I}\setminus\mathcal{J}} \bar{a}_{i,j} + \sum_{j\in\mathcal{J}} \bar{a}_{i,j} = 1$, $\partial f(0) > 0$ (Assumption F), and $\alpha_i > \beta f(0)$ (Condition 1.16.3). The preceding inequality contradicts (D.52). Consequently, $\tilde{\boldsymbol{y}}^\star$ cannot be a boundary NE of Γ of type B-2. ■

Proof of Proposition 1.17

Let $\Gamma := (\mathcal{I}, G, \mathbb{R}_+, \{(\alpha_i, \beta, \gamma)\}_{i\in\mathcal{I}}, f)$ be a NALA game, let $\boldsymbol{\alpha}$ be defined as in Section 1.3.2, and let $\alpha_{\min}$ and $\alpha_{\max}$ be defined as in Section 1.3.3. Suppose Conditions 1.17.1 to 1.17.6 are satisfied. In what follows, the component in row i and column j of $\bar{\boldsymbol{A}}(G)$, $\bar{a}_{i,j}(G)$, is abbreviated to $\bar{a}_{i,j}$.

First, note that $f(\mathbb{R}_+) = [f(0), +\infty)$ and $\operatorname{int}(f(\mathbb{R}_+)^n) = (f(0), +\infty)^n$ because $f(\mathbb{R}_+)$ is an interval (Lemma 1.1) and f is strictly increasing (Assumption F) and not bounded above (Condition 1.17.6). Second, note that $\rho(\bar{\boldsymbol{A}}(G)) = 1$ (Lemma B.8). Third, note that $\rho(\bar{\boldsymbol{A}}(G)^2) = \rho(\bar{\boldsymbol{A}}(G))^2$, which is a consequence of the *spectral mapping theorem* (see, for example, Kubrusly 2011, Theorem 6.19 and Corollary 6.20). Fourth, note that Conditions 1.17.1 and 1.17.2 imply that $\beta + \gamma > 0$ and $-1 < \gamma/(\beta+\gamma) < 0$. Fifth, note that Conditions 1.17.1 to 1.17.4 imply that

$$
\frac{\beta(\beta+2\gamma)}{\beta+\gamma} f(0)\mathbf{1}_n <_c \left(\boldsymbol{I}_n + \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)\boldsymbol{\alpha}. \tag{D.54}
$$

Indeed, for all $i \in \mathcal{I}_0^+(G)$,

$$
\beta f(0) < \alpha_{\min} \quad \Rightarrow \quad \beta f(0) < \alpha_i \quad \Leftrightarrow \quad \frac{\beta(\beta+2\gamma)}{\beta+\gamma} f(0) < \alpha_i + \frac{\gamma}{\beta+\gamma}\sum_{j\in\mathcal{I}} \bar{a}_{i,j}\alpha_j,
$$

and for all $i \in \mathcal{I} \setminus \mathcal{I}_0^+(G)$,

$$
\frac{\beta(\beta+2\gamma)}{\beta+\gamma} f(0) < \alpha_{\min} + \frac{\gamma}{\beta+\gamma}\alpha_{\max} \leq \alpha_i + \frac{\gamma}{\beta+\gamma}\sum_{j\in\mathcal{I}} \bar{a}_{i,j}\alpha_j,
$$

where the first inequality is according to Condition 1.17.4.

Let $\mathcal{Y}^\star \subset \mathbb{R}_{++}^n$ denote the set of all interior NEs in pure strategies of Γ. I show that $|\mathcal{Y}^\star| = 1$. To this end, let $\boldsymbol{y}^\star := (y_1^\star, \ldots, y_n^\star) \in \mathbb{R}_+^n$. Note that $\boldsymbol{y}^\star \in \mathcal{Y}^\star$ if and only if

$$
\boldsymbol{y}^\star \in \mathbb{R}_{++}^n, \tag{D.55}
$$

$$\forall i \in \mathcal{I} \quad \frac{\partial u_i(\boldsymbol{y}^\star)}{\partial y_i} = 0, \tag{D.56}$$

and

$$\forall i \in \mathcal{I} \quad \frac{\partial^2 u_i(\boldsymbol{y}^\star)}{\partial {y_i}^2} < 0. \tag{D.57}$$

Analogous to the proof of Proposition 1.13, we find

$$(\text{D.56}) \quad \Leftrightarrow \quad \left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)\boldsymbol{f}(\boldsymbol{y}^\star) = \frac{1}{\beta+\gamma}\boldsymbol{\alpha} \tag{D.58}$$

and

$$(\text{D.57}) \quad \Leftrightarrow \quad \forall i \in \mathcal{I} \quad \beta + \gamma \mathbb{1}_{\mathcal{I}\setminus\mathcal{I}_0^+(G)}(i) > 0.$$

The matrix $\boldsymbol{I}_n - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G)$ is nonsingular because $|\gamma/(\beta+\gamma)| < 1$ and $\rho(\bar{\boldsymbol{A}}(G)) = 1$ (Lemma B.3). It follows that the system of linear equations (D.6) has a unique solution $\boldsymbol{z}^\star \in \mathbb{R}^n$, which is given by (D.7). Next, I show that $\boldsymbol{z}^\star \in (f(0), +\infty)^n$. The matrix $\boldsymbol{I}_n - \gamma^2/(\beta+\gamma)^2\bar{\boldsymbol{A}}(G)^2$ is nonsingular with a nonnegative inverse that is bounded below by $\boldsymbol{I}_n$ because $0 < \gamma^2/(\beta+\gamma)^2\rho(\bar{\boldsymbol{A}}(G)^2) < 1$ (Lemma B.6). It follows that $\boldsymbol{z}^\star$ has the representation (D.34). The equality (D.35) and the two inequalities $\boldsymbol{I}_n \leq_c (\boldsymbol{I}_n - \gamma^2/(\beta+\gamma)^2\bar{\boldsymbol{A}}(G)^2)^{-1}$ and (D.54) imply that $\boldsymbol{z}^\star \in (f(0), +\infty)^n$. It is clear from a comparison of the right system of equations of (D.58) and the system of linear equations (D.6) that $\boldsymbol{y}^\star$ is a solution to the system of equations $\boldsymbol{z}^\star = \boldsymbol{f}(\boldsymbol{y}^\star)$. This system has a unique solution $\boldsymbol{y}^\star \in \mathbb{R}^n_{++}$ because f is bijective (Lemma 1.1) and $\boldsymbol{z}^\star \in (f(0), +\infty)^n$. In summary, the system of equations (D.56) has a unique solution $\boldsymbol{y}^\star$, which is given by (1.7). Finally, note that, for all $i \in \mathcal{I}$, $\beta + \gamma \mathbb{1}_{\mathcal{I}\setminus\mathcal{I}_0^+(G)}(i) > 0$, because $\beta > 0$ (Condition 1.17.1) and $\beta + \gamma > 0$ (Conditions 1.17.1 and 1.17.2). The preceding arguments demonstrate that there exists a unique $\boldsymbol{y}^\star \in \mathbb{R}^n_+$ that satisfies (D.55), (D.56), and (D.57). To sum up, I have shown that $\boldsymbol{y}^\star$ as given by (1.7) lies in $\mathcal{Y}^\star$ and that $|\mathcal{Y}^\star| = 1$.

In the remainder of the proof, I show that Γ has no boundary NEs. Table D.2 gives an overview of all possible types of boundary NEs of Γ.

First, $\boldsymbol{0}_n$ (this is referred to as a boundary NE of type B-1 in Table D.2) cannot be a boundary NE of Γ. See the proof of Proposition 1.14 for the precise argument.

Second, I show that a situation where some players play zero and the remaining players play a positive action (this is referred to as a boundary NE of type B-2 in Table D.2) cannot be a boundary NE of Γ. Suppose, for the sake of contradiction, $\tilde{\boldsymbol{y}}^\star := (\tilde{y}_1^\star, \ldots, \tilde{y}_n^\star) \in \mathbb{R}^n_+$ is a boundary NE of Γ of type B-2. Specifically, suppose there exists a $\mathcal{J} \subset \mathcal{I}$ with $0 < |\mathcal{J}| < n$ such that (i) for all $i \in \mathcal{I} \setminus \mathcal{J}$, $\tilde{y}_i^\star = 0$, and (ii) for all $i \in \mathcal{J}$, $\tilde{y}_i^\star \in \mathbb{R}_{++}$. Then we must have

$$\forall i \in \mathcal{I} \setminus \mathcal{J} \quad \frac{\partial u_i(\tilde{\boldsymbol{y}}^\star)}{\partial y_i} \leq 0 \tag{D.59}$$

and

$$\forall i \in \mathcal{J} \quad \frac{\partial u_i(\tilde{\boldsymbol{y}}^\star)}{\partial y_i} = 0. \tag{D.60}$$

In the rest of the proof, I show that, for all $i \in \mathcal{I} \setminus \mathcal{J}$, $\partial u_i(\tilde{\boldsymbol{y}}^\star)/\partial y_i > 0$. To this end, I establish an upper bound for $\{f(\tilde{y}_i^\star) \mid i \in \mathcal{J}\}$. Simple algebra shows that the system of equations (D.60) is equivalent to

$$\left(\boldsymbol{I}_{|\mathcal{J}|} - \frac{\gamma}{\beta+\gamma}[\bar{\boldsymbol{A}}(G)]_{\mathcal{J},\mathcal{J}}\right)[\boldsymbol{f}(\tilde{\boldsymbol{y}}^\star)]_{\mathcal{J}} = \frac{1}{\beta+\gamma}[\boldsymbol{\alpha}]_{\mathcal{J}} + \frac{\gamma}{\beta+\gamma}f(0)[\bar{\boldsymbol{A}}(G)]_{\mathcal{J},\mathcal{I}\setminus\mathcal{J}}\mathbf{1}_{|\mathcal{I}\setminus\mathcal{J}|}$$

because $\partial f > 0$ (Assumption F). Note that $\rho([\bar{\boldsymbol{A}}(G)]_{\mathcal{J},\mathcal{J}}) \leq \rho(\bar{\boldsymbol{A}}(G))$ (see, for example, Berman and Plemmons 1994, Corollary 1.6 on p. 28). It follows that for all $p \in \{1,2\}$, the matrix $\boldsymbol{I}_{|\mathcal{J}|} - \gamma^p/(\beta+\gamma)^p[\bar{\boldsymbol{A}}(G)]^p_{\mathcal{J},\mathcal{J}}$ is nonsingular because $|\gamma/(\beta+\gamma)| < 1$ and $\rho(\bar{\boldsymbol{A}}(G)) = 1$ (Lemma B.3). We find

$$\begin{aligned}
[\boldsymbol{f}(\tilde{\boldsymbol{y}}^\star)]_{\mathcal{J}} &= \frac{1}{\beta+\gamma}\left(\boldsymbol{I}_{|\mathcal{J}|} - \frac{\gamma}{\beta+\gamma}[\bar{\boldsymbol{A}}(G)]_{\mathcal{J},\mathcal{J}}\right)^{-1} \\
&\quad \times \left([\boldsymbol{\alpha}]_{\mathcal{J}} + \gamma f(0)[\bar{\boldsymbol{A}}(G)]_{\mathcal{J},\mathcal{I}\setminus\mathcal{J}}\mathbf{1}_{|\mathcal{I}\setminus\mathcal{J}|}\right) \\
&= \frac{1}{\beta+\gamma}\left(\boldsymbol{I}_{|\mathcal{J}|} - \frac{\gamma^2}{(\beta+\gamma)^2}[\bar{\boldsymbol{A}}(G)]^2_{\mathcal{J},\mathcal{J}}\right)^{-1}\left(\boldsymbol{I}_{|\mathcal{J}|} + \frac{\gamma}{\beta+\gamma}[\bar{\boldsymbol{A}}(G)]_{\mathcal{J},\mathcal{J}}\right) \\
&\quad \times \left([\boldsymbol{\alpha}]_{\mathcal{J}} + \gamma f(0)[\bar{\boldsymbol{A}}(G)]_{\mathcal{J},\mathcal{I}\setminus\mathcal{J}}\mathbf{1}_{|\mathcal{I}\setminus\mathcal{J}|}\right).
\end{aligned} \tag{D.61}$$

Condition 1.17.3 implies that

$$[\boldsymbol{\alpha}]_{\mathcal{J}} + \gamma f(0)[\bar{\boldsymbol{A}}(G)]_{\mathcal{J},\mathcal{I}\setminus\mathcal{J}}\mathbf{1}_{|\mathcal{I}\setminus\mathcal{J}|} >_c (\beta+\gamma)f(0)\mathbf{1}_{|\mathcal{J}|} - \gamma f(0)[\bar{\boldsymbol{A}}(G)]_{\mathcal{J},\mathcal{J}}\mathbf{1}_{|\mathcal{J}|},$$

which, together with Conditions 1.17.1 and 1.17.2, in turn implies that

$$\begin{aligned}
&\left(\boldsymbol{I}_{|\mathcal{J}|} + \frac{\gamma}{\beta+\gamma}[\bar{\boldsymbol{A}}(G)]_{\mathcal{J},\mathcal{J}}\right)\left([\boldsymbol{\alpha}]_{\mathcal{J}} + \gamma f(0)[\bar{\boldsymbol{A}}(G)]_{\mathcal{J},\mathcal{I}\setminus\mathcal{J}}\mathbf{1}_{|\mathcal{I}\setminus\mathcal{J}|}\right) \\
&\qquad <_c (\alpha_{\max} - \beta f(0))\mathbf{1}_{|\mathcal{J}|} + (\beta+\gamma)f(0)\left(\boldsymbol{I}_{|\mathcal{J}|} - \frac{\gamma^2}{(\beta+\gamma)^2}[\bar{\boldsymbol{A}}(G)]^2_{\mathcal{J},\mathcal{J}}\right)\mathbf{1}_{|\mathcal{J}|}.
\end{aligned} \tag{D.62}$$

Let

$$\boldsymbol{c} := \left(\boldsymbol{I}_{|\mathcal{J}|} - \frac{\gamma^2}{(\beta+\gamma)^2}[\bar{\boldsymbol{A}}(G)]^2_{\mathcal{J},\mathcal{J}}\right)^{-1}\mathbf{1}_{|\mathcal{J}|}, \tag{D.63}$$

and let $c_{\max}$ denote the largest component of $\boldsymbol{c}$. I show that

$$\mathbf{1}_{|\mathcal{J}|} \leq_c \boldsymbol{c} \leq_c \frac{(\beta+\gamma)^2}{\beta(\beta+2\gamma)}\mathbf{1}_{|\mathcal{J}|}, \tag{D.64}$$

where $(\beta+\gamma)^2/(\beta(\beta+2\gamma)) > 1$. First, I establish the lower bound. The inverse of $\boldsymbol{I}_{|\mathcal{J}|} - \gamma^2/(\beta+\gamma)^2[\bar{\boldsymbol{A}}(G)]^2_{\mathcal{J},\mathcal{J}}$ is nonnegative and bounded below by $\boldsymbol{I}_{|\mathcal{J}|}$ because

$0 < \gamma^2/(\beta+\gamma)^2\rho([\bar{\boldsymbol{A}}(G)]^2_{\mathcal{J},\mathcal{J}}) < 1$ (Lemma B.6). It follows that $\mathbf{1}_{|\mathcal{J}|} \leq_c \boldsymbol{c}$. Second, I establish the upper bound. We find

$$\begin{aligned}
\text{(D.63)} \quad &\Leftrightarrow \quad \boldsymbol{c} = \mathbf{1}_{|\mathcal{J}|} + \frac{\gamma^2}{(\beta+\gamma)^2}[\bar{\boldsymbol{A}}(G)]^2_{\mathcal{J},\mathcal{J}}\boldsymbol{c} \\
&\Rightarrow \quad \boldsymbol{c} \leq_c \mathbf{1}_{|\mathcal{J}|} + \frac{\gamma^2}{(\beta+\gamma)^2}c_{\max}[\bar{\boldsymbol{A}}(G)]^2_{\mathcal{J},\mathcal{J}}\mathbf{1}_{|\mathcal{J}|} \\
&\Rightarrow \quad \boldsymbol{c} \leq_c \left(1 + \frac{\gamma^2}{(\beta+\gamma)^2}c_{\max}\right)\mathbf{1}_{|\mathcal{J}|} \\
&\Leftrightarrow \quad c_{\max} \leq 1 + \frac{\gamma^2}{(\beta+\gamma)^2}c_{\max} \\
&\Leftrightarrow \quad c_{\max} \leq \frac{(\beta+\gamma)^2}{\beta(\beta+2\gamma)} \\
&\Leftrightarrow \quad \boldsymbol{c} \leq_c \frac{(\beta+\gamma)^2}{\beta(\beta+2\gamma)}\mathbf{1}_{|\mathcal{J}|}.
\end{aligned}$$

This concludes the proof of (D.64). Next, I establish that

$$\forall i \in \mathcal{J} \quad f(\tilde{y}_i^\star) \leq \frac{\beta+\gamma}{\beta(\beta+2\gamma)}\big(\alpha_{\max} - \beta f(0)\big) + f(0). \tag{D.65}$$

We find

$$\begin{aligned}
[\boldsymbol{f}(\tilde{\boldsymbol{y}}^\star)]_{\mathcal{J}} &= \frac{1}{\beta+\gamma}\left(\boldsymbol{I}_{|\mathcal{J}|} - \frac{\gamma^2}{(\beta+\gamma)^2}[\bar{\boldsymbol{A}}(G)]^2_{\mathcal{J},\mathcal{J}}\right)^{-1} \\
&\qquad \times \left(\boldsymbol{I}_{|\mathcal{J}|} + \frac{\gamma}{\beta+\gamma}[\bar{\boldsymbol{A}}(G)]_{\mathcal{J},\mathcal{J}}\right)\Big([\boldsymbol{\alpha}]_{\mathcal{J}} + \gamma f(0)[\bar{\boldsymbol{A}}(G)]_{\mathcal{J},\mathcal{I}\setminus\mathcal{J}}\mathbf{1}_{|\mathcal{I}\setminus\mathcal{J}|}\Big) \\
&<_c \frac{\alpha_{\max} - \beta f(0)}{\beta+\gamma}\left(\boldsymbol{I}_{|\mathcal{J}|} - \frac{\gamma^2}{(\beta+\gamma)^2}[\bar{\boldsymbol{A}}(G)]^2_{\mathcal{J},\mathcal{J}}\right)^{-1}\mathbf{1}_{|\mathcal{J}|} + f(0)\mathbf{1}_{|\mathcal{J}|} \\
&\leq_c \frac{\beta+\gamma}{\beta(\beta+2\gamma)}\big(\alpha_{\max} - \beta f(0)\big)\mathbf{1}_{|\mathcal{J}|} + f(0)\mathbf{1}_{|\mathcal{J}|}.
\end{aligned}$$

The equality is according to (D.61). The first inequality follows from the two inequalities (D.62) and $\boldsymbol{I}_{|\mathcal{J}|} \leq_c (\boldsymbol{I}_{|\mathcal{J}|} - \gamma^2/(\beta+\gamma)^2[\bar{\boldsymbol{A}}(G)]^2_{\mathcal{J},\mathcal{J}})^{-1}$ (Lemma B.1). The second inequality follows from (D.63) and (D.64). This concludes the proof of (D.65). Finally, I show that, for all $i \in \mathcal{I} \setminus \mathcal{J}$, $\partial u_i(\tilde{\boldsymbol{y}}^\star)/\partial y_i > 0$. Let $i \in \mathcal{I} \setminus \mathcal{J}$. We find

$$\begin{aligned}
\frac{\partial u_i(\tilde{\boldsymbol{y}}^\star)}{\partial y_i} &= \left(\alpha_i - \beta f(0) - \gamma\left(f(0) - \sum_{j\in\mathcal{I}\setminus\mathcal{J}}\bar{a}_{i,j}f(0) - \sum_{j\in\mathcal{J}}\bar{a}_{i,j}f(\tilde{y}_j^\star)\right)\right)\partial f(0) \\
&= \begin{cases}\big(\alpha_i - \beta f(0)\big)\partial f(0) & \text{if } \deg_G^+(i) = 0, \\ \big(\alpha_i - \beta f(0) - \gamma\sum_{j\in\mathcal{J}}\bar{a}_{i,j}\big(f(0) - f(\tilde{y}_j^\star)\big)\big)\partial f(0) & \text{if } \deg_G^+(i) > 0,\end{cases} \\
&> 0
\end{aligned}$$

because $\sum_{j\in\mathcal{I}\setminus\mathcal{J}} \bar{a}_{i,j} + \sum_{j\in\mathcal{J}} \bar{a}_{i,j} = 1$, $\partial f(0) > 0$ (Assumption F), $\alpha_i > \beta f(0)$ (Condition 1.17.3), and

$$-\gamma \sum_{j\in\mathcal{J}} \bar{a}_{i,j}\big(f(0) - f(\tilde{y}_j^\star)\big) \geq \frac{\gamma(\beta+\gamma)}{\beta(\beta+2\gamma)}\big(\alpha_{\max} - \beta f(0)\big) > \beta f(0) - \alpha_{\min},$$

where the first inequality follows from (D.65) and the second from Condition 1.17.5. This concludes the proof that $\tilde{\boldsymbol{y}}^\star$ cannot be a boundary NE of Γ of type B-2. ∎

Example 1.18: Proof of (1.11)

Consider the setup of Example 1.18. The NALA game Γ has a unique and interior NE $\boldsymbol{y}^\star$ (Proposition 1.14), which is given by (1.7). We find

$$\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G) = \frac{(n-1)\beta + n\gamma}{(n-1)(\beta+\gamma)}\boldsymbol{I}_n - \frac{\gamma}{(n-1)(\beta+\gamma)}\mathbf{1}_n\mathbf{1}_n^\top.$$

Note that $(n-1)\beta + n\gamma \neq 0$. Also note that $\gamma\mathbf{1}_n\mathbf{1}_n^\top$ has rank 1 because $\gamma \neq 0$. It follows that the matrix $\boldsymbol{I}_n - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G)$ is nonsingular. Its inverse is given by Sherman and Morrison's (1949) formula (see, for example, Bartlett 1951, p. 107), which yields

$$\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1} = \frac{\beta+\gamma}{(n-1)\beta+n\gamma}\left((n-1)\boldsymbol{I}_n + \frac{\gamma}{\beta}\mathbf{1}_n\mathbf{1}_n^\top\right).$$

Using (1.7) and the preceding result, we find

$$\boldsymbol{y}^\star = \frac{n}{(n-1)\beta+n\gamma}\left(\frac{\gamma}{\beta}\frac{\langle\mathbf{1}_n,\boldsymbol{\alpha}\rangle}{n}\mathbf{1}_n + \left(1-\frac{1}{n}\right)\boldsymbol{\alpha}\right).$$

For an alternative proof of (1.11) see the proof of Proposition 1.50. ∎

Proof of Lemma 1.19

Let $\Gamma := (\mathcal{I}, G, [0,\bar{v}], \{(\alpha_i,\beta,\gamma)\}_{i\in\mathcal{I}}, f)$ be a NALA game, let $\boldsymbol{\alpha}$ be defined as in Section 1.3.2, and let $\alpha_\wedge := \alpha_{\min}$ and $\alpha_\vee := \alpha_{\max}$ be defined as in Section 1.3.3. Suppose Conditions 1.15.1 to 1.15.4 are satisfied. Note that $\beta+\gamma > 0$.

The proof of (1.17) proceeds in two steps. First, I show that

$$\left(\alpha_\wedge + \frac{\gamma}{\beta+\gamma}\alpha_\vee\right)\mathbf{1}_n \leq_c \boldsymbol{\alpha} + \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\boldsymbol{\alpha} \leq_c \left(\alpha_\vee + \frac{\gamma}{\beta+\gamma}\alpha_\wedge\right)\mathbf{1}_n. \tag{D.66}$$

Second, I show that

$$\frac{\beta(\beta+2\gamma)}{\beta+\gamma}f(0) < \alpha_\wedge + \frac{\gamma}{\beta+\gamma}\alpha_\vee \leq \alpha_\vee + \frac{\gamma}{\beta+\gamma}\alpha_\wedge < \frac{\beta(\beta+2\gamma)}{\beta+\gamma}f(\bar{v}). \tag{D.67}$$

As regards the proof of (D.66), note that, for all $i \in \mathcal{I}$, $\gamma\alpha_\vee \le \gamma\alpha_i \le \gamma\alpha_\wedge$, because $\gamma < 0$. Using this result, we find

$$\forall i \in \mathcal{I}_0^+(G) \quad \alpha_\wedge + \frac{\gamma}{\beta+\gamma}\alpha_\vee \le \alpha_i + \frac{\gamma}{\beta+\gamma}\alpha_i \le \alpha_\vee + \frac{\gamma}{\beta+\gamma}\alpha_\wedge \tag{D.68}$$

and

$$\forall i \in \mathcal{I} \setminus \mathcal{I}_0^+(G) \quad \alpha_\wedge + \frac{\gamma}{\beta+\gamma}\alpha_\vee \le \alpha_i + \frac{\gamma}{\beta+\gamma}\frac{\sum_{j\in\mathcal{N}_G^+(i)}\alpha_j}{\deg_G^+(i)} \le \alpha_\vee + \frac{\gamma}{\beta+\gamma}\alpha_\wedge. \tag{D.69}$$

Combining (D.68) and (D.69), we find

$$\begin{aligned} &\forall i \in \mathcal{I} \quad \alpha_\wedge + \frac{\gamma}{\beta+\gamma}\alpha_\vee \le [\boldsymbol{\alpha}]_i + \frac{\gamma}{\beta+\gamma}[\bar{\boldsymbol{A}}(G)\boldsymbol{\alpha}]_i \le \alpha_\vee + \frac{\gamma}{\beta+\gamma}\alpha_\wedge \\ \Leftrightarrow \quad &\left(\alpha_\wedge + \frac{\gamma}{\beta+\gamma}\alpha_\vee\right)\mathbf{1}_n \le_c \boldsymbol{\alpha} + \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\boldsymbol{\alpha} \le_c \left(\alpha_\vee + \frac{\gamma}{\beta+\gamma}\alpha_\wedge\right)\mathbf{1}_n. \end{aligned}$$

As regards the proof of (D.67), note that Conditions 1.15.3 and 1.15.4 imply that

$$\begin{aligned} \alpha_\vee - \alpha_\wedge &< \beta f(\bar{v}) + \gamma\big(f(\bar{v}) - f(0)\big) - \beta f(0) - \gamma\big(f(0) - f(\bar{v})\big) \\ &= (\beta + 2\gamma)\big(f(\bar{v}) - f(0)\big). \end{aligned} \tag{D.70}$$

We find

$$\begin{aligned} (\beta+\gamma)\alpha_\wedge + \gamma\alpha_\vee &= (\beta+2\gamma)\alpha_\wedge + \gamma(\alpha_\vee - \alpha_\wedge) \\ &> (\beta+2\gamma)\alpha_\wedge + \gamma(\beta+2\gamma)\big(f(\bar{v}) - f(0)\big) \\ &= (\beta+2\gamma)\big(\alpha_\wedge + \gamma\big(f(\bar{v}) - f(0)\big)\big) \\ &> \beta(\beta+2\gamma)f(0), \end{aligned}$$

where the first inequality is according to (D.70) and the second inequality is according to Condition 1.15.3, and

$$\begin{aligned} (\beta+\gamma)\alpha_\vee + \gamma\alpha_\wedge &= (\beta+2\gamma)\alpha_\vee - \gamma(\alpha_\vee - \alpha_\wedge) \\ &< (\beta+2\gamma)\alpha_\vee - \gamma(\beta+2\gamma)\big(f(\bar{v}) - f(0)\big) \\ &= (\beta+2\gamma)\big(\alpha_\vee - \gamma\big(f(\bar{v}) - f(0)\big)\big) \\ &< \beta(\beta+2\gamma)f(\bar{v}), \end{aligned}$$

where the first inequality is according to (D.70) and the second inequality is according to Condition 1.15.4. Combining the preceding two results, we find

$$\beta(\beta+2\gamma)f(0) < (\beta+\gamma)\alpha_\wedge + \gamma\alpha_\vee \le (\beta+\gamma)\alpha_\vee + \gamma\alpha_\wedge < \beta(\beta+2\gamma)f(\bar{v}),$$

which is equivalent to (D.67). This concludes the proof of (1.17).

Finally, I show that $\{\bar{\boldsymbol{\alpha}} \in \mathbb{R}^n \mid \bar{\boldsymbol{\alpha}} \text{ satisfies (1.17)}\}$ is a convex set. Let $\{\bar{\boldsymbol{\alpha}}_1, \bar{\boldsymbol{\alpha}}_2\} \subset \{\bar{\boldsymbol{\alpha}} \in \mathbb{R}^n \mid \bar{\boldsymbol{\alpha}} \text{ satisfies (1.17)}\}$. Let $\theta \in (0,1)$. We find

$$\frac{\beta(\beta+2\gamma)}{\beta+\gamma}f(0)\mathbf{1}_n = \theta\frac{\beta(\beta+2\gamma)}{\beta+\gamma}f(0)\mathbf{1}_n + (1-\theta)\frac{\beta(\beta+2\gamma)}{\beta+\gamma}f(0)\mathbf{1}_n$$

$$\begin{aligned}&<_c \theta\left(\boldsymbol{I}_n - \frac{|\gamma|}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)\bar{\boldsymbol{\alpha}}_1 + (1-\theta)\left(\boldsymbol{I}_n - \frac{|\gamma|}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)\bar{\boldsymbol{\alpha}}_2\\ &= \left(\boldsymbol{I}_n - \frac{|\gamma|}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)(\theta\bar{\boldsymbol{\alpha}}_1 + (1-\theta)\bar{\boldsymbol{\alpha}}_2)\end{aligned}$$

and similarly

$$\left(\boldsymbol{I}_n - \frac{|\gamma|}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)(\theta\bar{\boldsymbol{\alpha}}_1 + (1-\theta)\bar{\boldsymbol{\alpha}}_2) <_c \frac{\beta(\beta+2\gamma)}{\beta+\gamma} f(\bar{v})\mathbf{1}_n.$$

Thus, $(\theta\bar{\boldsymbol{\alpha}}_1 + (1-\theta)\bar{\boldsymbol{\alpha}}_2) \in \{\bar{\boldsymbol{\alpha}} \in \mathbb{R}^n \mid \bar{\boldsymbol{\alpha}} \text{ satisfies } (1.17)\}$. ■

Example 1.21: Proofs of (1.20), (1.21), and (1.24)

Consider the setup of Example 1.21. In what follows, I write $\alpha_\wedge$ for $\alpha_{\min,\mathcal{I}_k}$ and $\alpha_\vee$ for $\alpha_{\max,\mathcal{I}_k}$.

First, I show that $\sum_{j\in\mathcal{I}_k} \alpha_j = (d+1)(\alpha_\wedge + \alpha_\vee)/2$. We find

$$\begin{aligned}\sum_{j\in\mathcal{I}_k} \alpha_j &= \sum_{j=1}^{d+1}\left(\alpha_\wedge + \frac{\alpha_\vee - \alpha_\wedge}{d}(j-1)\right)\\ &= (d+1)\alpha_\wedge + \frac{\alpha_\vee - \alpha_\wedge}{d}\left(\frac{(d+1)(d+2)}{2} - (d+1)\right)\\ &= (d+1)\frac{\alpha_\wedge + \alpha_\vee}{2}.\end{aligned}$$

Second, I show that (1.20) and (1.21) are true. Using the preceding result, we find

$$\forall i \in \mathcal{I}_k \quad \frac{\sum_{j\in\mathcal{N}_G^+(i)} \alpha_j}{\deg_G^+(i)} = \frac{1}{d}\sum_{j\in\mathcal{I}_k\setminus\{i\}} \alpha_j = \frac{d+1}{d}\frac{\alpha_\wedge + \alpha_\vee}{2} - \frac{1}{d}\alpha_i \tag{D.71}$$

and

$$\begin{aligned}&\forall i \in \mathcal{I}_k & \alpha_i &> \frac{|\gamma|}{\beta+\gamma}\frac{\sum_{j\in\mathcal{N}_G^+(i)} \alpha_j}{\deg_G^+(i)}\\ \Leftrightarrow\ &\forall i \in \mathcal{I}_k & \alpha_i &> \frac{|\gamma|}{\beta+\gamma}\frac{d+1}{d}\frac{\alpha_\wedge+\alpha_\vee}{2} - \frac{|\gamma|}{\beta+\gamma}\frac{1}{d}\alpha_i\\ \Leftrightarrow\ &\forall i \in \mathcal{I}_k & 2\big((\beta+\gamma)d + |\gamma|\big)\alpha_i &> |\gamma|(d+1)(\alpha_\wedge + \alpha_\vee)\\ \Leftrightarrow\ & & 2\big((\beta+\gamma)d + |\gamma|\big)\alpha_\wedge &> |\gamma|(d+1)(\alpha_\wedge + \alpha_\vee)\\ \Leftrightarrow\ & & 2\big((\beta+\gamma)d - |\gamma|d\big)\alpha_\wedge &> |\gamma|(d+1)(\alpha_\vee - \alpha_\wedge)\\ \Leftrightarrow\ & & \alpha_\vee - \alpha_\wedge &< \frac{\beta+2\gamma}{|\gamma|}\frac{2d}{d+1}\alpha_\wedge\\ \Leftrightarrow\ & & \frac{\alpha_\vee}{\alpha_\wedge} &< 1 + \frac{\beta+2\gamma}{|\gamma|}\frac{2d}{d+1}\end{aligned}$$

because $\beta > 0$, $-\beta/2 < \gamma < 0$, $d > 0$, and $\alpha_\wedge > 0$.

Third, I show that (1.24) is true. Using (D.71), we find

$$
\begin{aligned}
& \forall i \in \mathcal{I}_k & \alpha_i &< \frac{|\gamma|}{\beta+\gamma} \frac{\sum_{j \in \mathcal{N}_G^+(i)} \alpha_j}{\deg_G^+(i)} + \frac{\beta(\beta+2\gamma)}{\beta+\gamma}\bar{v} \\
\Leftrightarrow \quad & \forall i \in \mathcal{I}_k & \alpha_i + \frac{|\gamma|}{\beta+\gamma}\frac{1}{d}\alpha_i &< \frac{|\gamma|}{\beta+\gamma}\frac{d+1}{d}\frac{\alpha_\wedge+\alpha_\vee}{2} + \frac{\beta(\beta+2\gamma)}{\beta+\gamma}\bar{v} \\
\Leftrightarrow \quad & \forall i \in \mathcal{I}_k & 2\big((\beta+\gamma)d+|\gamma|\big)\alpha_i &< |\gamma|(d+1)(\alpha_\wedge+\alpha_\vee) + 2\beta(\beta+2\gamma)d\bar{v} \\
\Leftrightarrow \quad & & 2\big((\beta+\gamma)d+|\gamma|\big)\alpha_\vee &< |\gamma|(d+1)(\alpha_\wedge+\alpha_\vee) + 2\beta(\beta+2\gamma)d\bar{v} \\
\Leftrightarrow \quad & & 2\big((\beta+\gamma)d-|\gamma|d\big)\alpha_\vee &< |\gamma|(d+1)(\alpha_\wedge-\alpha_\vee) + 2\beta(\beta+2\gamma)d\bar{v} \\
\Leftrightarrow \quad & & \alpha_\vee - \alpha_\wedge &< \frac{\beta+2\gamma}{|\gamma|}\frac{2d}{d+1}(\beta\bar{v}-\alpha_\vee).
\end{aligned}
$$

■

Proof of Lemma 1.22

Let $\Gamma := (\mathcal{I}, G, [0,\bar{v}], \{(\alpha_i,\beta,\gamma)\}_{i\in\mathcal{I}}, f)$ be a NALA game, and let $\boldsymbol{\alpha}$ be defined as in Section 1.3.2. Suppose Conditions 1.15.1 and 1.15.2 and the two inequalities (1.17) are satisfied.

First, note that the matrix $\boldsymbol{I}_n - |\gamma|/(\beta+\gamma)\bar{\boldsymbol{A}}(G)$ is nonsingular with a non-negative inverse that is bounded below by $\boldsymbol{I}_n$ because $\rho(\bar{\boldsymbol{A}}(G)) = 1$ and $0 < |\gamma|/(\beta+\gamma) < 1$ (Lemma B.6). Second, note that

$$
\left(\boldsymbol{I}_n - \frac{|\gamma|}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1}\boldsymbol{1}_n = \frac{\beta+\gamma}{\beta+2\gamma}\boldsymbol{1}_n \tag{D.72}
$$

because $\bar{\boldsymbol{A}}(G)\boldsymbol{1}_n = \boldsymbol{1}_n$ implies that

$$
\left(\boldsymbol{I}_n - \frac{|\gamma|}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)\boldsymbol{1}_n = \frac{\beta+2\gamma}{\beta+\gamma}\boldsymbol{1}_n.
$$

Using (D.72), premultiplying $(\boldsymbol{I}_n - |\gamma|/(\beta+\gamma)\bar{\boldsymbol{A}}(G))\boldsymbol{\alpha}$ and each occurrence of $\boldsymbol{1}_n$ in (1.17) by the inverse of $\boldsymbol{I}_n - |\gamma|/(\beta+\gamma)\bar{\boldsymbol{A}}(G)$ yields $\beta f(0)\boldsymbol{1}_n <_c \boldsymbol{\alpha} <_c \beta f(\bar{v})\boldsymbol{1}_n$ (Lemma B.1). ■

Proof of Proposition 1.23

Let $\Gamma(G) := (\mathcal{I}, G, \mathcal{Y}, \{(\alpha_i,\beta,\gamma)\}_{i\in\mathcal{I}}, f)$ be a generic NALA game, and let $\boldsymbol{\alpha}$ be defined as in Section 1.3.2. Suppose $\beta > 0$ and $\gamma > -\beta/2$, so that $\beta+\gamma > 0$ and $|\gamma/(\beta+\gamma)| < 1$. In addition, suppose $\Gamma(G)$ has a unique and interior NE $\boldsymbol{y}^\star(\boldsymbol{\alpha},\beta,\gamma,f,G) = (y_1^\star(\boldsymbol{\alpha},\beta,\gamma,f,G),\ldots,y_n^\star(\boldsymbol{\alpha},\beta,\gamma,f,G))$, which is given by (1.7).

Let $(i,j) \in \mathcal{I}^2$, and let θ be α_j, β, or γ. We find

$$
\frac{\partial y_i^\star(\boldsymbol{\alpha},\beta,\gamma,f,G)}{\partial\theta} = \frac{\partial f^{-1}\big(y_i^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_{\mathcal{Y}},G)\big)}{\partial\theta}
$$

$$= \frac{1}{\partial f\big(f^{-1}\big(y_i^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_{\mathcal{Y}},G)\big)\big)} \frac{\partial y_i^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_{\mathcal{Y}},G)}{\partial\theta}$$
$$= \frac{1}{\partial f\big(y_i^\star(\boldsymbol{\alpha},\beta,\gamma,f,G)\big)} \frac{\partial y_i^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_{\mathcal{Y}},G)}{\partial\theta},$$

where $\partial f(y_i^\star(\boldsymbol{\alpha},\beta,\gamma,f,G)) > 0$ because $\partial f > 0$ (Assumption F). It follows that the partial derivatives of $y_i^\star(\boldsymbol{\alpha},\beta,\gamma,f,G)$ and $y_i^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_{\mathcal{Y}},G)$ with respect to θ have the same sign.

Next, I compute the partial derivatives of $\boldsymbol{y}^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_{\mathcal{Y}},G)$ with respect to $\boldsymbol{\alpha}$, β, and γ. To this end, note that

$$\boldsymbol{y}^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_{\mathcal{Y}},G) = \frac{1}{\beta+\gamma}\boldsymbol{\alpha} + \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\boldsymbol{y}^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_{\mathcal{Y}},G). \tag{D.73}$$

First, I compute the partial derivative of $\boldsymbol{y}^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_{\mathcal{Y}},G)$ with respect to $\boldsymbol{\alpha}$. Using (1.7), we find

$$\boldsymbol{J}(\beta,\gamma,G) := \frac{\partial \boldsymbol{y}^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_{\mathcal{Y}},G)}{\partial\boldsymbol{\alpha}} = \frac{1}{\beta+\gamma}\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1}.$$

Second, I compute the partial derivative of $\boldsymbol{y}^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_{\mathcal{Y}},G)$ with respect to β. Using (D.73), we find

$$\begin{aligned}\frac{\partial \boldsymbol{y}^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_{\mathcal{Y}},G)}{\partial\beta} &= -\frac{1}{(\beta+\gamma)^2}\left(\boldsymbol{\alpha} + \gamma\bar{\boldsymbol{A}}(G)\boldsymbol{y}^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_{\mathcal{Y}},G)\right) \\ &\quad + \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\frac{\partial \boldsymbol{y}^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_{\mathcal{Y}},G)}{\partial\beta},\end{aligned}$$

which implies that

$$\begin{aligned}\frac{\partial \boldsymbol{y}^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_{\mathcal{Y}},G)}{\partial\beta} &= -\frac{1}{\beta+\gamma}\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1} \\ &\quad \times \left(\frac{1}{\beta+\gamma}\boldsymbol{\alpha} + \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\boldsymbol{y}^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_{\mathcal{Y}},G)\right) \\ &= -\frac{1}{\beta+\gamma}\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1}\boldsymbol{y}^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_{\mathcal{Y}},G) \\ &= -\boldsymbol{J}(\beta,\gamma,G)\boldsymbol{y}^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_{\mathcal{Y}},G).\end{aligned}$$

Third, I compute the partial derivative of $\boldsymbol{y}^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_{\mathcal{Y}},G)$ with respect to γ. Using (D.73), we find

$$\begin{aligned}\frac{\partial \boldsymbol{y}^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_{\mathcal{Y}},G)}{\partial\gamma} &= -\frac{1}{(\beta+\gamma)^2}\boldsymbol{\alpha} + \frac{\beta}{(\beta+\gamma)^2}\bar{\boldsymbol{A}}(G)\boldsymbol{y}^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_{\mathcal{Y}},G) \\ &\quad + \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\frac{\partial \boldsymbol{y}^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_{\mathcal{Y}},G)}{\partial\gamma},\end{aligned}$$

which implies that

$$\begin{aligned}
\frac{\partial \boldsymbol{y}^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_\mathcal{Y},G)}{\partial\gamma} &= \frac{1}{\beta+\gamma}\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1} \\
&\quad\times\left(\frac{\beta}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\boldsymbol{y}^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_\mathcal{Y},G) - \frac{1}{\beta+\gamma}\boldsymbol{\alpha}\right) \\
&= \frac{1}{\beta+\gamma}\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1} \\
&\quad\times\left(\bar{\boldsymbol{A}}(G)\boldsymbol{y}^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_\mathcal{Y},G) - \boldsymbol{y}^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_\mathcal{Y},G)\right) \\
&= \boldsymbol{J}(\beta,\gamma,G)\left(\bar{\boldsymbol{A}}(G)-\boldsymbol{I}_n\right)\boldsymbol{y}^\star(\boldsymbol{\alpha},\beta,\gamma,\mathrm{id}_\mathcal{Y},G).
\end{aligned}$$

Proof of Result 1.23.1 Suppose $-\beta/2 < \gamma < 0$. Note that $\boldsymbol{I}_n - \gamma^2/(\beta+\gamma)^2\bar{\boldsymbol{A}}(G)^2$ is a nonsingular M-matrix with a nonnegative inverse that is bounded below by $\boldsymbol{I}_n$ because $\bar{\boldsymbol{A}}(G)$ is nonnegative and $0 < \gamma^2/(\beta+\gamma)^2\rho(\bar{\boldsymbol{A}}(G)^2) < 1$ (Lemma B.6).

First, I establish a lower bound for $\boldsymbol{J}(\beta,\gamma,G)$. We find

$$\begin{aligned}
\boldsymbol{J}(\beta,\gamma,G) &= \frac{1}{\beta+\gamma}\left(\boldsymbol{I}_n - \frac{\gamma^2}{(\beta+\gamma)^2}\bar{\boldsymbol{A}}(G)^2\right)^{-1}\left(\boldsymbol{I}_n - \frac{|\gamma|}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right) \\
&\geq_c \frac{1}{\beta+\gamma}\boldsymbol{I}_n - \frac{|\gamma|}{(\beta+\gamma)^2}\left(\boldsymbol{I}_n - \frac{\gamma^2}{(\beta+\gamma)^2}\bar{\boldsymbol{A}}(G)^2\right)^{-1}\boldsymbol{1}_n\boldsymbol{1}_n^\top \\
&= \frac{1}{\beta+\gamma}\boldsymbol{I}_n - \frac{|\gamma|}{\beta(\beta+2\gamma)}\boldsymbol{1}_n\boldsymbol{1}_n^\top.
\end{aligned}$$

The first equality is according to (1.25) and (D.33). The inequality follows from three inequalities: $\beta+\gamma > 0$, $\boldsymbol{I}_n \leq_c (\boldsymbol{I}_n - \gamma^2/(\beta+\gamma)^2\bar{\boldsymbol{A}}(G)^2)^{-1}$, and $\bar{\boldsymbol{A}}(G) \leq_c \boldsymbol{1}_n\boldsymbol{1}_n^\top$. The second equality follows from (D.35).

Second, I establish an upper bound for $\boldsymbol{J}(\beta,\gamma,G)$. We find

$$\begin{aligned}
\boldsymbol{J}(\beta,\gamma,G) &= \frac{1}{\beta+\gamma}\left(\boldsymbol{I}_n - \frac{\gamma^2}{(\beta+\gamma)^2}\bar{\boldsymbol{A}}(G)^2\right)^{-1}\left(\boldsymbol{I}_n - \frac{|\gamma|}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right) \\
&\leq_c \frac{1}{\beta+\gamma}\left(\boldsymbol{I}_n - \frac{\gamma^2}{(\beta+\gamma)^2}\bar{\boldsymbol{A}}(G)^2\right)^{-1} \\
&\leq_c \frac{\beta+\gamma}{\beta(\beta+2\gamma)}\boldsymbol{1}_n\boldsymbol{1}_n^\top.
\end{aligned}$$

The first inequality follows from three inequalities: $\beta+\gamma > 0$, $\boldsymbol{O}_n \leq_c \bar{\boldsymbol{A}}(G)$, and $\boldsymbol{I}_n \leq_c (\boldsymbol{I}_n - \gamma^2/(\beta+\gamma)^2\bar{\boldsymbol{A}}(G)^2)^{-1}$. The second inequality follows from the two inequalities $\beta+\gamma > 0$ and $\boldsymbol{I}_n \leq_c (\boldsymbol{I}_n - \gamma^2/(\beta+\gamma)^2\bar{\boldsymbol{A}}(G)^2)^{-1}$ and (D.35).

Third, I show that, for all $i \in \mathcal{I}$, $[\boldsymbol{J}(\beta,\gamma,G)]_{i,i} > 0$. To this end, let $i \in \mathcal{I}$, and let $\boldsymbol{Q}_i$ denote the square matrix of order n that results from $\bar{\boldsymbol{A}}(G)$ by setting all components in row i and column i to zero, that is, $\boldsymbol{Q}_i := (\boldsymbol{I}_n - \boldsymbol{e}_i\boldsymbol{e}_i^\top)\bar{\boldsymbol{A}}(G)(\boldsymbol{I}_n - \boldsymbol{e}_i\boldsymbol{e}_i^\top)$. The definition of $\boldsymbol{Q}_i$ implies that $\boldsymbol{O}_n \leq_c \boldsymbol{Q}_i \leq_c \bar{\boldsymbol{A}}(G)$, which in turn implies that $\rho(\boldsymbol{Q}_i) \leq \rho(\bar{\boldsymbol{A}}(G))$ (see, for example, Varga 2000, Theorem 2.21). It follows that

$I_n - \gamma/(\beta+\gamma)Q_i$ and $I_n - \gamma^2/(\beta+\gamma)^2 Q_i^2$ are nonsingular because $|\gamma/(\beta+\gamma)| < 1$ and $\rho(\bar{A}(G)) = 1$ (Lemma B.3). The Laplace expansion of $\det(I_n - \gamma/(\beta+\gamma)Q_i)$ along the ith column of $I_n - \gamma/(\beta+\gamma)Q_i$ yields

$$\begin{aligned}
\det\left(I_n - \frac{\gamma}{\beta+\gamma}Q_i\right) &= \sum_{k=1}^{n}(-1)^{k+i}\det\left(\left[I_n - \frac{\gamma}{\beta+\gamma}Q_i\right]_{-k,-i}\right)\left[I_n - \frac{\gamma}{\beta+\gamma}Q_i\right]_{k,i} \\
&= (-1)^{i+i}\det\left(\left[I_n - \frac{\gamma}{\beta+\gamma}Q_i\right]_{-i,-i}\right) \\
&= (-1)^{i+i}\det\left(\left[I_n - \frac{\gamma}{\beta+\gamma}\bar{A}(G)\right]_{-i,-i}\right) \\
&= \left[\operatorname{adj}\left(I_n - \frac{\gamma}{\beta+\gamma}\bar{A}(G)\right)\right]_{i,i}. \qquad \text{(D.74)}
\end{aligned}$$

The first equality is the Laplace expansion. The second equality follows from, for all $k \in \mathcal{I}$, $[I_n - \gamma/(\beta+\gamma)Q_i]_{k,i} = \delta_{k,i}$. The third equality follows from the definition of Q_i. The last equality follows from the definition of the adjugate of $I_n - \gamma/(\beta+\gamma)\bar{A}(G)$. We find

$$\begin{aligned}
\left[\left(I_n - \frac{\gamma}{\beta+\gamma}\bar{A}(G)\right)^{-1}\right]_{i,i} &= \frac{\left[\operatorname{adj}\left(I_n - \frac{\gamma}{\beta+\gamma}\bar{A}(G)\right)\right]_{i,i}}{\det\left(I_n - \frac{\gamma}{\beta+\gamma}\bar{A}(G)\right)} \\
&= \frac{\det\left(I_n - \frac{\gamma}{\beta+\gamma}Q_i\right)}{\det\left(I_n - \frac{\gamma}{\beta+\gamma}\bar{A}(G)\right)} \\
&= \frac{\det\left(I_n - \frac{\gamma^2}{(\beta+\gamma)^2}Q_i^2\right)}{\det\left(I_n - \frac{|\gamma|}{\beta+\gamma}Q_i\right)}\frac{\det\left(I_n - \frac{|\gamma|}{\beta+\gamma}\bar{A}(G)\right)}{\det\left(I_n - \frac{\gamma^2}{(\beta+\gamma)^2}\bar{A}(G)^2\right)} \\
&> 0.
\end{aligned}$$

The first equality is clear. The second equality is according to (D.74). The third equality follows from (D.32) and an analogous equality with Q_i substituted for $\bar{A}(G)$. The inequality follows from the fact that the determinant of a nonsingular M-matrix is positive (see, for example, Plemmons 1977, Theorem 1, especially Condition A_1). ■

Proof of Result 1.23.2 Evidently, if $\gamma = 0$, then $J(\beta, \gamma, G) = (1/\beta)I_n$. ■

Proof of Result 1.23.3 Suppose $\gamma > 0$. Note that $I_n - \gamma/(\beta+\gamma)\bar{A}(G)$ is a nonsingular M-matrix with a nonnegative inverse that is bounded below by I_n because $\bar{A}(G)$ is nonnegative and $0 < \gamma/(\beta+\gamma)\rho(\bar{A}(G)) < 1$ (Lemma B.6). First, I establish a lower bound for $J(\beta, \gamma, G)$. The two inequalities $\beta + \gamma > 0$ and

$\boldsymbol{I}_n \leq_c (\boldsymbol{I}_n - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G))^{-1}$ imply that $1/(\beta+\gamma)\boldsymbol{I}_n \leq_c \boldsymbol{J}(\beta,\gamma,G)$. Second, I establish an upper bound for $\boldsymbol{J}(\beta,\gamma,G)$. The two inequalities $\beta+\gamma > 0$ and $\boldsymbol{I}_n \leq_c (\boldsymbol{I}_n - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G))^{-1}$ and (D.12) imply that $\boldsymbol{J}(\beta,\gamma,G) \leq_c (1/\beta)\mathbf{1}_n\mathbf{1}_n^{\mathsf{T}}$. ■

Proof of Result 1.23.4 First, note that $\mathrm{sl}(G) = G$ if G is strongly connected. Second, note that G is strongly connected if and only if $\dot{\boldsymbol{A}}(G)$ is irreducible (see, for example, Berman and Plemmons 1994, Theorem 2.1 on p. 29). Third, note that $\bar{\boldsymbol{A}}(G)$ is irreducible if and only if $\dot{\boldsymbol{A}}(G)$ is irreducible.

Suppose G is strongly connected and $\gamma > 0$. It follows that $\bar{\boldsymbol{A}}(G)$ is irreducible. Since $0 < \gamma/(\beta+\gamma) < 1$, $\rho(\bar{\boldsymbol{A}}(G)) = 1$, and $\bar{\boldsymbol{A}}(G)$ is irreducible, we have $\boldsymbol{O}_n <_c (\beta+\gamma)\boldsymbol{J}(\beta,\gamma,G)$ (Lemma B.6), which in turn implies that $\boldsymbol{O}_n <_c \boldsymbol{J}(\beta,\gamma,G)$ because $\beta+\gamma > 0$. ■

Proof of Result 1.23.5 Suppose $\gamma > 0$. Let $(i,j) \in \mathcal{I}^2$ with $i \neq j$. Note that

$$\forall p \in \mathbb{Z}_+ \qquad \left[\dot{\boldsymbol{A}}\big(\mathrm{sl}(G)\big)^p\right]_{i,j} > 0 \quad \Leftrightarrow \quad \left[\bar{\boldsymbol{A}}(G)^p\right]_{i,j} > 0. \tag{D.75}$$

There exists a walk in G from i to j if and only if there exists a walk in $\mathrm{sl}(G)$ from i to j. There exists a walk in $\mathrm{sl}(G)$ from i to j if and only if there exists a $p \in \mathbb{Z}_{++}$ such that $[\dot{\boldsymbol{A}}(\mathrm{sl}(G))^p]_{i,j} > 0$ (see, for example, Festinger 1949, pp. 154–55). According to (D.75), for all $p \in \mathbb{Z}_{++}$, $[\dot{\boldsymbol{A}}(\mathrm{sl}(G))^p]_{i,j} > 0$ if and only if $[\bar{\boldsymbol{A}}(G)^p]_{i,j} > 0$. Since $0 < \gamma/(\beta+\gamma) < 1$ and $\rho(\bar{\boldsymbol{A}}(G)) = 1$, the definition of $\boldsymbol{J}(\beta,\gamma,G)$ implies (Lemma B.2) that

$$\boldsymbol{J}(\beta,\gamma,G) = \frac{1}{\beta+\gamma}\sum_{m=0}^{\infty}\left(\frac{\gamma}{\beta+\gamma}\right)^m \bar{\boldsymbol{A}}(G)^m,$$

which in turn implies that

$$[\boldsymbol{J}(\beta,\gamma,G)]_{i,j} = \frac{1}{\beta+\gamma}\sum_{m=0}^{\infty}\left(\frac{\gamma}{\beta+\gamma}\right)^m \left[\bar{\boldsymbol{A}}(G)^m\right]_{i,j}. \tag{D.76}$$

Note that all summands in (D.76) are nonnegative because $\beta > 0$, $\gamma > 0$, and $\boldsymbol{O}_n \leq_c \bar{\boldsymbol{A}}(G)$. Thus, $[\boldsymbol{J}(\beta,\gamma,G)]_{i,j} > 0$ if and only if there exists a $p \in \mathbb{Z}_{++}$ such that $[\bar{\boldsymbol{A}}(G)^p]_{i,j} > 0$. ■

Proof of Result 1.23.6 Suppose $\beta > \gamma \geq 0$. Note that

$$\forall k \in \mathcal{I} \quad [\boldsymbol{J}(\beta,\gamma,G)]_{k,k} \geq \frac{1}{\beta+\gamma} \tag{D.77}$$

according to Results 1.23.2 and 1.23.3. I show that

$$\forall (i,j) \in \mathcal{I}^2 \text{ with } i \neq j \quad [\boldsymbol{J}(\beta,\gamma,G)]_{i,j} < \frac{1}{\beta+\gamma}.$$

Let $(i,j) \in \mathcal{I}^2$ with $i \neq j$. According to (D.12), we have

$$\boldsymbol{J}(\beta,\gamma,G)\mathbf{1}_n = \frac{1}{\beta}\mathbf{1}_n. \tag{D.78}$$

Using (D.78), we find

$$[\boldsymbol{J}(\beta,\gamma,G)]_{i,i} + [\boldsymbol{J}(\beta,\gamma,G)]_{i,j} + \sum_{k=1,k\notin\{i,j\}}^{n} [\boldsymbol{J}(\beta,\gamma,G)]_{i,k} = \frac{1}{\beta},$$

which implies that

$$\begin{aligned}
[\boldsymbol{J}(\beta,\gamma,G)]_{i,j} &= \frac{1}{\beta} - [\boldsymbol{J}(\beta,\gamma,G)]_{i,i} - \sum_{k=1,k\notin\{i,j\}}^{n} [\boldsymbol{J}(\beta,\gamma,G)]_{i,k} \\
&\leq \frac{1}{\beta} - \frac{1}{\beta+\gamma} \\
&= \frac{\gamma}{\beta}\frac{1}{\beta+\gamma} \\
&< \frac{1}{\beta+\gamma},
\end{aligned}$$

where the first inequality follows from (D.77) and the fact that for all $k \in \mathcal{I}$, $[\boldsymbol{J}(\beta,\gamma,G)]_{i,k} \geq 0$ (Results 1.23.2 and 1.23.3), and the second inequality follows from the assumption that $\beta > \gamma$. ■

Proof of Corollary 1.26

Let $(\mathcal{I}, G, \mathcal{Y}, \{(\alpha_i,\beta,\gamma)\}_{i\in\mathcal{I}}, f)$ be a generic NALA game. Suppose $\beta > 0$ and $\gamma > -\beta/2$, so that $\beta+\gamma > 0$ and $|\gamma/(\beta+\gamma)| < 1$. Let $x \in \mathcal{I}$, and let $\boldsymbol{j}_x(\beta,\gamma,G)$ be defined as in (1.30).

Results 1.26.1 to 1.26.6 follow directly from Results 1.23.1 to 1.23.6. In the rest of the proof, I show that Result 1.26.7 is true. Suppose $\deg_G^+(x) = 0$. The definition of $\boldsymbol{J}(\beta,\gamma,G)$ implies that

$$\boldsymbol{j}_x(\beta,\gamma,G) = \frac{1}{\beta+\gamma}\boldsymbol{e}_x + \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\boldsymbol{j}_x(\beta,\gamma,G),$$

which in turn implies that

$$\begin{aligned}
[\boldsymbol{j}_x(\beta,\gamma,G)]_x &= \frac{1}{\beta+\gamma} + \frac{\gamma}{\beta+\gamma}\sum_{j=1}^{n}[\bar{\boldsymbol{A}}(G)]_{x,j}[\boldsymbol{j}_x(\beta,\gamma,G)]_j \\
&= \frac{1}{\beta+\gamma} + \frac{\gamma}{\beta+\gamma}[\boldsymbol{j}_x(\beta,\gamma,G)]_x
\end{aligned}$$

because $[\bar{\boldsymbol{A}}(G)]_{x,j} = \delta_{j,x}$. It follows that $[\boldsymbol{j}_x(\beta,\gamma,G)]_x = 1/\beta > 0$. ■

Proof of Proposition 1.27

Let $\Gamma(G) := (\mathcal{I}, G, \mathcal{Y}, \{(\alpha_i,\beta,\gamma)\}_{i\in\mathcal{I}}, f)$ be a generic NALA game, $\boldsymbol{\alpha}$ be defined as in Section 1.3.2, and $(x,y) \in \mathcal{I}^2 \setminus \mathcal{A}(G)$ with $x \neq y$. The digraph $G \boxplus (x,y)$ induces the generic NALA game $\mathit{\Gamma}(G \boxplus (x,y)) := (\mathcal{I}, G \boxplus (x,y), \mathcal{Y}, \{(\alpha_i, \beta, \gamma)\}_{i \in \mathcal{I}}, f)$. Suppose $\beta > 0$ and $\gamma > -\beta/2$, so that $\beta + \gamma > 0$ and $|\gamma/(\beta+\gamma)| < 1$. In addition, suppose $\mathit{\Gamma}(G)$ has a unique and interior NE $\boldsymbol{y}^\star(G)$, which is given by (1.7), and $\mathit{\Gamma}(G \boxplus (x,y))$ has a unique and interior NE $\boldsymbol{y}^\star(G \boxplus (x,y))$, which is of the form (1.7).

First, I show that $1 - \gamma\langle \boldsymbol{\Delta}(x,y,G), \boldsymbol{j}_x(\beta,\gamma,G)\rangle > 0$. The inequality is trivial if $\gamma = 0$. Suppose $\gamma \neq 0$ in what follows. Note that $\gamma/(\beta+\gamma)\boldsymbol{e}_x\boldsymbol{\Delta}(x,y,G)^\mathsf{T}$ is a matrix with rank 1 because $\gamma \neq 0$ and $\boldsymbol{\Delta}(x,y,G) \neq \boldsymbol{0}_n$. This fact and (1.29) imply that $\boldsymbol{I}_n - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G \boxplus (x,y))$ is a rank-one-perturbation of $\boldsymbol{I}_n - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G)$, both matrices being nonsingular. Cauchy's formula for the determinant of a rank-one perturbation (see, for example, Horn and Johnson 2012, formula (0.8.5.11)) yields

$$1 - \gamma\langle \boldsymbol{\Delta}(x,y,G), \boldsymbol{j}_x(\beta,\gamma,G)\rangle = \frac{\det\left(\boldsymbol{I}_n - \dfrac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G \boxplus (x,y))\right)}{\det\left(\boldsymbol{I}_n - \dfrac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)}. \tag{D.79}$$

Both determinants in (D.79) are positive. To see this, consider two cases. First, the case $\gamma > 0$. Both $\boldsymbol{I}_n - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G)$ and $\boldsymbol{I}_n - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G \boxplus (x,y))$ are nonsingular M-matrices, which have positive determinants (see, for example, Plemmons 1977, Theorem 1, especially Condition A_1). Second, the case $-\beta/2 < \gamma < 0$. Using (D.32) and an analogous equality with $\bar{\boldsymbol{A}}(G \boxplus (x,y))$ substituted for $\bar{\boldsymbol{A}}(G)$ gives

$$\det\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right) = \frac{\det\left(\boldsymbol{I}_n - \dfrac{\gamma^2}{(\beta+\gamma)^2}\bar{\boldsymbol{A}}(G)^2\right)}{\det\left(\boldsymbol{I}_n - \dfrac{|\gamma|}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)}$$

and

$$\det\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}\big(G \boxplus (x,y)\big)\right) = \frac{\det\left(\boldsymbol{I}_n - \dfrac{\gamma^2}{(\beta+\gamma)^2}\bar{\boldsymbol{A}}\big(G \boxplus (x,y)\big)^2\right)}{\det\left(\boldsymbol{I}_n - \dfrac{|\gamma|}{\beta+\gamma}\bar{\boldsymbol{A}}\big(G \boxplus (x,y)\big)\right)},$$

where the nonsingular M-matrices $\boldsymbol{I}_n - |\gamma|/(\beta+\gamma)\bar{\boldsymbol{A}}(G)$, $\boldsymbol{I}_n - \gamma^2/(\beta+\gamma)^2\bar{\boldsymbol{A}}(G)^2$, $\boldsymbol{I}_n - |\gamma|/(\beta+\gamma)\bar{\boldsymbol{A}}(G \boxplus (x,y))$, and $\boldsymbol{I}_n - \gamma^2/(\beta+\gamma)^2\bar{\boldsymbol{A}}(G \boxplus (x,y))^2$ have positive determinants. This concludes the proof of $1 - \gamma\langle \boldsymbol{\Delta}(x,y,G), \boldsymbol{j}_x(\beta,\gamma,G)\rangle > 0$.

Second, I prove (1.31). Note that, according to (1.7),

$$\boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big) = \frac{1}{\beta+\gamma}\boldsymbol{\alpha} + \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big) \tag{D.80}$$

and

$$\boldsymbol{f}\big(\boldsymbol{y}^\star(G \boxplus (x,y))\big) = \frac{1}{\beta+\gamma}\boldsymbol{\alpha} + \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}\big(G \boxplus (x,y)\big)\boldsymbol{f}\big(\boldsymbol{y}^\star(G \boxplus (x,y))\big). \tag{D.81}$$

If $\gamma = 0$, then $\boldsymbol{f}(\boldsymbol{y}^\star(G \boxplus (x,y))) - \boldsymbol{f}(\boldsymbol{y}^\star(G)) = \mathbf{0}_n$, which is a special case of (1.31). Suppose $\gamma \neq 0$ in what follows. Using (D.80) and (D.81), we find

$$\begin{aligned}
&\boldsymbol{f}\big(\boldsymbol{y}^\star\big(G \boxplus (x,y)\big)\big) - \boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big) \\
&\quad = \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}\big(G \boxplus (x,y)\big)\boldsymbol{f}\big(\boldsymbol{y}^\star\big(G \boxplus (x,y)\big)\big) - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big) \qquad (D.82)\\
&\quad = \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}\big(G \boxplus (x,y)\big)\Big(\boldsymbol{f}\big(\boldsymbol{y}^\star\big(G \boxplus (x,y)\big)\big) - \boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big)\Big) \\
&\qquad + \frac{\gamma}{\beta+\gamma}\Big(\bar{\boldsymbol{A}}\big(G \boxplus (x,y)\big) - \bar{\boldsymbol{A}}(G)\Big)\boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big),
\end{aligned}$$

which implies that

$$\begin{aligned}
\boldsymbol{f}\big(\boldsymbol{y}^\star\big(G \boxplus (x,y)\big)\big) - \boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big) &= \frac{\gamma}{\beta+\gamma}\Big(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}\big(G \boxplus (x,y)\big)\Big)^{-1} \\
&\quad \times \Big(\bar{\boldsymbol{A}}\big(G \boxplus (x,y)\big) - \bar{\boldsymbol{A}}(G)\Big)\boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big). \qquad (D.83)
\end{aligned}$$

Using (1.29) and (D.83), we find

$$\begin{aligned}
&\boldsymbol{f}\big(\boldsymbol{y}^\star\big(G \boxplus (x,y)\big)\big) - \boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big) \\
&\quad = \frac{\gamma}{\beta+\gamma}\bigg(\Big(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\Big) - \frac{\gamma}{\beta+\gamma}\boldsymbol{e}_x\boldsymbol{\Delta}(x,y,G)^\top\bigg)^{-1} \\
&\qquad \times \big(\boldsymbol{e}_x\boldsymbol{\Delta}(x,y,G)^\top\big)\boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big). \qquad (D.84)
\end{aligned}$$

The inverse of $(\boldsymbol{I}_n - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G)) - \gamma/(\beta+\gamma)\boldsymbol{e}_x\boldsymbol{\Delta}(x,y,G)^\top$ is given by Sherman and Morrison's (1949) formula (see, for example, Bartlett 1951, p. 107):

$$\begin{aligned}
&\bigg(\Big(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\Big) - \frac{\gamma}{\beta+\gamma}\boldsymbol{e}_x\boldsymbol{\Delta}(x,y,G)^\top\bigg)^{-1} \\
&\quad = \Big(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\Big)^{-1} \\
&\qquad + \frac{\frac{\gamma}{\beta+\gamma}\Big(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\Big)^{-1}\big(\boldsymbol{e}_x\boldsymbol{\Delta}(x,y,G)^\top\big)\Big(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\Big)^{-1}}{1 - \frac{\gamma}{\beta+\gamma}\Big\langle \boldsymbol{\Delta}(x,y,G), \Big(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\Big)^{-1}\boldsymbol{e}_x\Big\rangle} \\
&\quad = \bigg(\boldsymbol{I}_n + \frac{\gamma\boldsymbol{j}_x(\beta,\gamma,G)\boldsymbol{\Delta}(x,y,G)^\top}{1 - \gamma\langle\boldsymbol{\Delta}(x,y,G), \boldsymbol{j}_x(\beta,\gamma,G)\rangle}\bigg)\Big(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\Big)^{-1}. \qquad (D.85)
\end{aligned}$$

Combining (D.84) and (D.85) gives

$$\begin{aligned}
&\boldsymbol{f}\big(\boldsymbol{y}^\star\big(G \boxplus (x,y)\big)\big) - \boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big) \\
&\quad = \frac{\gamma}{\beta+\gamma}\bigg(\boldsymbol{I}_n + \frac{\gamma\boldsymbol{j}_x(\beta,\gamma,G)\boldsymbol{\Delta}(x,y,G)^\top}{1 - \gamma\langle\boldsymbol{\Delta}(x,y,G), \boldsymbol{j}_x(\beta,\gamma,G)\rangle}\bigg)\Big(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\Big)^{-1}
\end{aligned}$$

$$
\begin{aligned}
&\times \big(\boldsymbol{e}_x \boldsymbol{\Delta}(x,y,G)^{\mathsf{T}}\big)\boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big) \\
&= \gamma\left(\boldsymbol{I}_n + \frac{\gamma \boldsymbol{j}_x(\beta,\gamma,G)\boldsymbol{\Delta}(x,y,G)^{\mathsf{T}}}{1-\gamma\langle \boldsymbol{\Delta}(x,y,G), \boldsymbol{j}_x(\beta,\gamma,G)\rangle}\right)\big(\boldsymbol{j}_x(\beta,\gamma,G)\boldsymbol{\Delta}(x,y,G)^{\mathsf{T}}\big)\boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big) \\
&= \gamma\big\langle \boldsymbol{\Delta}(x,y,G), \boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big)\big\rangle\left(\boldsymbol{I}_n + \frac{\gamma \boldsymbol{j}_x(\beta,\gamma,G)\boldsymbol{\Delta}(x,y,G)^{\mathsf{T}}}{1-\gamma\langle \boldsymbol{\Delta}(x,y,G), \boldsymbol{j}_x(\beta,\gamma,G)\rangle}\right)\boldsymbol{j}_x(\beta,\gamma,G) \\
&= \frac{\gamma\big\langle \boldsymbol{\Delta}(x,y,G), \boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big)\big\rangle}{1-\gamma\langle \boldsymbol{\Delta}(x,y,G), \boldsymbol{j}_x(\beta,\gamma,G)\rangle}\boldsymbol{j}_x(\beta,\gamma,G).
\end{aligned}
$$

This concludes the proof of (1.31).

Third, I prove (1.32). Using (D.82), we find

$$
\begin{aligned}
&\boldsymbol{f}\big(\boldsymbol{y}^\star\big(G \boxplus (x,y)\big)\big) - \bar{\boldsymbol{A}}\big(G \boxplus (x,y)\big)\boldsymbol{f}\big(\boldsymbol{y}^\star\big(G \boxplus (x,y)\big)\big) \\
&\quad = \boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big) - \bar{\boldsymbol{A}}(G)\boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big) \\
&\qquad + \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}\big(G \boxplus (x,y)\big)\boldsymbol{f}\big(\boldsymbol{y}^\star\big(G \boxplus (x,y)\big)\big) \\
&\qquad - \bar{\boldsymbol{A}}\big(G \boxplus (x,y)\big)\boldsymbol{f}\big(\boldsymbol{y}^\star\big(G \boxplus (x,y)\big)\big) \\
&\qquad + \bar{\boldsymbol{A}}(G)\boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big) - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big) \\
&\quad = \boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big) - \bar{\boldsymbol{A}}(G)\boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big) - \boldsymbol{d}(\boldsymbol{\alpha},\beta,\gamma,f,G,x,y),
\end{aligned}
$$

where

$$
\begin{aligned}
\boldsymbol{d}(\boldsymbol{\alpha},\beta,\gamma,f,G,x,y) &:= \frac{\beta}{\beta+\gamma}\bar{\boldsymbol{A}}\big(G \boxplus (x,y)\big)\boldsymbol{f}\big(\boldsymbol{y}^\star\big(G \boxplus (x,y)\big)\big) \\
&\qquad - \frac{\beta}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big).
\end{aligned}
$$

If $\gamma = 0$, then $\boldsymbol{f}(\boldsymbol{y}^\star(G)) = \boldsymbol{f}(\boldsymbol{y}^\star(G \boxplus (x,y))) = (1/\beta)\boldsymbol{\alpha}$ and $\boldsymbol{j}_x(\beta,\gamma,G) = (1/\beta)\boldsymbol{e}_x$ (Result 1.26.2), so that

$$
\begin{aligned}
\boldsymbol{d}(\boldsymbol{\alpha},\beta,\gamma,f,G,x,y) &= \Big(\bar{\boldsymbol{A}}\big(G \boxplus (x,y)\big) - \bar{\boldsymbol{A}}(G)\Big)\frac{1}{\beta}\boldsymbol{\alpha} \\
&= \big(\boldsymbol{e}_x \boldsymbol{\Delta}(x,y,G)^{\mathsf{T}}\big)\frac{1}{\beta}\boldsymbol{\alpha} \\
&= \beta\langle \boldsymbol{\Delta}(x,y,G), (1/\beta)\boldsymbol{\alpha}\rangle \frac{1}{\beta}\boldsymbol{e}_x \\
&= \frac{\beta\big\langle \boldsymbol{\Delta}(x,y,G), \boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big)\big\rangle}{1-\gamma\langle \boldsymbol{\Delta}(x,y,G), \boldsymbol{j}_x(\beta,\gamma,G)\rangle}\boldsymbol{j}_x(\beta,\gamma,G).
\end{aligned}
$$

If $\gamma \neq 0$, then

$$
\begin{aligned}
\boldsymbol{d}(\boldsymbol{\alpha},\beta,\gamma,f,G,x,y) &= \frac{\beta}{\gamma}\left(\boldsymbol{f}\big(\boldsymbol{y}^\star\big(G \boxplus (x,y)\big)\big) - \frac{1}{\beta+\gamma}\boldsymbol{\alpha}\right) \\
&\qquad - \frac{\beta}{\gamma}\left(\boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big) - \frac{1}{\beta+\gamma}\boldsymbol{\alpha}\right)
\end{aligned}
$$

$$= \frac{\beta}{\gamma}\Big(\boldsymbol{f}\big(\boldsymbol{y}^\star(G \boxplus (x,y))\big) - \boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big)\Big)$$
$$= \frac{\beta\langle \boldsymbol{\Delta}(x,y,G), \boldsymbol{f}\big(\boldsymbol{y}^\star(G)\big)\rangle}{1-\gamma\langle \boldsymbol{\Delta}(x,y,G), \boldsymbol{j}_x(\beta,\gamma,G)\rangle}\boldsymbol{j}_x(\beta,\gamma,G),$$

where the first equality follows from (D.80) and (D.81) and the last equality is according to (1.31). This concludes the proof of (1.32). ■

Proofs of Propositions 1.29, 1.31, 1.33, 1.34, and 1.35

Let $\Gamma(G) := (\mathcal{I}, G, \mathcal{Y}, \{(\alpha_i, \beta, \gamma)\}_{i\in\mathcal{I}}, f)$ be a generic NALA game, and let $\boldsymbol{\alpha}$ be defined as in Section 1.3.2. Suppose Condition G is satisfied, $\beta > 0$, and $\gamma > -\beta/2$, so that $\beta+\gamma > 0$ and $|\gamma/(\beta+\gamma)| < 1$. In addition, suppose $\Gamma(G)$ has a unique and interior NE $\boldsymbol{y}^\star := (y_1^\star, \ldots, y_n^\star)$, which is given by (1.7). Let G_κ be a weakly connected component of G of order $n_\kappa > 1$, and let $\mathcal{I}_\kappa := \mathcal{V}(G_\kappa)$. Let $h\colon \mathcal{I}_\kappa \to \{1, \ldots, n_\kappa\}$ be the unique order isomorphism. The component G_κ induces the generic NALA game $\Gamma(G_\kappa) := (\mathcal{I}_\kappa, G_\kappa, \mathcal{Y}, \{(\alpha_i, \beta, \gamma)\}_{i\in\mathcal{I}_\kappa}, f)$. Let $\alpha_{\min,\mathcal{I}_\kappa} := \min\{\alpha_i \mid i \in \mathcal{I}_\kappa\}$ and $\alpha_{\max,\mathcal{I}_\kappa} := \max\{\alpha_i \mid i \in \mathcal{I}_\kappa\}$. Finally, let $\boldsymbol{\alpha}_{\mathcal{I}_\kappa} := (\alpha_{h^{-1}(1)}, \ldots, \alpha_{h^{-1}(n_\kappa)})$, and $\boldsymbol{y}^\star_{\mathcal{I}_\kappa} := (y^\star_{h^{-1}(1)}, \ldots, y^\star_{h^{-1}(n_\kappa)})$.

Note that $\boldsymbol{I}_{n_\kappa} - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G_\kappa)$ is nonsingular because $|\gamma/(\beta+\gamma)| < 1$ and $\rho(\bar{\boldsymbol{A}}(G_\kappa)) = 1$ (Lemma B.3).

Proof of Proposition 1.29

Let g be the unique order isomorphism from $\mathcal{I} \setminus \mathcal{I}_\kappa$ to $\{n_\kappa+1, \ldots, n\}$. Let π be the permutation of $\mathcal{I}$ defined by

$$\pi(i) := \begin{cases} h^{-1}(i) & \text{if } i \le n_\kappa, \\ g^{-1}(i) & \text{if } i > n_\kappa. \end{cases}$$

Let $\boldsymbol{P}_\pi$ denote the permutation matrix of π. The matrix $\boldsymbol{P}_\pi$ is nonsingular with $\boldsymbol{P}_\pi^{-1} = \boldsymbol{P}_\pi^\intercal$. We find

$$\boldsymbol{P}_\pi\boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\alpha}_{\mathcal{I}_\kappa} \\ [\boldsymbol{\alpha}]_{\mathcal{I}\setminus\mathcal{I}_\kappa} \end{pmatrix}, \tag{D.86}$$

$$\boldsymbol{P}_\pi\boldsymbol{f}(\boldsymbol{y}^\star) = \begin{pmatrix} \boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}) \\ [\boldsymbol{f}(\boldsymbol{y}^\star)]_{\mathcal{I}\setminus\mathcal{I}_\kappa} \end{pmatrix}, \tag{D.87}$$

and

$$\boldsymbol{P}_\pi\bar{\boldsymbol{A}}(G)\boldsymbol{P}_\pi^{-1} = \begin{pmatrix} \bar{\boldsymbol{A}}(G_\kappa) & \boldsymbol{0}_{n_\kappa}\boldsymbol{0}^\intercal_{n-n_\kappa} \\ \boldsymbol{0}_{n-n_\kappa}\boldsymbol{0}^\intercal_{n_\kappa} & [\bar{\boldsymbol{A}}(G)]_{\mathcal{I}\setminus\mathcal{I}_\kappa,\mathcal{I}\setminus\mathcal{I}_\kappa} \end{pmatrix}, \tag{D.88}$$

where $\boldsymbol{P}_\pi\bar{\boldsymbol{A}}(G)\boldsymbol{P}_\pi^{-1}$ is block diagonal because G_κ is a weakly connected component of G. According to (1.7), $\boldsymbol{y}^\star$ satisfies

$$\boldsymbol{f}(\boldsymbol{y}^\star) = \frac{1}{\beta+\gamma}\boldsymbol{\alpha} + \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star). \tag{D.89}$$

Using (D.86) to (D.89), we find

$$
\begin{aligned}
\begin{pmatrix} \boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}) \\ [\boldsymbol{f}(\boldsymbol{y}^\star)]_{\mathcal{I}\setminus\mathcal{I}_\kappa} \end{pmatrix} &= \boldsymbol{P}_\pi \boldsymbol{f}(\boldsymbol{y}^\star) \\
&= \frac{1}{\beta+\gamma}\boldsymbol{P}_\pi\boldsymbol{\alpha} + \frac{\gamma}{\beta+\gamma}\boldsymbol{P}_\pi\bar{\boldsymbol{A}}(G)\boldsymbol{P}_\pi^{-1}\boldsymbol{P}_\pi\boldsymbol{f}(\boldsymbol{y}^\star) \\
&= \frac{1}{\beta+\gamma}\begin{pmatrix} \boldsymbol{\alpha}_{\mathcal{I}_\kappa} \\ [\boldsymbol{\alpha}]_{\mathcal{I}\setminus\mathcal{I}_\kappa} \end{pmatrix} \\
&\quad + \frac{\gamma}{\beta+\gamma}\begin{pmatrix} \bar{\boldsymbol{A}}(G_\kappa) & \boldsymbol{0}_{n_\kappa}\boldsymbol{0}^\top_{n-n_\kappa} \\ \boldsymbol{0}_{n-n_\kappa}\boldsymbol{0}^\top_{n_\kappa} & [\bar{\boldsymbol{A}}(G)]_{\mathcal{I}\setminus\mathcal{I}_\kappa,\mathcal{I}\setminus\mathcal{I}_\kappa} \end{pmatrix}\begin{pmatrix} \boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}) \\ [\boldsymbol{f}(\boldsymbol{y}^\star)]_{\mathcal{I}\setminus\mathcal{I}_\kappa} \end{pmatrix},
\end{aligned}
$$

which implies that

$$
\boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}) = \frac{1}{\beta+\gamma}\boldsymbol{\alpha}_{\mathcal{I}_\kappa} + \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G_\kappa)\boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}),
$$

which in turn is equivalent to (1.36). It is straightforward to show that $\boldsymbol{y}^\star_{\mathcal{I}_\kappa}$ is the unique and interior NE of $\Gamma(G_\kappa)$. ■

Proof of Proposition 1.31

Proof of Result 1.31.1 Suppose $-\beta/2 < \gamma < 0$. First, note that $\boldsymbol{I}_{n_\kappa} - \gamma^2/(\beta+\gamma)^2\bar{\boldsymbol{A}}(G_\kappa)^2$ is a nonsingular M-matrix, which has a nonnegative inverse that is bounded below by $\boldsymbol{I}_{n_\kappa}$, because $0 < \gamma^2/(\beta+\gamma)^2 < 1$, $\rho(\bar{\boldsymbol{A}}(G_\kappa)^2) = 1$, and $\boldsymbol{O}_n \leq_c \bar{\boldsymbol{A}}(G_\kappa)$ (Lemma B.6). Second, note that

$$
\left(\boldsymbol{I}_{n_\kappa} - \frac{\gamma^2}{(\beta+\gamma)^2}\bar{\boldsymbol{A}}(G_\kappa)^2\right)^{-1}\boldsymbol{1}_{n_\kappa} = \frac{(\beta+\gamma)^2}{\beta(\beta+2\gamma)}\boldsymbol{1}_{n_\kappa}. \tag{D.90}
$$

Third, note that

$$
\boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}) = \frac{1}{\beta+\gamma}\left(\boldsymbol{I}_{n_\kappa} - \frac{\gamma^2}{(\beta+\gamma)^2}\bar{\boldsymbol{A}}(G_\kappa)^2\right)^{-1}\left(\boldsymbol{I}_{n_\kappa} + \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G_\kappa)\right)\boldsymbol{\alpha}_{\mathcal{I}_\kappa}.
$$

Analogous to the proof of (D.66), which holds true if $\beta > 0$ and $-\beta < \gamma < 0$, we find

$$
\begin{aligned}
\frac{(\beta+\gamma)\alpha_{\min,\mathcal{I}_\kappa} + \gamma\alpha_{\max,\mathcal{I}_\kappa}}{\beta+\gamma}\boldsymbol{1}_{n_\kappa} &\leq_c \left(\boldsymbol{I}_{n_\kappa} + \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G_\kappa)\right)\boldsymbol{\alpha}_{\mathcal{I}_\kappa} \\
&\leq_c \frac{(\beta+\gamma)\alpha_{\max,\mathcal{I}_\kappa} + \gamma\alpha_{\min,\mathcal{I}_\kappa}}{\beta+\gamma}\boldsymbol{1}_{n_\kappa}.
\end{aligned} \tag{D.91}
$$

Using (D.90), premultiplying each vector in (D.91) by the inverse of $\boldsymbol{I}_{n_\kappa} - \gamma^2/(\beta+\gamma)^2\bar{\boldsymbol{A}}(G_\kappa)^2$ yields (Lemma B.1)

$$
(\beta+\gamma)\frac{(\beta+\gamma)\alpha_{\min,\mathcal{I}_\kappa} + \gamma\alpha_{\max,\mathcal{I}_\kappa}}{\beta(\beta+2\gamma)}\boldsymbol{1}_{n_\kappa}
$$

$$\begin{aligned}&\leq_c \left(\boldsymbol{I}_{n_\kappa} - \frac{\gamma^2}{(\beta+\gamma)^2}\bar{\boldsymbol{A}}(G_\kappa)^2\right)^{-1}\left(\boldsymbol{I}_{n_\kappa} + \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G_\kappa)\right)\boldsymbol{\alpha}_{\mathcal{I}_\kappa}\\ &\leq_c (\beta+\gamma)\frac{(\beta+\gamma)\alpha_{\max,\mathcal{I}_\kappa} + \gamma\alpha_{\min,\mathcal{I}_\kappa}}{\beta(\beta+2\gamma)}\boldsymbol{1}_{n_\kappa},\end{aligned}$$

which is equivalent to (1.37) because

$$\frac{(\beta+\gamma)\alpha_{\min,\mathcal{I}_\kappa} + \gamma\alpha_{\max,\mathcal{I}_\kappa}}{\beta+2\gamma} = \alpha_{\min,\mathcal{I}_\kappa} - \frac{|\gamma|}{\beta+2\gamma}(\alpha_{\max,\mathcal{I}_\kappa} - \alpha_{\min,\mathcal{I}_\kappa})$$

and

$$\frac{(\beta+\gamma)\alpha_{\max,\mathcal{I}_\kappa} + \gamma\alpha_{\min,\mathcal{I}_\kappa}}{\beta+2\gamma} = \alpha_{\max,\mathcal{I}_\kappa} + \frac{|\gamma|}{\beta+2\gamma}(\alpha_{\max,\mathcal{I}_\kappa} - \alpha_{\min,\mathcal{I}_\kappa}).$$

Suppose Conditions 1.15.1 to 1.15.4 are satisfied. Using (D.67) and the three inequalities $\alpha_{\min} \leq \alpha_{\min,\mathcal{I}_\kappa} \leq \alpha_{\max,\mathcal{I}_\kappa} \leq \alpha_{\max}$, we find

$$f(0) < \frac{(\beta+\gamma)\alpha_{\min} + \gamma\alpha_{\max}}{\beta(\beta+2\gamma)} \leq \frac{(\beta+\gamma)\alpha_{\min,\mathcal{I}_\kappa} + \gamma\alpha_{\max,\mathcal{I}_\kappa}}{\beta(\beta+2\gamma)}$$

and

$$f(\bar{v}) > \frac{(\beta+\gamma)\alpha_{\max} + \gamma\alpha_{\min}}{\beta(\beta+2\gamma)} \geq \frac{(\beta+\gamma)\alpha_{\max,\mathcal{I}_\kappa} + \gamma\alpha_{\min,\mathcal{I}_\kappa}}{\beta(\beta+2\gamma)}.$$

It follows that $f(0) < (1/\beta)(\alpha_{\min,\mathcal{I}_\kappa} - |\gamma|/(\beta+2\gamma)(\alpha_{\max,\mathcal{I}_\kappa} - \alpha_{\min,\mathcal{I}_\kappa}))$ and $(1/\beta)(\alpha_{\max,\mathcal{I}_\kappa} + |\gamma|/(\beta+2\gamma)(\alpha_{\max,\mathcal{I}_\kappa} - \alpha_{\min,\mathcal{I}_\kappa})) < f(\bar{v})$. ■

Proof of Result 1.31.2 Suppose $\gamma = 0$. According to (1.36), $\boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}) = (1/\beta)\boldsymbol{\alpha}_{\mathcal{I}_\kappa}$. ■

Proof of Result 1.31.3 Suppose $\gamma > 0$. First, note that $\boldsymbol{I}_{n_\kappa} - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G_\kappa)$ is a nonsingular M-matrix, which has a nonnegative inverse that is bounded below by $\boldsymbol{I}_{n_\kappa}$, because $0 < \gamma/(\beta+\gamma) < 1$, $\rho(\bar{\boldsymbol{A}}(G_\kappa)) = 1$, and $\boldsymbol{O}_n \leq_c \bar{\boldsymbol{A}}(G_\kappa)$ (Lemma B.6). Second, note that

$$\left(\boldsymbol{I}_{n_\kappa} - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G_\kappa)\right)^{-1}\boldsymbol{1}_{n_\kappa} = \frac{\beta+\gamma}{\beta}\boldsymbol{1}_{n_\kappa}. \tag{D.92}$$

Third, note that

$$\alpha_{\min,\mathcal{I}_\kappa}\boldsymbol{1}_{n_\kappa} \leq_c \boldsymbol{\alpha}_{\mathcal{I}_\kappa} \leq_c \alpha_{\max,\mathcal{I}_\kappa}\boldsymbol{1}_{n_\kappa}. \tag{D.93}$$

Using (D.92), premultiplying each vector in (D.93) by the inverse of $\boldsymbol{I}_{n_\kappa} - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G_\kappa)$ yields (Lemma B.1)

$$\alpha_{\min,\mathcal{I}_\kappa}\frac{\beta+\gamma}{\beta}\boldsymbol{1}_{n_\kappa} \leq_c \left(\boldsymbol{I}_{n_\kappa} - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G_\kappa)\right)^{-1}\boldsymbol{\alpha}_{\mathcal{I}_\kappa} \leq_c \alpha_{\max,\mathcal{I}_\kappa}\frac{\beta+\gamma}{\beta}\boldsymbol{1}_{n_\kappa},$$

which is equivalent to $(\alpha_{\min,\mathcal{I}_\kappa}/\beta)\boldsymbol{1}_{n_\kappa} \leq_c \boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}) \leq_c (\alpha_{\max,\mathcal{I}_\kappa}/\beta)\boldsymbol{1}_{n_\kappa}$. ■

Proof of Proposition 1.33

Proof of Result 1.33.1 Suppose $\gamma > 0$. Note that $\boldsymbol{I}_{n_\kappa} - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G_\kappa)$ is a nonsingular M-matrix, which has a nonnegative inverse that is bounded below by $\boldsymbol{I}_{n_\kappa}$ (Lemma B.6). Premultiplying both sides of the inequality $\boldsymbol{0}_{n_\kappa} \leq_c (<_c)\ \boldsymbol{\alpha}_{\mathcal{I}_\kappa}$ by the inverse of $\boldsymbol{I}_{n_\kappa} - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G_\kappa)$ and dividing by $\beta+\gamma$ yields $\boldsymbol{0}_{n_\kappa} \leq_c (<_c)\ \boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa})$ (Lemma B.1 and (1.36)), which implies that $\boldsymbol{0}_{n_\kappa} \leq_c (<_c)\ \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G_\kappa)\boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa})$, which in turn implies that

$$\frac{1}{\beta+\gamma}\boldsymbol{\alpha}_{\mathcal{I}_\kappa} \leq_c (<_c)\ \frac{1}{\beta+\gamma}\boldsymbol{\alpha}_{\mathcal{I}_\kappa} + \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G_\kappa)\boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}) = \boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}). \qquad \blacksquare$$

Proof of Result 1.33.2 Suppose $\gamma \neq 0$. The system of equations (1.36) is equivalent to $\boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}) - \bar{\boldsymbol{A}}(G_\kappa)\boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}) = (\beta/\gamma)((1/\beta)\boldsymbol{\alpha}_{\mathcal{I}_\kappa} - \boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}))$, from which it follows that $\operatorname{sgn}(\gamma)\boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}) \leq_c (<_c) \operatorname{sgn}(\gamma)\bar{\boldsymbol{A}}(G_\kappa)\boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa})$ is both necessary and sufficient for $(1/\beta)\boldsymbol{\alpha}_{\mathcal{I}_\kappa} \leq_c (<_c)\ \boldsymbol{f}_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa})$. ■

Proof of Proposition 1.34

Proof of Result 1.34.1 Suppose $\boldsymbol{\alpha}_{\mathcal{I}_\kappa} = \bar{\alpha}\boldsymbol{1}_{n_\kappa}$ for some $\bar{\alpha} \in \mathbb{R}$. We find

$$\begin{aligned}
\boldsymbol{f}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}) &= \frac{1}{\beta+\gamma}\left(\boldsymbol{I}_{n_\kappa} - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G_\kappa)\right)^{-1}\boldsymbol{\alpha}_{\mathcal{I}_\kappa} \\
&= \frac{\bar{\alpha}}{\beta+\gamma}\left(\boldsymbol{I}_{n_\kappa} - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G_\kappa)\right)^{-1}\boldsymbol{1}_{n_\kappa} \\
&= \frac{\bar{\alpha}}{\beta+\gamma}\,\frac{\beta+\gamma}{\beta}\boldsymbol{1}_{n_\kappa} \\
&= \frac{\bar{\alpha}}{\beta}\boldsymbol{1}_{n_\kappa},
\end{aligned}$$

where the first equality is according to (1.36) and the third equality is according to (D.92). ■

Proof of Result 1.34.2 Suppose $\boldsymbol{y}^\star_{\mathcal{I}_\kappa} = \bar{y}\boldsymbol{1}_{n_\kappa}$ for some $\bar{y} \in \operatorname{int}(\mathcal{Y})$. We find

$$\begin{aligned}
\boldsymbol{\alpha}_{\mathcal{I}_\kappa} &= (\beta+\gamma)\left(\boldsymbol{I}_{n_\kappa} - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G_\kappa)\right)\boldsymbol{f}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}) \\
&= (\beta+\gamma)f(\bar{y})\left(\boldsymbol{I}_{n_\kappa} - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G_\kappa)\right)\boldsymbol{1}_{n_\kappa} \\
&= (\beta+\gamma)f(\bar{y})\left(1 - \frac{\gamma}{\beta+\gamma}\right)\boldsymbol{1}_{n_\kappa} \\
&= \beta f(\bar{y})\boldsymbol{1}_{n_\kappa},
\end{aligned}$$

where the first equality follows from (1.36), the second equality is according to $\boldsymbol{f}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}) = \boldsymbol{f}(\bar{y}\boldsymbol{1}_{n_\kappa}) = f(\bar{y})\boldsymbol{1}_{n_\kappa}$, and the third equality follows from $\bar{\boldsymbol{A}}(G_\kappa)\boldsymbol{1}_{n_\kappa} = \boldsymbol{1}_{n_\kappa}$. ■

Proof of Proposition 1.35

Proof of Result 1.35.1 Suppose $\boldsymbol{\alpha}_{\mathcal{I}_\kappa} = \bar{\boldsymbol{A}}(G_\kappa)\boldsymbol{\alpha}_{\mathcal{I}_\kappa}$. Note that

$$\left(\boldsymbol{I}_{n_\kappa} - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G_\kappa)\right)^{-1} = \boldsymbol{I}_{n_\kappa} + \frac{\gamma}{\beta+\gamma}\left(\boldsymbol{I}_{n_\kappa} - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G_\kappa)\right)^{-1}\bar{\boldsymbol{A}}(G_\kappa). \quad \text{(D.94)}$$

Indeed, premultiplying both sides of

$$\boldsymbol{I}_{n_\kappa} = \left(\boldsymbol{I}_{n_\kappa} - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G_\kappa)\right) + \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G_\kappa)$$

by the inverse of $\boldsymbol{I}_{n_\kappa} - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G_\kappa)$ gives (D.94). We find

$$\begin{aligned}
f_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}) &= \frac{1}{\beta+\gamma}\left(\boldsymbol{I}_{n_\kappa} - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G_\kappa)\right)^{-1}\boldsymbol{\alpha}_{\mathcal{I}_\kappa} \\
&= \frac{1}{\beta+\gamma}\left(\boldsymbol{I}_{n_\kappa} + \frac{\gamma}{\beta+\gamma}\left(\boldsymbol{I}_{n_\kappa} - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G_\kappa)\right)^{-1}\bar{\boldsymbol{A}}(G_\kappa)\right)\boldsymbol{\alpha}_{\mathcal{I}_\kappa} \\
&= \frac{1}{\beta+\gamma}\boldsymbol{\alpha}_{\mathcal{I}_\kappa} + \frac{\gamma}{\beta+\gamma}\frac{1}{\beta+\gamma}\left(\boldsymbol{I}_{n_\kappa} - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G_\kappa)\right)^{-1}\boldsymbol{\alpha}_{\mathcal{I}_\kappa} \\
&= \frac{1}{\beta+\gamma}\boldsymbol{\alpha}_{\mathcal{I}_\kappa} + \frac{\gamma}{\beta+\gamma}f_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}), \qquad \text{(D.95)}
\end{aligned}$$

where the first and the last equality are according to (1.36), the second equality is according to (D.94), and the third equality is according to the assumption that $\boldsymbol{\alpha}_{\mathcal{I}_\kappa} = \bar{\boldsymbol{A}}(G_\kappa)\boldsymbol{\alpha}_{\mathcal{I}_\kappa}$. Finally, note that (D.95) is equivalent to $f_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}) = (1/\beta)\boldsymbol{\alpha}_{\mathcal{I}_\kappa}$. ■

Proof of Result 1.35.2 Suppose $\gamma \neq 0$ and $f_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}) = (1/\beta)\boldsymbol{\alpha}_{\mathcal{I}_\kappa}$. Using (1.36), we find

$$\begin{aligned}
& & f_{n_\kappa}(\boldsymbol{y}^\star_{\mathcal{I}_\kappa}) &= \frac{1}{\beta+\gamma}\left(\boldsymbol{I}_{n_\kappa} - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G_\kappa)\right)^{-1}\boldsymbol{\alpha}_{\mathcal{I}_\kappa} \\
&\Leftrightarrow & \frac{1}{\beta}\left(\boldsymbol{I}_{n_\kappa} - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G_\kappa)\right)\boldsymbol{\alpha}_{\mathcal{I}_\kappa} &= \frac{1}{\beta+\gamma}\boldsymbol{\alpha}_{\mathcal{I}_\kappa} \\
&\Leftrightarrow & \frac{\gamma}{\beta(\beta+\gamma)}\boldsymbol{\alpha}_{\mathcal{I}_\kappa} &= \frac{\gamma}{\beta(\beta+\gamma)}\bar{\boldsymbol{A}}(G_\kappa)\boldsymbol{\alpha}_{\mathcal{I}_\kappa} \\
&\Leftrightarrow & \boldsymbol{\alpha}_{\mathcal{I}_\kappa} &= \bar{\boldsymbol{A}}(G_\kappa)\boldsymbol{\alpha}_{\mathcal{I}_\kappa}. \qquad ■
\end{aligned}$$

Proof of Lemma 1.37

Let G be a digraph on $\mathcal{I}$. Let $\boldsymbol{S}(G)$ be defined as in (1.38).

Proof of Result 1.37.1 Suppose Condition G is satisfied, which implies that $\mathcal{I} \setminus \mathcal{I}_0^+(G) \neq \emptyset$. Evidently, $\boldsymbol{S}(G)$ is symmetric and nonnegative definite.

First, I show that $\boldsymbol{S}(G) \neq \boldsymbol{O}_n$. Suppose, for the sake of contradiction, $\boldsymbol{S}(G) = \boldsymbol{O}_n$, that is, $\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G) = \bar{\boldsymbol{A}}(G)^\mathsf{T}(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G))$. Let $i \in \mathcal{I} \setminus \mathcal{I}_0^+(G)$. We find

$$\begin{aligned}
1 &= [\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)]_{i,i} \\
&= \big[\bar{\boldsymbol{A}}(G)^\mathsf{T}\big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)\big]_{i,i} \\
&= [\bar{\boldsymbol{A}}(G)]_{i,i}[\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)]_{i,i} + \sum_{j \in \mathcal{I} \setminus \{i\}} [\bar{\boldsymbol{A}}(G)]_{j,i}[\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)]_{j,i} \\
&= -\sum_{j \in \mathcal{I} \setminus \{i\}} [\bar{\boldsymbol{A}}(G)]_{j,i}^2 \\
&\leq 0,
\end{aligned}$$

a contradiction.

Second, I show that $\rho(\boldsymbol{S}(G)) > 0$. Note that $\sigma(\boldsymbol{S}(G)) \subset \mathbb{R}_+$ because $\boldsymbol{S}(G)$ is symmetric and nonnegative definite. It is therefore sufficient to show that $\sigma(\boldsymbol{S}(G)) \neq \{0\}$. Suppose, for the sake of contradiction, $\sigma(\boldsymbol{S}(G)) = \{0\}$. The matrix $\boldsymbol{S}(G)$ is diagonalizable because it is symmetric. It follows that there exists a nonsingular matrix $\boldsymbol{P}$ of order n such that $\boldsymbol{P}^{-1}\boldsymbol{S}(G)\boldsymbol{P} = \boldsymbol{O}_n$, from which $\boldsymbol{S}(G) = \boldsymbol{O}_n$ follows, a contradiction.

Third, I show that

$$\rho\big(\boldsymbol{S}(G)\big) \leq 2\max\left\{ \mathbb{1}_{\mathcal{I} \setminus \mathcal{I}_0^+(G)}(j) + \frac{\deg_G^-(j)}{\min\{\deg_G^+(i) \mid i \in \mathcal{I} \setminus \mathcal{I}_0^+(G)\}} \;\middle|\; j \in \mathcal{I} \right\}.$$

To this end, I establish two auxiliary results:

$$\|\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\|_\infty = 2 \tag{D.96}$$

and

$$\|\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\|_1 \leq \max\left\{ \mathbb{1}_{\mathcal{I} \setminus \mathcal{I}_0^+(G)}(j) + \frac{\deg_G^-(j)}{\min\{\deg_G^+(i) \mid i \in \mathcal{I} \setminus \mathcal{I}_0^+(G)\}} \;\middle|\; j \in \mathcal{I} \right\}. \tag{D.97}$$

First, I prove (D.96). We have, according to (1.3),

$$\forall (i,j) \in \mathcal{I}^2 \quad [\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)]_{i,j} = \begin{cases} 0 & \text{if } i = j \text{ and } \deg_G^+(i) = 0, \\ 1 & \text{if } i = j \text{ and } \deg_G^+(i) > 0, \\ 0 & \text{if } i \neq j \text{ and } j \notin \mathcal{N}_G^+(i), \\ -\dfrac{1}{\deg_G^+(i)} & \text{if } i \neq j \text{ and } j \in \mathcal{N}_G^+(i), \end{cases}$$

which implies that

$$\forall i \in \mathcal{I} \quad \sum_{j=1}^{n} \big|[\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)]_{i,j}\big| = \begin{cases} 0 & \text{if } \deg_G^+(i) = 0, \\ 2 & \text{if } \deg_G^+(i) > 0. \end{cases}$$

Since $\mathcal{I} \setminus \mathcal{I}_0^+(G) \neq \emptyset$, we find

$$\|\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\|_\infty = \max\Big\{\sum_{j=1}^n \big|[\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)]_{i,j}\big| \;\Big|\; i \in \mathcal{I}\Big\} = 2. \tag{D.98}$$

Second, I prove (D.97). We have, for all $j \in \mathcal{I}$,

$$\begin{aligned}
&\sum_{i=1}^n \big|[\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)]_{i,j}\big| \\
&\quad = \big|1 - [\bar{\boldsymbol{A}}(G)]_{j,j}\big| + \sum_{i\in\mathcal{I}\setminus\{j\}} [\bar{\boldsymbol{A}}(G)]_{i,j} \\
&\quad = \big|1 - \mathbb{1}_{\mathcal{I}_0^+(G)}(j)\big| + \sum_{i\in\mathcal{I}_0^+(G)\setminus\{j\}} [\bar{\boldsymbol{A}}(G)]_{i,j} + \sum_{i\in\mathcal{I}\setminus(\mathcal{I}_0^+(G)\cup\{j\})} [\bar{\boldsymbol{A}}(G)]_{i,j} \\
&\quad = \big|\mathbb{1}_{\mathcal{I}\setminus\mathcal{I}_0^+(G)}(j)\big| + \sum_{i\in\mathcal{I}_0^+(G)\setminus\{j\}} \delta_{i,j} + \sum_{i\in\mathcal{I}\setminus(\mathcal{I}_0^+(G)\cup\{j\})} \frac{\mathbb{1}_{\mathcal{N}_G^+(i)}(j)}{\deg_G^+(i)} \\
&\quad = \mathbb{1}_{\mathcal{I}\setminus\mathcal{I}_0^+(G)}(j) + \sum_{i\in\mathcal{I}\setminus\mathcal{I}_0^+(G)} \frac{\mathbb{1}_{\mathcal{N}_G^-(j)}(i)}{\deg_G^+(i)} \\
&\quad \leq \mathbb{1}_{\mathcal{I}\setminus\mathcal{I}_0^+(G)}(j) + \frac{1}{\min\{\deg_G^+(i) \mid i \in \mathcal{I}\setminus\mathcal{I}_0^+(G)\}} \sum_{i\in\mathcal{I}\setminus\mathcal{I}_0^+(G)} \mathbb{1}_{\mathcal{N}_G^-(j)}(i) \\
&\quad = \mathbb{1}_{\mathcal{I}\setminus\mathcal{I}_0^+(G)}(j) + \frac{|\mathcal{N}_G^-(j)|}{\min\{\deg_G^+(i) \mid i \in \mathcal{I}\setminus\mathcal{I}_0^+(G)\}} \\
&\quad = \mathbb{1}_{\mathcal{I}\setminus\mathcal{I}_0^+(G)}(j) + \frac{\deg_G^-(j)}{\min\{\deg_G^+(i) \mid i \in \mathcal{I}\setminus\mathcal{I}_0^+(G)\}}.
\end{aligned} \tag{D.99}$$

The first equality is obvious. The second equality follows from (1.3), which implies that $[\bar{\boldsymbol{A}}(G)]_{j,j} = \mathbb{1}_{\mathcal{I}_0^+(G)}(j)$. The third equality is according to (1.3). The forth equality follows from the fact that $j \notin \mathcal{N}_G^+(j)$ and the fact that $j \in \mathcal{N}_G^+(i)$ if and only if $i \in \mathcal{N}_G^-(j)$. The inequality is obvious. The fifth equality follows from the fact that for all $i \in \mathcal{I}_0^+(G)$, $i \notin \mathcal{N}_G^-(j)$. The last equality is according to the definition of $\deg_G^-(j)$. Using (D.99), we find

$$\begin{aligned}
\|\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\|_1 &= \max\Big\{\sum_{i=1}^n \big|[\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)]_{i,j}\big| \;\Big|\; j \in \mathcal{I}\Big\} \\
&\leq \max\left\{\mathbb{1}_{\mathcal{I}\setminus\mathcal{I}_0^+(G)}(j) + \frac{\deg_G^-(j)}{\min\{\deg_G^+(i) \mid i \in \mathcal{I}\setminus\mathcal{I}_0^+(G)\}} \;\middle|\; j \in \mathcal{I}\right\}.
\end{aligned}$$

Finally, we find

$$\begin{aligned}
\rho(\boldsymbol{S}) &= \rho\Big(\big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)^\mathsf{T}\big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)\Big) \\
&\leq \big\|\big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)^\mathsf{T}\big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)\big\|_\infty
\end{aligned}$$

$$
\begin{aligned}
&\le \|(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G))^{\mathsf{T}}\|_\infty \|\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\|_\infty \\
&= \|\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\|_1 \|\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\|_\infty \\
&\le 2\max\left\{ \mathbb{1}_{\mathcal{I}\setminus\mathcal{I}_0^+(G)}(j) + \frac{\deg_G^-(j)}{\min\{\deg_G^+(i) \mid i \in \mathcal{I}\setminus\mathcal{I}_0^+(G)\}} \,\middle|\, j \in \mathcal{I} \right\}.
\end{aligned}
$$

The first equality is trivial. The first inequality is according to Lemma B.7 because $\|\cdot\|_\infty$ is sub-multiplicative, which also justifies the second inequality. The second equality is straightforward to see. The third inequality follows from (D.96) and (D.97). ■

Proof of Result 1.37.2 Suppose Condition G is not satisfied. It follows that $\bar{\boldsymbol{A}}(G) = \boldsymbol{I}_n$ and $\boldsymbol{S}(G) = \boldsymbol{O}_n$. ■

Proof of Proposition 1.38

Let $\Gamma := (\mathcal{I}, G, \mathcal{Y}, \{(\alpha_i, \beta, \gamma)\}_{i\in\mathcal{I}}, f)$ be a generic NALA game, and let $\boldsymbol{\alpha}$ be defined as in Section 1.3.2. Suppose Γ has a unique and interior social optimum $\boldsymbol{y}_S^\star := (y_{S,1}^\star, \dots, y_{S,n}^\star)$. Let $\boldsymbol{S}(G)$ be defined as in (1.38). In what follows, the component in row i and column j of $\bar{\boldsymbol{A}}(G)$, $\bar{a}_{i,j}(G)$, is abbreviated to $\bar{a}_{i,j}$.

It follows from the definition of a social optimum of Γ that $\boldsymbol{y}_S^\star$ is the unique, global, and interior maximum point of $w\colon \mathcal{Y}^n \to \mathbb{R}$. Since $\boldsymbol{y}_S^\star$ is an interior global maximum point of w, by Fermat's theorem, $\boldsymbol{y}_S^\star$ is a stationary point of w, that is, it satisfies

$$
\forall i \in \mathcal{I} \quad \frac{\partial w(\boldsymbol{y}_S^\star)}{\partial y_i} = 0. \tag{D.100}
$$

Let $i \in \mathcal{I}$. For all $\boldsymbol{y} := (y_1, \dots, y_n) \in \mathcal{Y}^n$, we have

$$
\begin{aligned}
\frac{\partial w(\boldsymbol{y})}{\partial y_i} &= \alpha_i \partial f(y_i) - \beta f(y_i)\partial f(y_i) \\
&\quad - \gamma\Big(f(y_i) - \sum_{j\in\mathcal{I}} (\bar{a}_{j,i} + \bar{a}_{i,j}) f(y_j) + \sum_{j\in\mathcal{I}}\sum_{k\in\mathcal{I}} \bar{a}_{j,i}\bar{a}_{j,k} f(y_k) \Big)\partial f(y_i)
\end{aligned}
$$

because

$$
w(\boldsymbol{y}) = \sum_{j\in\mathcal{I}} u_j(\boldsymbol{y}) = \sum_{j\in\mathcal{I}} \alpha_j f(y_j) - \frac{\beta}{2}\sum_{j\in\mathcal{I}} f(y_j)^2 - \frac{\gamma}{2}\sum_{j\in\mathcal{I}}\Big(f(y_j) - \sum_{k\in\mathcal{I}} \bar{a}_{j,k} f(y_k) \Big)^2.
$$

Since $\partial f > 0$ (Assumption F), the preceding result about the partial derivative implies that the system of equations (D.100) is equivalent to

$$
\forall i \in \mathcal{I} \quad \beta f(y_{S,i}^\star) + \gamma\Big(f(y_{S,i}^\star) - \sum_{j\in\mathcal{I}} (\bar{a}_{j,i} + \bar{a}_{i,j}) f(y_{S,j}^\star) + \sum_{j\in\mathcal{I}}\sum_{k\in\mathcal{I}} \bar{a}_{j,i}\bar{a}_{j,k} f(y_{S,k}^\star) \Big) = \alpha_i,
$$

which in turn is equivalent to

$$\big(\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G)\big)\boldsymbol{f}(\boldsymbol{y}_S^\star) = \boldsymbol{\alpha}. \tag{D.101}$$

The assumption that $\boldsymbol{y}_S^\star$ is the *unique* and interior social optimum of Γ implies that the matrix $\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G)$ is nonsingular. Suppose, for the sake of contradiction, this matrix is singular. It follows that there exists a $\boldsymbol{c} := (c_1, \ldots, c_n) \in \mathbb{R}^n \setminus \{\boldsymbol{0}_n\}$ such that

$$\big(\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G)\big)\boldsymbol{c} = \boldsymbol{0}_n. \tag{D.102}$$

Since $\boldsymbol{y}_S^\star$ is an *interior* social optimum of Γ, $\boldsymbol{f}(\boldsymbol{y}_S^\star)$ lies in $\operatorname{int}(f(\mathcal{Y})^n)$, from which it follows that for all $i \in \mathcal{I}$, there exists an open ball with radius $r_i > 0$ and center $f(y_{S,i}^\star)$ that is contained in $\operatorname{int}(f(\mathcal{Y}))$. For all $i \in \mathcal{I}$, let

$$\lambda_i := \begin{cases} 1 & \text{if } c_i = 0, \\ \dfrac{r_i}{2|c_i|} & \text{if } c_i \neq 0. \end{cases}$$

Let $\lambda := \min\{\lambda_i \mid i \in \mathcal{I}\} > 0$. The definition of λ implies that for all $i \in \mathcal{I}$, the point $f(y_{S,i}^\star) + \lambda c_i$ lies in $\operatorname{int}(f(\mathcal{Y}))$, from which it follows that $y_{T,i}^\star := f^{-1}(f(y_{S,i}^\star) + \lambda c_i)$ lies in $\operatorname{int}(\mathcal{Y})$. Let $\boldsymbol{y}_T^\star := (y_{T,1}^\star, \ldots, y_{T,n}^\star)$. Note that $\boldsymbol{f}(\boldsymbol{y}_T^\star) = \boldsymbol{f}(\boldsymbol{y}_S^\star) + \lambda \boldsymbol{c}$. Evidently, $\boldsymbol{y}_T^\star \in \operatorname{int}(\mathcal{Y}^n)$ and $\boldsymbol{f}(\boldsymbol{y}_T^\star) \in \operatorname{int}(f(\mathcal{Y})^n)$. Note that $\boldsymbol{y}_T^\star \neq \boldsymbol{y}_S^\star$ because $\lambda > 0$ and $\boldsymbol{c} \neq \boldsymbol{0}_n$. We find

$$\begin{aligned}
w(\boldsymbol{y}_T^\star) &= \langle \boldsymbol{\alpha}, \boldsymbol{f}(\boldsymbol{y}_T^\star)\rangle - \frac{1}{2}\big\langle \boldsymbol{f}(\boldsymbol{y}_T^\star), \big(\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G)\big)\boldsymbol{f}(\boldsymbol{y}_T^\star)\big\rangle \\
&= \langle \boldsymbol{\alpha}, \boldsymbol{f}(\boldsymbol{y}_S^\star) + \lambda \boldsymbol{c}\rangle - \frac{1}{2}\big\langle \boldsymbol{f}(\boldsymbol{y}_S^\star) + \lambda \boldsymbol{c}, \big(\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G)\big)\big(\boldsymbol{f}(\boldsymbol{y}_S^\star) + \lambda \boldsymbol{c}\big)\big\rangle \\
&= \langle \boldsymbol{\alpha}, \boldsymbol{f}(\boldsymbol{y}_S^\star)\rangle + \lambda\langle \boldsymbol{\alpha}, \boldsymbol{c}\rangle - \frac{1}{2}\big\langle \boldsymbol{f}(\boldsymbol{y}_S^\star), \big(\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G)\big)\boldsymbol{f}(\boldsymbol{y}_S^\star)\big\rangle \\
&\quad - \lambda\big\langle \boldsymbol{f}(\boldsymbol{y}_S^\star), \big(\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G)\big)\boldsymbol{c}\big\rangle - \frac{1}{2}\lambda^2\big\langle \boldsymbol{c}, \big(\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G)\big)\boldsymbol{c}\big\rangle \\
&= \langle \boldsymbol{\alpha}, \boldsymbol{f}(\boldsymbol{y}_S^\star)\rangle - \frac{1}{2}\big\langle \boldsymbol{f}(\boldsymbol{y}_S^\star), \big(\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G)\big)\boldsymbol{f}(\boldsymbol{y}_S^\star)\big\rangle \\
&= w(\boldsymbol{y}_S^\star),
\end{aligned}$$

where $\langle \boldsymbol{\alpha}, \boldsymbol{c}\rangle = 0$ according to (D.101) and (D.102) and $\langle \boldsymbol{f}(\boldsymbol{y}_S^\star), (\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G))\boldsymbol{c}\rangle = 0$ and $\langle \boldsymbol{c}, (\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G))\boldsymbol{c}\rangle = 0$ according to (D.102). Since $\boldsymbol{y}_S^\star$ is the *unique*, global, and interior maximum point of w, $w(\boldsymbol{y}_T^\star) = w(\boldsymbol{y}_S^\star)$ implies that $\boldsymbol{y}_T^\star = \boldsymbol{y}_S^\star$, which contradicts $\boldsymbol{y}_T^\star \neq \boldsymbol{y}_S^\star$. This concludes the proof of (1.39). ∎

Proof of Proposition 1.39

Let $\Gamma := (\mathcal{I}, G, \mathbb{R}, \{(\alpha_i, \beta, \gamma)\}_{i\in\mathcal{I}}, f)$ be a NALA game, and let $\boldsymbol{\alpha}$ be defined as in Section 1.3.2. Suppose Conditions 1.39.1, 1.39.2, and 1.39.3 are satisfied. Let $\boldsymbol{S}(G)$ be defined as in (1.38).

First, note that $f(\mathbb{R}) = \mathbb{R}$ because $f(\mathbb{R})$ is an interval (Lemma 1.1) and f is strictly increasing (Assumption F) and not bounded below and above (Condition 1.39.3). Second, note that $f(\mathbb{R})$ is convex. Third, note that $f(\mathbb{R})$ is open, so that $\operatorname{int}(f(\mathbb{R})) = f(\mathbb{R})$. It follows that $f(\mathbb{R})^n$ is convex and open.

Let $\mathcal{M}^\star \subset \mathbb{R}^n$ denote the set of all global maximum points of w. I show that $|\mathcal{M}^\star| = 1$. Let the function $\widetilde{w}\colon f(\mathbb{R})^n \to \mathbb{R}$ be defined by $\widetilde{w} := w \circ (f^{-1}, \dots, f^{-1})$, that is, for all $\boldsymbol{z} \in f(\mathbb{R})^n$,

$$\widetilde{w}(\boldsymbol{z}) = \langle \boldsymbol{\alpha}, \boldsymbol{z} \rangle - \frac{1}{2}\langle \boldsymbol{z}, \big(\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G)\big)\boldsymbol{z} \rangle.$$

I show that $\widetilde{w}$ has a unique global maximum point. The function $\widetilde{w}$ is twice continuously differentiable on $f(\mathbb{R})^n$, its Hessian (matrix) satisfies

$$\forall \boldsymbol{z} \in f(\mathbb{R})^n \quad (\operatorname{Hess} \widetilde{w})(\boldsymbol{z}) = -\big(\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G)\big).$$

Since $(\operatorname{Hess} \widetilde{w})(\boldsymbol{z})$ does not depend on $\boldsymbol{z} \in f(\mathbb{R})^n$, I write $\operatorname{Hess} \widetilde{w}$ for $(\operatorname{Hess} \widetilde{w})(\boldsymbol{z})$ in what follows. The symmetric matrix $\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G)$ is positive definite (and therefore nonsingular) because $\beta > 0$ (Condition 1.39.1) and $\gamma \rho(\boldsymbol{S}(G)) > -\beta$ (Condition 1.39.2). Indeed, $\min \sigma(\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G)) > 0$ (see the proof of Lemma 1.43 for details). It follows that $\widetilde{w}$ is strictly concave on $f(\mathbb{R})^n$. Since $\widetilde{w}$ is strictly concave, it has at most a global maximum point, and a local maximum point of $\widetilde{w}$ is a global maximum point. It is therefore enough to show that $\widetilde{w}$ has a local maximum point. Since $\widetilde{w}$ is strictly concave and differentiable, it has a local maximum point at $\boldsymbol{z} \in f(\mathbb{R})^n$ if and only if $(\operatorname{grad} \widetilde{w})(\boldsymbol{z}) = \boldsymbol{0}_n$. We find

$$\forall \boldsymbol{z} \in f(\mathbb{R})^n \quad (\operatorname{grad} \widetilde{w})(\boldsymbol{z}) = \boldsymbol{\alpha} + (\operatorname{Hess} \widetilde{w})\boldsymbol{z}.$$

Since $\operatorname{Hess} \widetilde{w}$ is nonsingular, there exists a unique $\boldsymbol{z}^\star := (z_1^\star, \dots, z_n^\star) \in \mathbb{R}^n$ (sic) such that $(\operatorname{grad} \widetilde{w})(\boldsymbol{z}^\star) = \boldsymbol{0}_n$; in particular,

$$\boldsymbol{z}^\star = \big(-(\operatorname{Hess} \widetilde{w})\big)^{-1}\boldsymbol{\alpha} = \big(\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G)\big)^{-1}\boldsymbol{\alpha} = \boldsymbol{Q}(\beta, \gamma, G)\boldsymbol{\alpha}, \tag{D.103}$$

where $\boldsymbol{Q}(\beta, \gamma, G)$ is defined as in (1.41). We have $\boldsymbol{z}^\star \in f(\mathbb{R})^n$ because $f(\mathbb{R})^n = \mathbb{R}^n$. This concludes the proof that $\boldsymbol{z}^\star$ is the unique global maximum point of $\widetilde{w}$.

Since f is bijective (Lemma 1.1), there exists a unique $\boldsymbol{y}_S^\star := (y_{S,1}^\star, \dots, y_{S,n}^\star) \in \mathbb{R}^n$ such that $\boldsymbol{z}^\star = \boldsymbol{f}(\boldsymbol{y}_S^\star)$; in particular, for all $i \in \mathcal{I}$, $y_{S,i}^\star := f^{-1}(z_i^\star)$.

The vector $\boldsymbol{y}_S^\star$ is the unique global maximum point of w. This can be seen as follows. Suppose, for the sake of contradiction, that there exists a $\hat{\boldsymbol{y}} \in \mathbb{R}^n$ with $\hat{\boldsymbol{y}} \neq \boldsymbol{y}_S^\star$ and $w(\hat{\boldsymbol{y}}) \geq w(\boldsymbol{y}_S^\star)$. Let $\hat{\boldsymbol{z}} := \boldsymbol{f}(\hat{\boldsymbol{y}})$. Since f is injective (Lemma 1.1), $\hat{\boldsymbol{y}} \neq \boldsymbol{y}_S^\star$ implies that $\hat{\boldsymbol{z}} = \boldsymbol{f}(\hat{\boldsymbol{y}}) \neq \boldsymbol{f}(\boldsymbol{y}_S^\star) = \boldsymbol{z}^\star$. Since $\boldsymbol{z}^\star$ is the unique global maximum point of $\widetilde{w}$, $\hat{\boldsymbol{z}} \neq \boldsymbol{z}^\star$ implies that $\widetilde{w}(\hat{\boldsymbol{z}}) < \widetilde{w}(\boldsymbol{z}^\star)$. We find

$$\widetilde{w}(\hat{\boldsymbol{z}}) = \widetilde{w}\big(\boldsymbol{f}(\hat{\boldsymbol{y}})\big) = w(\hat{\boldsymbol{y}}) \geq w(\boldsymbol{y}_S^\star) = \widetilde{w}\big(\boldsymbol{f}(\boldsymbol{y}_S^\star)\big) = \widetilde{w}(\boldsymbol{z}^\star),$$

a contradiction to $\widetilde{w}(\hat{\boldsymbol{z}}) < \widetilde{w}(\boldsymbol{z}^\star)$. This concludes the proof that $\boldsymbol{y}_S^\star$ is the unique global maximum point of w. It follows that Γ has a unique and interior social optimum, which is given by (1.39). ∎

Proof of Proposition 1.40

Let $\Gamma := (\mathcal{I}, G, \mathbb{R}_+, \{(\alpha_i, \beta, \gamma)\}_{i\in\mathcal{I}}, f)$ be a NALA game, and let $\boldsymbol{\alpha}$ be defined as in Section 1.3.2 and $\bar{\alpha}_{\mathcal{I}} := (1/n)\sum_{i\in\mathcal{I}} \alpha_i$. Suppose Conditions 1.40.1 to 1.40.4 are satisfied. Let $\boldsymbol{S}(G)$ be defined as in (1.38).

First, note that $f(\mathbb{R}_+) = [f(0), +\infty)$ because $f(\mathbb{R}_+)$ is an interval (Lemma 1.1) and f is strictly increasing (Assumption F) and not bounded above (Condition 1.40.4). Second, note that $f(\mathbb{R}_+)$ is convex. Third, note that $\operatorname{int}(f(\mathbb{R}_+)) = (f(0), +\infty)$.

Let $\mathcal{M}^\star \subset \mathbb{R}^n_+$ denote the set of all global maximum points of w. I show that $|\mathcal{M}^\star| = 1$. Let the function $\widetilde{w}\colon f(\mathbb{R}_+)^n \to \mathbb{R}$ be defined by $\widetilde{w} := w \circ (f^{-1}, \dots, f^{-1})$. I show that $\widetilde{w}$ has a unique, global, and interior maximum point. The function $\widetilde{w}$ is twice continuously differentiable and strictly concave on $f(\mathbb{R}_+)^n$ (Conditions 1.40.1 and 1.40.2). It is therefore enough to show that $\widetilde{w}$ has a local maximum point in the interior of $f(\mathbb{R}_+)^n$. The function $\boldsymbol{z} \mapsto \widetilde{w}(\boldsymbol{z})$ with domain $\mathbb{R}^n$ (sic) has a unique global maximum point $\boldsymbol{z}^\star := (z_1^\star, \dots, z_n^\star) \in \mathbb{R}^n$, which is given by (D.103). I show that $\boldsymbol{z}^\star \in \operatorname{int}(f(\mathbb{R}_+)^n) = (f(0), +\infty)^n$. To this end, let

$$\forall i \in \mathcal{I} \quad \boldsymbol{a}_i := \big(\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G)\big)^{-1} \boldsymbol{e}_i. \tag{D.104}$$

We find

$$\begin{aligned}
\boldsymbol{z}^\star - f(0)\mathbf{1}_n &= \big(\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G)\big)^{-1}\boldsymbol{\alpha} - f(0)\mathbf{1}_n \\
&= \big(\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G)\big)^{-1}\Big(\bar{\alpha}_{\mathcal{I}}\mathbf{1}_n + \sum_{i=1}^n (\alpha_i - \bar{\alpha}_{\mathcal{I}})\boldsymbol{e}_i\Big) - f(0)\mathbf{1}_n \\
&= \frac{1}{\beta}\big(\bar{\alpha}_{\mathcal{I}} - \beta f(0)\big)\mathbf{1}_n + \sum_{i=1}^n (\alpha_i - \bar{\alpha}_{\mathcal{I}})\big(\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G)\big)^{-1}\boldsymbol{e}_i \\
&= \frac{1}{\beta}\big(\bar{\alpha}_{\mathcal{I}} - \beta f(0)\big)\mathbf{1}_n + \sum_{i=1}^n (\alpha_i - \bar{\alpha}_{\mathcal{I}})\boldsymbol{a}_i \\
&= \frac{1}{\beta}\big(\bar{\alpha}_{\mathcal{I}} - \beta f(0)\big)\mathbf{1}_n + \sum_{i=1}^n |\alpha_i - \bar{\alpha}_{\mathcal{I}}|\, \mathbb{1}_{(-\infty,0)}(\alpha_i - \bar{\alpha}_{\mathcal{I}})(-1)\boldsymbol{a}_i \\
&\quad + \sum_{i=1}^n |\alpha_i - \bar{\alpha}_{\mathcal{I}}|\, \mathbb{1}_{(0,+\infty)}(\alpha_i - \bar{\alpha}_{\mathcal{I}})\boldsymbol{a}_i \\
&\geq_c \frac{1}{\beta}\big(\bar{\alpha}_{\mathcal{I}} - \beta f(0)\big)\mathbf{1}_n \\
&\quad + \sum_{i=1}^n |\alpha_i - \bar{\alpha}_{\mathcal{I}}|\, \mathbb{1}_{(-\infty,0)}(\alpha_i - \bar{\alpha}_{\mathcal{I}})\Big(-\frac{1}{\beta s(\beta,\gamma,G)}\mathbf{1}_n\Big) \\
&\quad + \sum_{i=1}^n |\alpha_i - \bar{\alpha}_{\mathcal{I}}|\, \mathbb{1}_{(0,+\infty)}(\alpha_i - \bar{\alpha}_{\mathcal{I}})\Big(-\frac{1}{\beta s(\beta,\gamma,G)}\mathbf{1}_n\Big) \\
&= \frac{1}{\beta}\Big(\bar{\alpha}_{\mathcal{I}} - \beta f(0) - \frac{1}{s(\beta,\gamma,G)}\sum_{i=1}^n |\alpha_i - \bar{\alpha}_{\mathcal{I}}|\Big)\mathbf{1}_n \\
&>_c \mathbf{0}_n.
\end{aligned}$$

The first equality is according to (D.103). The second equality follows from the identity $\boldsymbol{\alpha} = \bar{\alpha}_{\mathcal{I}} \mathbf{1}_n + \sum_{i=1}^{n} (\alpha_i - \bar{\alpha}_{\mathcal{I}}) \boldsymbol{e}_i$. The third equality follows from Result 1.43.5. The forth equality is according to (D.104). The fifth and the sixth equality are obvious. The first inequality follows from Results 1.43.1 to 1.43.4, which together imply that for all $i \in \mathcal{I}$, $-(1/(\beta s(\beta,\gamma,G)))\mathbf{1}_n \leq_c \boldsymbol{a}_i \leq_c (1/(\beta s(\beta,\gamma,G)))\mathbf{1}_n$. The second inequality follows from Conditions 1.40.1 and 1.40.3. This concludes the proof that $\boldsymbol{z}^\star \in \operatorname{int}(f(\mathbb{R}_+)^n)$ and the proof that $\boldsymbol{z}^\star$ is the unique global maximum point of $\widetilde{w}$. Finally, since f is bijective (Lemma 1.1), there exists a unique $\boldsymbol{y}_S^\star := (y_{S,1}^\star, \ldots, y_{S,n}^\star) \in \operatorname{int}(\mathbb{R}_+^n)$ such that $\boldsymbol{z}^\star = \boldsymbol{f}(\boldsymbol{y}_S^\star)$; in particular, for all $i \in \mathcal{I}$, $y_{S,i}^\star := f^{-1}(z_i^\star)$. The vector $\boldsymbol{y}_S^\star$ is the unique, global, and interior maximum point of w. It follows that Γ has a unique and interior social optimum, which is given by (1.39).■

Proof of Proposition 1.41

Let $\Gamma := (\mathcal{I}, G, [0,\bar{v}], \{(\alpha_i, \beta, \gamma)\}_{i \in \mathcal{I}}, f)$ be a NALA game, and let $\boldsymbol{\alpha}$ be defined as in Section 1.3.2 and $\bar{\alpha}_{\mathcal{I}} := (1/n) \sum_{i \in \mathcal{I}} \alpha_i$. Suppose Conditions 1.41.1, 1.41.2, and 1.41.3 are satisfied. Let $\boldsymbol{S}(G)$ be defined as in (1.38).

First, note that $f([0,\bar{v}]) = [f(0), f(\bar{v})]$ (Lemma 1.1). Second, note that $f([0,\bar{v}])$ is convex. Third, note that $\operatorname{int}(f([0,\bar{v}])) = (f(0), f(\bar{v}))$.

Let $\mathcal{M}^\star \subset [0,\bar{v}]^n$ denote the set of all global maximum points of w. I show that $|\mathcal{M}^\star| = 1$. Let the function $\widetilde{w}\colon f([0,\bar{v}])^n \to \mathbb{R}$ be defined by $\widetilde{w} := w \circ (f^{-1}, \ldots, f^{-1})$. I show that $\widetilde{w}$ has a unique, global, and interior maximum point. The function $\widetilde{w}$ is twice continuously differentiable and strictly concave on $f([0,\bar{v}])^n$ (Conditions 1.41.1 and 1.41.2). It is therefore enough to show that $\widetilde{w}$ has a local maximum point in the interior of $f([0,\bar{v}])^n$. The function $\boldsymbol{z} \mapsto \widetilde{w}(\boldsymbol{z})$ with domain $\mathbb{R}^n$ (sic) has a unique global maximum point $\boldsymbol{z}^\star := (z_1^\star, \ldots, z_n^\star) \in \mathbb{R}^n$, which is given by (D.103). I show that $\boldsymbol{z}^\star \in \operatorname{int}(f([0,\bar{v}])^n) = (f(0), f(\bar{v}))^n$. According to the proof of Proposition 1.40, we have

$$\boldsymbol{z}^\star - f(0)\mathbf{1}_n \geq_c \frac{1}{\beta}\Big(\bar{\alpha}_{\mathcal{I}} - \beta f(0) - \frac{1}{s(\beta,\gamma,G)} \sum_{i=1}^{n} |\alpha_i - \bar{\alpha}_{\mathcal{I}}|\Big)\mathbf{1}_n >_c \mathbf{0}_n,$$

where the second inequality follows from Conditions 1.41.1 and 1.41.3. Let $\{\boldsymbol{a}_i\}_{i \in \mathcal{I}}$ be defined as in (D.104). We find

$$\begin{aligned}
\boldsymbol{z}^\star - f(\bar{v})\mathbf{1}_n = {} & \frac{1}{\beta}\big(\bar{\alpha}_{\mathcal{I}} - \beta f(\bar{v})\big)\mathbf{1}_n + \sum_{i=1}^{n} |\alpha_i - \bar{\alpha}_{\mathcal{I}}|\, \mathbb{1}_{(-\infty,0)}(\alpha_i - \bar{\alpha}_{\mathcal{I}})(-1)\boldsymbol{a}_i \\
& + \sum_{i=1}^{n} |\alpha_i - \bar{\alpha}_{\mathcal{I}}|\, \mathbb{1}_{(0,+\infty)}(\alpha_i - \bar{\alpha}_{\mathcal{I}})\boldsymbol{a}_i \\
\leq_c {} & \frac{1}{\beta}\big(\bar{\alpha}_{\mathcal{I}} - \beta f(\bar{v})\big)\mathbf{1}_n + \sum_{i=1}^{n} |\alpha_i - \bar{\alpha}_{\mathcal{I}}|\, \mathbb{1}_{(-\infty,0)}(\alpha_i - \bar{\alpha}_{\mathcal{I}})\Big(\frac{1}{\beta s(\beta,\gamma,G)}\mathbf{1}_n\Big) \\
& + \sum_{i=1}^{n} |\alpha_i - \bar{\alpha}_{\mathcal{I}}|\, \mathbb{1}_{(0,+\infty)}(\alpha_i - \bar{\alpha}_{\mathcal{I}})\Big(\frac{1}{\beta s(\beta,\gamma,G)}\mathbf{1}_n\Big)
\end{aligned}$$

$$= \frac{1}{\beta}\left(\bar{\alpha}_{\mathcal{I}} - \beta f(\bar{v}) + \frac{1}{s(\beta,\gamma,G)}\sum_{i=1}^{n}|\alpha_i - \bar{\alpha}_{\mathcal{I}}|\right)\mathbf{1}_n$$
$$<_c \mathbf{0}_n,$$

where the second inequality follows from Conditions 1.41.1 and 1.41.3. This concludes the proof that $\boldsymbol{z}^\star \in (f(0), f(\bar{v}))^n$ and the proof that $\boldsymbol{z}^\star$ is the unique global maximum point of $\widetilde{w}$. Finally, since f is bijective (Lemma 1.1), there exists a unique $\boldsymbol{y}_S^\star := (y_{S,1}^\star, \ldots, y_{S,n}^\star) \in (0,\bar{v})^n$ such that $\boldsymbol{z}^\star = \boldsymbol{f}(\boldsymbol{y}_S^\star)$; in particular, for all $i \in \mathcal{I}$, $y_{S,i}^\star := f^{-1}(z_i^\star)$. The vector $\boldsymbol{y}_S^\star$ is the unique, global, and interior maximum point of w. It follows that Γ has a unique and interior social optimum, which is given by (1.39). ∎

Proof of Lemma 1.43

Let G be a digraph on $\mathcal{I}$. Let $\boldsymbol{S}(G)$ be defined as in (1.38). Note that $\sigma(\boldsymbol{S}(G)) \subset \mathbb{R}_+$ because $\boldsymbol{S}(G)$ is symmetric and nonnegative definite. Suppose $\beta \in \mathbb{R}_{++}$ and $\gamma \in \mathbb{R}$ are such that $\gamma\rho(\boldsymbol{S}(G)) > -\beta$.

Before proving Results 1.43.1 to 1.43.5, I establish the following two auxiliary results:

$$\min \sigma\big(\boldsymbol{S}(G)\big) = 0. \tag{D.105}$$

and

$$\sigma\big(\beta\boldsymbol{I}_n + \gamma\boldsymbol{S}(G)\big) = \beta + \gamma\sigma\big(\boldsymbol{S}(G)\big) := \big\{\beta + \gamma\mu \,\big|\, \mu \in \sigma\big(\boldsymbol{S}(G)\big)\big\}. \tag{D.106}$$

First, I prove (D.105). Note that $\min\sigma(\boldsymbol{S}(G)) \geq 0$ because $\sigma(\boldsymbol{S}(G)) \subset \mathbb{R}_+$. Also note that $0 \in \sigma(\boldsymbol{S}(G))$ because $\bar{\boldsymbol{A}}(G)\mathbf{1}_n = \mathbf{1}_n$ implies that $\boldsymbol{S}(G)\mathbf{1}_n = \mathbf{0}_n = 0\mathbf{1}_n$. Combining the preceding two results, we find $\min\sigma(\boldsymbol{S}(G)) = 0$. Second, I prove (D.106). The equality is trivial if $\gamma = 0$. Suppose $\gamma \neq 0$ in what follows. I show that $\sigma(\beta\boldsymbol{I}_n + \gamma\boldsymbol{S}(G)) \subset \beta + \gamma\sigma(\boldsymbol{S}(G))$ and $\beta + \gamma\sigma(\boldsymbol{S}(G)) \subset \sigma(\beta\boldsymbol{I}_n + \gamma\boldsymbol{S}(G))$. Let $(\lambda, \boldsymbol{v})$ be an eigenpair of $\beta\boldsymbol{I}_n + \gamma\boldsymbol{S}(G)$. We have $(\beta\boldsymbol{I}_n + \gamma\boldsymbol{S}(G))\boldsymbol{v} = \lambda\boldsymbol{v}$, which is equivalent to $\boldsymbol{S}(G)\boldsymbol{v} = (1/\gamma)(\lambda - \beta)\boldsymbol{v}$, which implies that $(1/\gamma)(\lambda-\beta) \in \sigma(\boldsymbol{S}(G))$, which in turn implies that $\lambda = \beta + \gamma(1/\gamma)(\lambda - \beta) \in \beta + \gamma\sigma(\boldsymbol{S}(G))$. This proves that $\sigma(\beta\boldsymbol{I}_n + \gamma\boldsymbol{S}(G)) \subset \beta + \gamma\sigma(\boldsymbol{S}(G))$. Let $\lambda \in \beta + \gamma\sigma(\boldsymbol{S}(G))$. We have $\lambda = \beta + \gamma\mu$ for some $\mu \in \sigma(\boldsymbol{S}(G))$ with corresponding eigenvector $\boldsymbol{w}$. We find $(\beta\boldsymbol{I}_n + \gamma\boldsymbol{S}(G))\boldsymbol{w} = \beta\boldsymbol{w} + \gamma\boldsymbol{S}(G)\boldsymbol{w} = \beta\boldsymbol{w} + \gamma\mu\boldsymbol{w} = (\beta + \gamma\mu)\boldsymbol{w}$, which implies that $\lambda = \beta + \gamma\mu \in \sigma(\beta\boldsymbol{I}_n + \gamma\boldsymbol{S}(G))$. This proves that $\beta + \gamma\sigma(\boldsymbol{S}(G)) \subset \sigma(\beta\boldsymbol{I}_n + \gamma\boldsymbol{S}(G))$.

The two auxiliary results (D.105) and (D.106) allow us to conclude that the symmetric matrix $\beta\boldsymbol{I}_n + \gamma\boldsymbol{S}(G)$ is positive definite and therefore nonsingular. Indeed, if $\gamma < 0$, then $\min\sigma(\beta\boldsymbol{I}_n + \gamma\boldsymbol{S}(G)) = \beta + \gamma\max\sigma(\boldsymbol{S}(G)) = \beta + \gamma\rho(\boldsymbol{S}(G)) > \beta - \beta = 0$, and if $\gamma \geq 0$, then $\min\sigma(\beta\boldsymbol{I}_n + \gamma\boldsymbol{S}(G)) = \beta + \gamma\min\sigma(\boldsymbol{S}(G)) = \beta + \gamma 0 = \beta > 0$.

Let $\boldsymbol{Q}(\beta,\gamma,G)$ be defined as in (1.41).

Proof of Result 1.43.1 The matrix $\boldsymbol{Q}(\beta,\gamma,G)$ is symmetric and positive definite because its inverse, $\beta\boldsymbol{I}_n + \gamma\boldsymbol{S}(G)$, is symmetric and positive definite. ∎

Proof of Result 1.43.2 Note that $\max \sigma(\boldsymbol{Q}(\beta,\gamma,G)) = \rho(\boldsymbol{Q}(\beta,\gamma,G))$ because $\boldsymbol{Q}(\beta,\gamma,G)$ is symmetric and positive definite. Using (D.105) and (D.106), we find

$$\begin{aligned}\max \sigma\big(\boldsymbol{Q}(\beta,\gamma,G)\big) &= \max\left\{\frac{1}{\lambda}\,\middle|\, \lambda \in \sigma\big(\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G)\big)\right\}\\ &= \frac{1}{\min \sigma\big(\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G)\big)}\\ &= \begin{cases}\dfrac{1}{\beta - |\gamma|\rho\big(\boldsymbol{S}(G)\big)} & \text{if } \gamma < 0,\\ \dfrac{1}{\beta} & \text{if } \gamma \geq 0.\end{cases}\end{aligned}$$ ∎

Proof of Result 1.43.3 Let $i \in \mathcal{I}$. According to the Rayleigh quotient theorem (see, for example, Horn and Johnson 2012, Theorem 4.2.2),

$$\min \sigma\big(\boldsymbol{Q}(\beta,\gamma,G)\big) \leq \langle \boldsymbol{e}_i, \boldsymbol{Q}(\beta,\gamma,G)\boldsymbol{e}_i\rangle \leq \max \sigma\big(\boldsymbol{Q}(\beta,\gamma,G)\big),$$

where $\langle \boldsymbol{e}_i, \boldsymbol{Q}(\beta,\gamma,G)\boldsymbol{e}_i\rangle = [\boldsymbol{Q}(\beta,\gamma,G)]_{i,i}$. ∎

Proof of Result 1.43.4 Let $(i,j) \in \mathcal{I}^2$ with $i \neq j$. We have

$$\begin{aligned}0 &< \big\langle \boldsymbol{e}_i - \operatorname{sgn}\big([\boldsymbol{Q}(\beta,\gamma,G)]_{i,j}\big)\boldsymbol{e}_j, \boldsymbol{Q}(\beta,\gamma,G)\big(\boldsymbol{e}_i - \operatorname{sgn}\big([\boldsymbol{Q}(\beta,\gamma,G)]_{i,j}\big)\boldsymbol{e}_j\big)\big\rangle\\ &= \langle \boldsymbol{e}_i, \boldsymbol{Q}(\beta,\gamma,G)\boldsymbol{e}_i\rangle - \operatorname{sgn}\big([\boldsymbol{Q}(\beta,\gamma,G)]_{i,j}\big)\big\langle \boldsymbol{e}_j, \boldsymbol{Q}(\beta,\gamma,G)\boldsymbol{e}_i\big\rangle\\ &\quad - \operatorname{sgn}\big([\boldsymbol{Q}(\beta,\gamma,G)]_{i,j}\big)\big\langle \boldsymbol{e}_i, \boldsymbol{Q}(\beta,\gamma,G)\boldsymbol{e}_j\big\rangle\\ &\quad + \operatorname{sgn}\big([\boldsymbol{Q}(\beta,\gamma,G)]_{i,j}\big)^2\big\langle \boldsymbol{e}_j, \boldsymbol{Q}(\beta,\gamma,G)\boldsymbol{e}_j\big\rangle\\ &= [\boldsymbol{Q}(\beta,\gamma,G)]_{i,i} - 2|[\boldsymbol{Q}(\beta,\gamma,G)]_{i,j}| + [\boldsymbol{Q}(\beta,\gamma,G)]_{j,j}.\end{aligned} \tag{D.107}$$

The inequality follows from the positive definiteness of $\boldsymbol{Q}(\beta,\gamma,G)$. The first equality is straightforward to show. The second equality follows from the symmetry of $\boldsymbol{Q}(\beta,\gamma,G)$. Finally, we find

$$\begin{aligned}|[\boldsymbol{Q}(\beta,\gamma,G)]_{i,j}| &< \frac{[\boldsymbol{Q}(\beta,\gamma,G)]_{i,i} + [\boldsymbol{Q}(\beta,\gamma,G)]_{j,j}}{2}\\ &\leq \max\{[\boldsymbol{Q}(\beta,\gamma,G)]_{k,k} \mid k \in \mathcal{I}\}\\ &\leq \max \sigma\big(\boldsymbol{Q}(\beta,\gamma,G)\big).\end{aligned}$$

The first inequality is according to (D.107). The second inequality is obvious. The last inequality is according to Result 1.43.3. ∎

Proof of Result 1.43.5 We have

$$\big(\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G)\big)\boldsymbol{1}_n = \beta \boldsymbol{1}_n \tag{D.108}$$

because $\bar{\boldsymbol{A}}(G)\boldsymbol{1}_n = \boldsymbol{1}_n$. Premultiplying both sides of (D.108) by $\boldsymbol{Q}(\beta,\gamma,G)$ gives $\boldsymbol{1}_n = \beta \boldsymbol{Q}(\beta,\gamma,G)\boldsymbol{1}_n$. ∎

Proof of Proposition 1.44

Let $\Gamma := (\mathcal{I}, G, \mathcal{Y}, \{(\alpha_i, \beta, \gamma)\}_{i \in \mathcal{I}}, f)$ be a generic NALA game, and let $\boldsymbol{\alpha}$ be defined as in Section 1.3.2. Suppose $\beta > 0$. In addition, suppose Γ has a unique and interior social optimum $\boldsymbol{y}_S^\star$, which is given by (1.39).

Result 1.44.1 follows from (1.39). Result 1.44.2 follows from (1.39) and Result 1.43.5. Result 1.44.3 follows from (1.39) and the fact that $\bar{\boldsymbol{A}}(G)\mathbf{1}_n = \mathbf{1}_n$. Result 1.44.4 follows from (1.39), the identity

$$\left(\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G)\right)^{-1} = \frac{1}{\beta}\left(\boldsymbol{I}_n + \frac{\gamma}{\beta}\boldsymbol{S}(G)\right)^{-1} = \frac{1}{\beta}\left(\boldsymbol{I}_n - \frac{\gamma}{\beta}\left(\boldsymbol{I}_n + \frac{\gamma}{\beta}\boldsymbol{S}(G)\right)^{-1}\boldsymbol{S}(G)\right),$$

and the fact that $\boldsymbol{S}(G)\boldsymbol{\alpha} = (\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G))^\top(\boldsymbol{\alpha} - \bar{\boldsymbol{A}}(G)\boldsymbol{\alpha}) = \mathbf{0}_n$ if $\boldsymbol{\alpha} = \bar{\boldsymbol{A}}(G)\boldsymbol{\alpha}$. ∎

Proof of Lemma 1.45

Let G be a digraph on $\mathcal{I}$. Suppose Condition G is satisfied. Definition A implies that

$$\forall (i,j) \in \mathcal{I}^2 \quad \left[\bar{\boldsymbol{A}}(G)^\top\right]_{i,j} = \begin{cases} \delta_{i,j} & \text{if } \deg_G^+(j) = 0, \\ \dfrac{\mathbb{1}_{\mathcal{N}_G^-(i)}(j)}{\deg_G^+(j)} & \text{if } \deg_G^+(j) > 0, \end{cases} \tag{D.109}$$

and

$$\forall (i,j) \in \mathcal{I}^2 \quad \left[\bar{\boldsymbol{A}}(G)^\top \bar{\boldsymbol{A}}(G)\right]_{i,j}$$
$$= \begin{cases} \delta_{i,j}\, \mathbb{1}_{\mathcal{I}_0^+(G)}(j) & \text{if } \mathcal{N}_G^-(i) \cap \mathcal{N}_G^-(j) = \varnothing, \\ \delta_{i,j}\, \mathbb{1}_{\mathcal{I}_0^+(G)}(j) + \displaystyle\sum_{k \in \mathcal{N}_G^-(i) \cap \mathcal{N}_G^-(j)} \frac{1}{\deg_G^+(k)^2} & \text{if } \mathcal{N}_G^-(i) \cap \mathcal{N}_G^-(j) \neq \varnothing. \end{cases} \tag{D.110}$$

Suppose, for the sake of contradiction,

$$\bar{\boldsymbol{A}}(G)^\top \bar{\boldsymbol{A}}(G) = \bar{\boldsymbol{A}}(G)^\top. \tag{D.111}$$

According to Condition G, there exists a $\hat{\jmath} \in \mathcal{I}$ such that $\mathcal{N}_G^+(\hat{\jmath}) \neq \varnothing$ or, equivalently, $\deg_G^+(\hat{\jmath}) > 0$. Choose $\hat{\imath} \in \mathcal{N}_G^+(\hat{\jmath})$. Because G has no self-loops, which implies that $\hat{\jmath} \notin \mathcal{N}_G^+(\hat{\jmath})$ and $\hat{\jmath} \notin \mathcal{N}_G^-(\hat{\jmath})$, we must have $\hat{\imath} \neq \hat{\jmath}$. Using (D.109), (D.110), and (D.111), we find

$$\left[\bar{\boldsymbol{A}}(G)^\top \bar{\boldsymbol{A}}(G)\right]_{\hat{\jmath},\hat{\jmath}} = \left[\bar{\boldsymbol{A}}(G)^\top\right]_{\hat{\jmath},\hat{\jmath}} = \frac{\mathbb{1}_{\mathcal{N}_G^-(\hat{\jmath})}(\hat{\jmath})}{\deg_G^+(\hat{\jmath})} = 0,$$

which implies that $\mathcal{N}_G^-(\hat{\jmath}) = \varnothing$ must be true. Using this result and (D.109), (D.110), and (D.111), we find

$$0 = \left[\bar{\boldsymbol{A}}(G)^\top \bar{\boldsymbol{A}}(G)\right]_{\hat{\imath},\hat{\jmath}} = \left[\bar{\boldsymbol{A}}(G)^\top\right]_{\hat{\imath},\hat{\jmath}} = \frac{\mathbb{1}_{\mathcal{N}_G^-(\hat{\imath})}(\hat{\jmath})}{\deg_G^+(\hat{\jmath})},$$

which implies that $\hat{\jmath} \notin \mathcal{N}_G^-(\hat{\imath})$ and therefore $\hat{\imath} \notin \mathcal{N}_G^+(\hat{\jmath})$ must be true. This contradicts our choice of $\hat{\imath}$. Consequently, (D.111) cannot be true. ■

Proof of Proposition 1.46

Let $\Gamma := (\mathcal{I}, G, \mathcal{Y}, \{(\alpha_i, \beta, \gamma)\}_{i \in \mathcal{I}}, f)$ be a generic NALA game, and let $\boldsymbol{\alpha}$ be defined as in Section 1.3.2. Suppose $\beta > 0$ and $\gamma > -\beta/2$. In addition, suppose Γ has a unique and interior NE $\boldsymbol{y}_N^\star$, which is given by (1.7), and a unique and interior social optimum $\boldsymbol{y}_S^\star$, which is given by (1.39).

First, note that $\boldsymbol{y}_N^\star = \boldsymbol{y}_S^\star$ if and only if $\boldsymbol{f}(\boldsymbol{y}_N^\star) = \boldsymbol{f}(\boldsymbol{y}_S^\star)$ because f is bijective (Lemma 1.1). Second, note that

$$(1.7) \quad \Leftrightarrow \quad \boldsymbol{\alpha} = \Big(\beta \boldsymbol{I}_n + \gamma\big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)\Big)\boldsymbol{f}(\boldsymbol{y}_N^\star) \tag{D.112}$$

$$\Leftrightarrow \quad \frac{1}{\beta}\boldsymbol{\alpha} - \boldsymbol{f}(\boldsymbol{y}_N^\star) = \frac{\gamma}{\beta}\big(\boldsymbol{f}(\boldsymbol{y}_N^\star) - \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}_N^\star)\big) \tag{D.113}$$

and

$$(1.39) \quad \Leftrightarrow \quad \boldsymbol{\alpha} = \big(\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G)\big)\boldsymbol{f}(\boldsymbol{y}_S^\star). \tag{D.114}$$

We find

$$\begin{aligned}
\boldsymbol{0}_n &= \Big(\beta \boldsymbol{I}_n + \gamma\big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)\Big)\boldsymbol{f}(\boldsymbol{y}_N^\star) - \big(\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G)\big)\boldsymbol{f}(\boldsymbol{y}_S^\star) \\
&= \Big(\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G) + \gamma \bar{\boldsymbol{A}}(G)^\top\big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)\Big)\boldsymbol{f}(\boldsymbol{y}_N^\star) - \big(\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G)\big)\boldsymbol{f}(\boldsymbol{y}_S^\star) \\
&= \big(\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G)\big)\big(\boldsymbol{f}(\boldsymbol{y}_N^\star) - \boldsymbol{f}(\boldsymbol{y}_S^\star)\big) + \gamma \bar{\boldsymbol{A}}(G)^\top\big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)\boldsymbol{f}(\boldsymbol{y}_N^\star) \\
&= \big(\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G)\big)\big(\boldsymbol{f}(\boldsymbol{y}_N^\star) - \boldsymbol{f}(\boldsymbol{y}_S^\star)\big) - \beta \bar{\boldsymbol{A}}(G)^\top\Big(\boldsymbol{f}(\boldsymbol{y}_N^\star) - \frac{1}{\beta}\boldsymbol{\alpha}\Big).
\end{aligned} \tag{D.115}$$

The first equality is according to (D.112) and (D.114). The last equality is according to (D.113). Using (D.115), we find

$$\boldsymbol{f}(\boldsymbol{y}_N^\star) - \boldsymbol{f}(\boldsymbol{y}_S^\star) = \beta\big(\beta \boldsymbol{I}_n + \gamma \boldsymbol{S}(G)\big)^{-1}\bar{\boldsymbol{A}}(G)^\top\Big(\boldsymbol{f}(\boldsymbol{y}_N^\star) - \frac{1}{\beta}\boldsymbol{\alpha}\Big).$$

We conclude that $\boldsymbol{y}_N^\star = \boldsymbol{y}_S^\star$ if and only if $\bar{\boldsymbol{A}}(G)^\top(\boldsymbol{f}(\boldsymbol{y}_N^\star) - (1/\beta)\boldsymbol{\alpha}) = \boldsymbol{0}_n$ or, equivalently, $\boldsymbol{f}(\boldsymbol{y}_N^\star) - (1/\beta)\boldsymbol{\alpha} \in \ker(\bar{\boldsymbol{A}}(G)^\top)$. ■

Proof of Corollary 1.47

Let $\Gamma := (\mathcal{I}, G, \mathcal{Y}, \{(\alpha_i, \beta, \gamma)\}_{i \in \mathcal{I}}, f)$ be a generic NALA game, and let $\boldsymbol{\alpha}$ be defined as in Section 1.3.2. Suppose $\beta > 0$ and $\gamma > -\beta/2$. In addition, suppose Γ has a unique and interior NE $\boldsymbol{y}_N^\star$, which is given by (1.7), and a unique and interior social optimum $\boldsymbol{y}_S^\star$, which is given by (1.39). Note that $\boldsymbol{0}_n \in \ker(\bar{\boldsymbol{A}}(G)^\top)$.

First, I show that each of the Conditions 1.47.1, 1.47.2, and 1.47.3 is sufficient for $\boldsymbol{f}(\boldsymbol{y}_N^\star) - (1/\beta)\boldsymbol{\alpha} \in \ker(\bar{\boldsymbol{A}}(G)^\top)$. If $\gamma = 0$, then $\boldsymbol{f}(\boldsymbol{y}_N^\star) = (1/\beta)\boldsymbol{\alpha}$ (Result 1.31.2

and Remark 1.36), which implies that $\boldsymbol{f}(\boldsymbol{y}_N^\star) - (1/\beta)\boldsymbol{\alpha} = \boldsymbol{0}_n \in \ker(\bar{\boldsymbol{A}}(G)^\top)$. If $\boldsymbol{\alpha} = \bar{\boldsymbol{A}}(G)\boldsymbol{\alpha}$, then $\boldsymbol{f}(\boldsymbol{y}_N^\star) = (1/\beta)\boldsymbol{\alpha}$ (Result 1.35.1 and Remark 1.36), which implies that $\boldsymbol{f}(\boldsymbol{y}_N^\star) - (1/\beta)\boldsymbol{\alpha} = \boldsymbol{0}_n \in \ker(\bar{\boldsymbol{A}}(G)^\top)$. Evidently, $\boldsymbol{f}(\boldsymbol{y}_N^\star) \neq (1/\beta)\boldsymbol{\alpha}$ and $\boldsymbol{f}(\boldsymbol{y}_N^\star) - (1/\beta)\boldsymbol{\alpha} \in \ker(\bar{\boldsymbol{A}}(G)^\top)$ is sufficient for $\boldsymbol{f}(\boldsymbol{y}_N^\star) - (1/\beta)\boldsymbol{\alpha} \in \ker(\bar{\boldsymbol{A}}(G)^\top)$.

Second, I show that at least one of the Conditions 1.47.1, 1.47.2, or 1.47.3 is necessary for $\boldsymbol{f}(\boldsymbol{y}_N^\star) - (1/\beta)\boldsymbol{\alpha} \in \ker(\bar{\boldsymbol{A}}(G)^\top)$. Suppose $\boldsymbol{f}(\boldsymbol{y}_N^\star) - (1/\beta)\boldsymbol{\alpha} \in \ker(\bar{\boldsymbol{A}}(G)^\top)$ is true. I consider two cases. First, suppose $\boldsymbol{f}(\boldsymbol{y}_N^\star) = (1/\beta)\boldsymbol{\alpha}$. I show by contraposition that either $\gamma = 0$ or both $\gamma \neq 0$ and $\boldsymbol{\alpha} = \bar{\boldsymbol{A}}(G)\boldsymbol{\alpha}$ must be true if $\boldsymbol{f}(\boldsymbol{y}_N^\star) = (1/\beta)\boldsymbol{\alpha}$. We have

$$
\begin{aligned}
\neg\Big(\gamma = 0 \lor \big(\gamma \neq 0 \land \boldsymbol{\alpha} = \bar{\boldsymbol{A}}(G)\boldsymbol{\alpha}\big)\Big) \quad &\Leftrightarrow \quad \gamma \neq 0 \land \big(\gamma = 0 \lor \boldsymbol{\alpha} \neq \bar{\boldsymbol{A}}(G)\boldsymbol{\alpha}\big) \\
&\Leftrightarrow \quad \gamma \neq 0 \land \boldsymbol{\alpha} \neq \bar{\boldsymbol{A}}(G)\boldsymbol{\alpha} \\
&\Rightarrow \quad \gamma \neq 0 \land \big(\gamma = 0 \lor \boldsymbol{f}(\boldsymbol{y}_N^\star) \neq (1/\beta)\boldsymbol{\alpha}\big) \\
&\Leftrightarrow \quad \gamma \neq 0 \land \boldsymbol{f}(\boldsymbol{y}_N^\star) \neq (1/\beta)\boldsymbol{\alpha} \\
&\Rightarrow \quad \boldsymbol{f}(\boldsymbol{y}_N^\star) \neq (1/\beta)\boldsymbol{\alpha},
\end{aligned}
$$

where the first implication follows from Result 1.35.2 and Remark 1.36. Second, suppose $\boldsymbol{f}(\boldsymbol{y}_N^\star) \neq (1/\beta)\boldsymbol{\alpha}$. Evidently, $\boldsymbol{f}(\boldsymbol{y}_N^\star) - (1/\beta)\boldsymbol{\alpha} \in \ker(\bar{\boldsymbol{A}}(G)^\top)$ and $\boldsymbol{f}(\boldsymbol{y}_N^\star) \neq (1/\beta)\boldsymbol{\alpha}$. ■

Example 1.48

Suppose G is complete. Let the polynomial function $p\colon \mathbb{C} \to \mathbb{C}$ be defined by $p(\lambda) := \det(\lambda \boldsymbol{I}_n - \bar{\boldsymbol{A}}(G))$. Note that $\sigma(\bar{\boldsymbol{A}}(G)) = \{\lambda \in \mathbb{C} \mid p(\lambda) = 0\}$. Since G is complete, we have

$$
\begin{aligned}
\forall \lambda \in \mathbb{C} \quad p(\lambda) &= \det\left(\lambda \boldsymbol{I}_n - \frac{1}{n-1}\left(\boldsymbol{1}_n\boldsymbol{1}_n^\top - \boldsymbol{I}_n\right)\right) \\
&= \frac{1}{(n-1)^n} \det\Big(\big(1 + (n-1)\lambda\big)\boldsymbol{I}_n - \boldsymbol{1}_n\boldsymbol{1}_n^\top\Big).
\end{aligned}
$$

We find $1/(1-n) \in \sigma(\bar{\boldsymbol{A}}(G))$ because

$$
p\left(\frac{1}{1-n}\right) = \frac{1}{(1-n)^n} \det\left(\boldsymbol{1}_n\boldsymbol{1}_n^\top\right) = 0.
$$

If $\lambda \in \mathbb{C} \setminus \{1/(1-n)\}$, then

$$
\begin{aligned}
p(\lambda) &= \left(\frac{1 + (n-1)\lambda}{n-1}\right)^n \det\left(\boldsymbol{I}_n - \frac{1}{1 + (n-1)\lambda}\boldsymbol{1}_n\boldsymbol{1}_n^\top\right) \\
&= \left(\frac{1 + (n-1)\lambda}{n-1}\right)^n \left(1 - \frac{1}{1 + (n-1)\lambda}\langle \boldsymbol{1}_n, \boldsymbol{1}_n\rangle\right) \\
&= \left(\frac{1 + (n-1)\lambda}{n-1}\right)^n \frac{(n-1)(\lambda - 1)}{1 + (n-1)\lambda},
\end{aligned}
$$

where the second equality follows from the matrix determinant lemma (see, for example, Harville 1997, Corollary 18.1.3). We conclude that $\sigma(\bar{A}(G)) = \{1/(1-n), 1\}$. ■

Proof of Proposition 1.50

Let $\Gamma := (\mathcal{I}, G, \mathcal{Y}, \{(\alpha_i, \beta, \gamma)\}_{i \in \mathcal{I}}, f)$ be a generic NALA game, and let $\boldsymbol{\alpha}$ be defined as in Section 1.3.2. Suppose G is complete, the players of Γ are ex ante heterogeneous, $\beta > 0$, and $\gamma > -\beta/2$. In addition, suppose Γ has a unique and interior NE $\boldsymbol{y}_N^\star$, which is given by (1.7), and a unique and interior social optimum $\boldsymbol{y}_S^\star$, which is given by (1.39).

Since G is complete, we have

$$\bar{\boldsymbol{A}}(G) = \frac{\mathbf{1}_n \mathbf{1}_n^\top - \boldsymbol{I}_n}{n-1}, \tag{D.116}$$

which implies that

$$\boldsymbol{S}(G) = \frac{n}{(n-1)^2}\left(n\boldsymbol{I}_n - \mathbf{1}_n \mathbf{1}_n^\top\right). \tag{D.117}$$

First, I derive an expression for $\boldsymbol{f}(\boldsymbol{y}_N^\star)$. Using (1.7) and (D.116), we find

$$\boldsymbol{f}(\boldsymbol{y}_N^\star) = \frac{1}{\beta+\gamma}\boldsymbol{\alpha} + \frac{\gamma}{\beta+\gamma}\frac{\mathbf{1}_n \mathbf{1}_n^\top - \boldsymbol{I}_n}{n-1}\boldsymbol{f}(\boldsymbol{y}_N^\star),$$

which is equivalent to

$$\left(1 + \frac{\gamma}{\beta+\gamma}\frac{1}{n-1}\right)\boldsymbol{f}(\boldsymbol{y}_N^\star) = \frac{1}{\beta+\gamma}\boldsymbol{\alpha} + \frac{\gamma}{\beta+\gamma}\frac{\langle \mathbf{1}_n, \boldsymbol{f}(\boldsymbol{y}_N^\star)\rangle}{n-1}\mathbf{1}_n. \tag{D.118}$$

Using (D.118), we find

$$\langle \mathbf{1}_n, \boldsymbol{f}(\boldsymbol{y}_N^\star)\rangle = \frac{1}{\beta}\langle \mathbf{1}_n, \boldsymbol{\alpha}\rangle. \tag{D.119}$$

Combining (D.118) and (D.119) gives

$$\boldsymbol{f}(\boldsymbol{y}_N^\star) = \frac{1}{n(\beta+\gamma)-\beta}\left(\frac{\gamma}{\beta}\langle \mathbf{1}_n, \boldsymbol{\alpha}\rangle \mathbf{1}_n + (n-1)\boldsymbol{\alpha}\right) \tag{D.120}$$

$$= \frac{n}{n(\beta+\gamma)-\beta}\left(\frac{\gamma}{\beta}\frac{\langle \mathbf{1}_n, \boldsymbol{\alpha}\rangle}{n}\mathbf{1}_n + \left(1 - \frac{1}{n}\right)\boldsymbol{\alpha}\right). \tag{D.121}$$

Second, I derive an expression for $\boldsymbol{f}(\boldsymbol{y}_S^\star)$. Using (1.39) and (D.117), we find

$$\beta\boldsymbol{f}(\boldsymbol{y}_S^\star) + \frac{n\gamma}{(n-1)^2}\left(n\boldsymbol{I}_n - \mathbf{1}_n \mathbf{1}_n^\top\right)\boldsymbol{f}(\boldsymbol{y}_S^\star) = \boldsymbol{\alpha},$$

which is equivalent to

$$\left(\beta + \frac{n^2\gamma}{(n-1)^2}\right)\boldsymbol{f}(\boldsymbol{y}_S^\star) = \boldsymbol{\alpha} + \frac{n\gamma}{(n-1)^2}\langle \mathbf{1}_n, \boldsymbol{f}(\boldsymbol{y}_S^\star)\rangle \mathbf{1}_n. \tag{D.122}$$

Using (D.122), we find

$$\langle \mathbf{1}_n, \boldsymbol{f}(\boldsymbol{y}_S^\star)\rangle = \frac{1}{\beta}\langle \mathbf{1}_n, \boldsymbol{\alpha}\rangle. \tag{D.123}$$

Combining (D.122) and (D.123) gives

$$\boldsymbol{f}(\boldsymbol{y}_S^\star) = \frac{1}{(n-1)^2\beta + n^2\gamma}\left(n\frac{\gamma}{\beta}\langle \mathbf{1}_n, \boldsymbol{\alpha}\rangle \mathbf{1}_n + (n-1)^2\boldsymbol{\alpha}\right) \tag{D.124}$$

$$= \frac{n^2}{(n-1)^2\beta + n^2\gamma}\left(\frac{\gamma}{\beta}\frac{\langle \mathbf{1}_n, \boldsymbol{\alpha}\rangle}{n}\mathbf{1}_n + \left(1-\frac{1}{n}\right)^2\boldsymbol{\alpha}\right). \tag{D.125}$$

Third, I prove Results 1.50.1 and 1.50.2. According to (D.119) and (D.123), $\langle \mathbf{1}_n, \boldsymbol{f}(\boldsymbol{y}_N^\star)\rangle = \langle \mathbf{1}_n, \boldsymbol{f}(\boldsymbol{y}_S^\star)\rangle = (1/\beta)\langle \mathbf{1}_n, \boldsymbol{\alpha}\rangle$. Using (D.120) and (D.124), we find

$$\boldsymbol{f}(\boldsymbol{y}_N^\star) - \boldsymbol{f}(\boldsymbol{y}_S^\star) = \frac{(n-1)n\gamma}{\big(n(\beta+\gamma)-\beta\big)\big((n-1)^2\beta + n^2\gamma\big)}\left(\boldsymbol{\alpha} - \frac{\langle \mathbf{1}_n, \boldsymbol{\alpha}\rangle}{n}\mathbf{1}_n\right). \tag{D.126}$$

Since $n > 1$, $\gamma \neq 0$, and $\boldsymbol{\alpha} \neq (1/n)\langle \mathbf{1}_n, \boldsymbol{\alpha}\rangle \mathbf{1}_n$ (because the players of Γ are ex ante heterogeneous), we have $\boldsymbol{f}(\boldsymbol{y}_N^\star) \neq \boldsymbol{f}(\boldsymbol{y}_S^\star)$, which is equivalent to $\boldsymbol{y}_N^\star \neq \boldsymbol{y}_S^\star$ because f is bijective (Lemma 1.1).

Fourth, I prove Results 1.50.3, 1.50.4, and 1.50.5, thereby making the dependence of $y_{N,i}^\star$ and $y_{S,i}^\star$ on n explicit. According to (D.126),

$$\forall i \in \mathcal{I}\ \forall n > 1 \quad f\big(y_{N,i}^\star(n)\big) - f\big(y_{S,i}^\star(n)\big) = r(n)\left(\alpha_i - \frac{\sum_{k=1}^n \alpha_k}{n}\right),$$

where

$$r(n) := \frac{(n-1)n\gamma}{\big(n(\beta+\gamma)-\beta\big)\big((n-1)^2\beta + n^2\gamma\big)} \in o(1) \text{ as } n \to \infty$$

because $\gamma > -\beta/2$. Consequently, if $\sum_{k=1}^n \alpha_k \in \mathcal{O}(n)$ as $n \to \infty$, then, for all $i \in \mathcal{I}$, $f(y_{N,i}^\star(n)) - f(y_{S,i}^\star(n)) \in o(1)$ as $n \to \infty$. If f^{-1} is Lipschitz continuous, then there exists a constant $L \geq 0$ such that

$$\forall i \in \mathcal{I}\ \forall n > 1 \quad \big|y_{N,i}^\star(n) - y_{S,i}^\star(n)\big| \leq L\big|f\big(y_{N,i}^\star(n)\big) - f\big(y_{S,i}^\star(n)\big)\big|.$$

Consequently, if f^{-1} is Lipschitz continuous and $\sum_{k=1}^n \alpha_k \in \mathcal{O}(n)$ as $n \to \infty$, then, for all $i \in \mathcal{I}$, $y_{N,i}^\star(n) - y_{S,i}^\star(n) \in o(1)$ as $n \to \infty$. According to (D.121) and (D.125), if there exists an $\bar{\alpha} \in \mathbb{R}$ such that $(1/n)\sum_{k=1}^n \alpha_k - \bar{\alpha} \in o(1)$ as $n \to \infty$, then, for all $i \in \mathcal{I}$, $f(y_{N,i}^\star(n)) - c_i \in o(1)$ as $n \to \infty$ and $f(y_{S,i}^\star(n)) - c_i \in o(1)$ as $n \to \infty$, where

$$\forall i \in \mathcal{I} \quad c_i := \frac{1}{\beta+\gamma}\left(\frac{\gamma}{\beta}\bar{\alpha} + \alpha_i\right).$$

Thus, since f^{-1} is continuous (Lemma 1.1), if there exists an $\bar{\alpha} \in \mathbb{R}$ such that $(1/n)\sum_{k=1}^n \alpha_k - \bar{\alpha} \in o(1)$ as $n \to \infty$, then, for all $i \in \mathcal{I}$, $y_{N,i}^\star(n) - f^{-1}(c_i) \in o(1)$ as $n \to \infty$ and $y_{S,i}^\star(n) - f^{-1}(c_i) \in o(1)$ as $n \to \infty$, which in turn implies that $y_{N,i}^\star(n) - y_{S,i}^\star(n) \in o(1)$ as $n \to \infty$. ■

Proof of Proposition 1.51

Let $\mathit{\Gamma} := (\mathcal{I}, G, \mathcal{Y}, \{(\alpha_i, \beta, \gamma)\}_{i\in\mathcal{I}}, f)$ be a generic NALA game, and let $\boldsymbol{\alpha}$ be defined as in Section 1.3.2. Suppose Condition G is satisfied, $\beta > 0$, and $\gamma > -\beta/2$, so that $\beta + \gamma > 0$ and $|\gamma/(\beta+\gamma)| < 1$. In addition, suppose $\mathit{\Gamma}$ has a unique and interior social optimum $\boldsymbol{y}_S^\star$, which is given by (1.39). Let

$$\boldsymbol{\alpha}_S := (\alpha_{S,1}, \ldots, \alpha_{S,n}) := \boldsymbol{\alpha} + \gamma \bar{\boldsymbol{A}}(G)^\mathsf{T} \big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big) \boldsymbol{f}(\boldsymbol{y}_S^\star) \tag{D.127}$$

and

$$N(G) := \max\left\{ \mathbb{1}_{\mathcal{I}_0^+(G)}(j) + \frac{\deg_G^-(j)}{\min\{\deg_G^+(i) \mid i \in \mathcal{I} \setminus \mathcal{I}_0^+(G)\}} \;\middle|\; j \in \mathcal{I} \right\}.$$

First, I show that $\boldsymbol{y}_S^\star$ can be decentralized as the unique and interior NE of the generic NALA game $(\mathcal{I}, G, \mathcal{Y}, \{(\alpha_{S,i}, \beta, \gamma)\}_{i\in\mathcal{I}}, f)$. To this end, note that the matrix $\boldsymbol{I}_n - \gamma/(\beta+\gamma)\bar{\boldsymbol{A}}(G)$ is nonsingular because $|\gamma/(\beta+\gamma)| < 1$ and $\rho(\bar{\boldsymbol{A}}(G)) = 1$ (Lemma B.3). Using (1.39), we find

$$\boldsymbol{f}(\boldsymbol{y}_S^\star) = \frac{1}{\beta+\gamma}\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G) - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)^\mathsf{T}\big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)\right)^{-1}\boldsymbol{\alpha},$$

which is equivalent to

$$\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)\boldsymbol{f}(\boldsymbol{y}_S^\star) = \frac{1}{\beta+\gamma}\Big(\boldsymbol{\alpha} + \gamma\bar{\boldsymbol{A}}(G)^\mathsf{T}\big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)\boldsymbol{f}(\boldsymbol{y}_S^\star)\Big),$$

which in turn is equivalent to

$$\boldsymbol{f}(\boldsymbol{y}_S^\star) = \frac{1}{\beta+\gamma}\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1}\boldsymbol{\alpha}_S. \tag{D.128}$$

Second, I show that $\boldsymbol{\alpha}_S = (\boldsymbol{I}_n - \gamma\bar{\boldsymbol{A}}(G)^\mathsf{T}(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G))\boldsymbol{J}(\beta,\gamma,G))^{-1}\boldsymbol{\alpha}$ if (1.47) is true. Suppose (1.47) is true. We have

$$\boldsymbol{\alpha}_S = \boldsymbol{\alpha} + \gamma\bar{\boldsymbol{A}}(G)^\mathsf{T}\big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)\boldsymbol{f}(\boldsymbol{y}_S^\star) = \boldsymbol{\alpha} + \gamma\bar{\boldsymbol{A}}(G)^\mathsf{T}\big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)\boldsymbol{J}(\beta,\gamma,G)\boldsymbol{\alpha}_S,$$

where the first equality is according to (D.127) and the second equality is according to (D.128) and the definition of $\boldsymbol{J}(\beta,\gamma,G)$ in (1.25). Using the preceding result, we find

$$\Big(\boldsymbol{I}_n - \gamma\bar{\boldsymbol{A}}(G)^\mathsf{T}\big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)\boldsymbol{J}(\beta,\gamma,G)\Big)\boldsymbol{\alpha}_S = \boldsymbol{\alpha},$$

where $\boldsymbol{I}_n - \gamma\bar{\boldsymbol{A}}(G)^\mathsf{T}(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G))\boldsymbol{J}(\beta,\gamma,G)$ is nonsingular (Lemma B.3).

Third, I show that

$$-\frac{\beta}{2+2N(G)} < \gamma < \frac{\beta}{2N(G)} \tag{D.129}$$

is sufficient for (1.47). To this end, I establish two auxiliary results:

$$\|\bar{\boldsymbol{A}}(G)\|_1 \le N(G) \tag{D.130}$$

and

$$\|\boldsymbol{J}(\beta,\gamma,G)\|_\infty \le \begin{cases} \dfrac{1}{\beta+2\gamma} & \text{if } \gamma < 0, \\ \dfrac{1}{\beta} & \text{if } \gamma \ge 0. \end{cases} \tag{D.131}$$

First, I prove (D.130). Similar to the proof of (D.99), we find, for all $j \in \mathcal{I}$,

$$\begin{aligned} \sum_{i=1}^{n} \big|[\bar{\boldsymbol{A}}(G)]_{i,j}\big| &= \sum_{i \in \mathcal{I}_0^+(G)} \delta_{i,j} + \sum_{i \in \mathcal{I} \setminus \mathcal{I}_0^+(G)} \frac{\mathbb{1}_{\mathcal{N}_G^+(i)}(j)}{\deg_G^+(i)} \\ &= \mathbb{1}_{\mathcal{I}_0^+(G)}(j) + \sum_{i \in \mathcal{I} \setminus \mathcal{I}_0^+(G)} \frac{\mathbb{1}_{\mathcal{N}_G^-(j)}(i)}{\deg_G^+(i)} \\ &\le \mathbb{1}_{\mathcal{I}_0^+(G)}(j) + \frac{1}{\min\{\deg_G^+(i) \mid i \in \mathcal{I} \setminus \mathcal{I}_0^+(G)\}} \sum_{i \in \mathcal{I} \setminus \mathcal{I}_0^+(G)} \mathbb{1}_{\mathcal{N}_G^-(j)}(i) \\ &= \mathbb{1}_{\mathcal{I}_0^+(G)}(j) + \frac{\deg_G^-(j)}{\min\{\deg_G^+(i) \mid i \in \mathcal{I} \setminus \mathcal{I}_0^+(G)\}}. \end{aligned}$$

The preceding result implies that

$$\|\bar{\boldsymbol{A}}(G)\|_1 = \max\Big\{\sum_{i=1}^{n} \big|[\bar{\boldsymbol{A}}(G)]_{i,j}\big| \,\Big|\, j \in \mathcal{I}\Big\} \le N(G).$$

This concludes the proof of (D.130). Second, I prove (D.131). First, consider the case $\gamma < 0$. I show that $\|\boldsymbol{J}(\beta,\gamma,G)\|_\infty \le 1/(\beta+2\gamma)$. To this end, note that

$$\boldsymbol{J}(\beta,\gamma,G) = \frac{1}{\beta+\gamma}\left(\boldsymbol{I}_n - \frac{\gamma^2}{(\beta+\gamma)^2}\bar{\boldsymbol{A}}(G)^2\right)^{-1}\left(\boldsymbol{I}_n - \frac{|\gamma|}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right). \tag{D.132}$$

The matrix $\boldsymbol{I}_n - \gamma^2/(\beta+\gamma)^2\bar{\boldsymbol{A}}(G)^2$ has a nonnegative inverse because it is a nonsingular M-matrix. This fact together with (D.35) implies that

$$\left\|\left(\boldsymbol{I}_n - \frac{\gamma^2}{(\beta+\gamma)^2}\bar{\boldsymbol{A}}(G)^2\right)^{-1}\right\|_\infty = \frac{(\beta+\gamma)^2}{\beta(\beta+2\gamma)}. \tag{D.133}$$

We have

$$\forall i \in \mathcal{I} \quad \sum_{j=1}^{n} \left|\left[\boldsymbol{I}_n - \frac{|\gamma|}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right]_{i,j}\right| = \begin{cases} \dfrac{\beta+2\gamma}{\beta+\gamma} & \text{if } \deg_G^+(i) = 0, \\ \dfrac{\beta}{\beta+\gamma} & \text{if } \deg_G^+(i) > 0. \end{cases}$$

Condition G and the preceding result imply that

$$\left\|\boldsymbol{I}_n - \frac{|\gamma|}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right\|_\infty = \frac{\beta}{\beta+\gamma}. \tag{D.134}$$

Using (D.132), we find

$$\begin{aligned}\|\boldsymbol{J}(\beta,\gamma,G)\|_\infty &\leq_c \frac{1}{\beta+\gamma}\left\|\left(\boldsymbol{I}_n-\frac{\gamma^2}{(\beta+\gamma)^2}\bar{\boldsymbol{A}}(G)^2\right)^{-1}\right\|_\infty\left\|\left(\boldsymbol{I}_n-\frac{|\gamma|}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)\right\|_\infty\\ &=\frac{1}{\beta+\gamma}\frac{(\beta+\gamma)^2}{\beta(\beta+2\gamma)}\frac{\beta}{\beta+\gamma}\\ &=\frac{1}{\beta+2\gamma},\end{aligned}$$

where the inequality follows from the fact that $\|\cdot\|_\infty$ is sub-multiplicative and the first equality is according to (D.133) and (D.134). Second, consider the case $\gamma \geq 0$. According to Results 1.23.2 and 1.23.3, for all $(i,j) \in \mathcal{I}^2$, $[\boldsymbol{J}(\beta,\gamma,G)]_{i,j} \geq 0$. This result and (D.12) imply that for all $i \in \mathcal{I}$, $\sum_{j=1}^{n}|[\boldsymbol{J}(\beta,\gamma,G)]_{i,j}| = 1/\beta$, from which $\|\boldsymbol{J}(\beta,\gamma,G)\|_\infty = 1/\beta$ follows. This concludes the proof of (D.131). Using the two auxiliary results, I show that (D.129) is sufficient for (1.47). First, consider the case $\gamma < 0$. We find

$$\begin{aligned}\rho\Big(\gamma\bar{\boldsymbol{A}}(G)^\mathsf{T}(\boldsymbol{I}_n-\bar{\boldsymbol{A}}(G))\boldsymbol{J}(\beta,\gamma,G)\Big) &\leq \|\gamma\bar{\boldsymbol{A}}(G)^\mathsf{T}(\boldsymbol{I}_n-\bar{\boldsymbol{A}}(G))\boldsymbol{J}(\beta,\gamma,G)\|_\infty\\ &\leq -\gamma\|\bar{\boldsymbol{A}}(G)\|_1\|\boldsymbol{I}_n-\bar{\boldsymbol{A}}(G)\|_\infty\|\boldsymbol{J}(\beta,\gamma,G)\|_\infty\\ &\leq -\frac{2\gamma N(G)}{\beta+2\gamma}.\end{aligned}$$

The first inequality is according to Lemma B.7 because $\|\cdot\|_\infty$ is sub-multiplicative. The second inequality is obvious. The third inequality follows from (D.98), (D.130), and (D.131). We have

$$-\frac{2\gamma N(G)}{\beta+2\gamma}<1 \quad\Leftrightarrow\quad -\frac{\beta}{2+2N(G)}<\gamma.$$

Second, consider the case $\gamma \geq 0$. We find

$$\rho\Big(\gamma\bar{\boldsymbol{A}}(G)^\mathsf{T}(\boldsymbol{I}_n-\bar{\boldsymbol{A}}(G))\boldsymbol{J}(\beta,\gamma,G)\Big) \leq \frac{2\gamma N(G)}{\beta}.$$

We have

$$\frac{2\gamma N(G)}{\beta}<1 \quad\Leftrightarrow\quad \gamma<\frac{\beta}{2N(G)}.$$

This concludes the proof that (D.129) is sufficient for (1.47). ∎

Proof of Proposition 1.52

Let $\tau \in \operatorname{int}(n\mathcal{Y})$ be a target for aggregate equilibrium action. Consider the generic NALA game $\Gamma_\tau := (\mathcal{I}, G, \mathcal{Y}, \{(\beta f(\tau/n),\beta,0)\}_{i\in\mathcal{I}}, f)$, where $\beta > 0$.

The generic NALA game Γ_τ has a unique and interior NE $\boldsymbol{y}^\star$, which is given by $\boldsymbol{y}^\star = (\tau/n)\mathbf{1}_n$. Consequently, $\langle \mathbf{1}_n, \boldsymbol{y}^\star\rangle = \tau$. We conclude that τ is attainable by $\{(\beta f(\tau/n),\beta,0)\}_{i\in\mathcal{I}}$. It follows that $\operatorname{int}(n\mathcal{Y}) \subset \mathcal{T}^\star(\mathcal{Y})$. Moreover, $\mathcal{T}^\star(\mathcal{Y}) \subset \operatorname{int}(n\mathcal{Y})$ according to the definition of $\mathcal{T}^\star(\mathcal{Y})$. ∎

Proof of Proposition 1.53

Let $\Gamma := (\mathcal{I}, G, \mathcal{Y}, \{(\alpha_i, \beta, \gamma)\}_{i\in\mathcal{I}}, f)$ be a generic NALA game, and let $\boldsymbol{\alpha}$ be defined as in Section 1.3.2. Suppose $\beta > 0$ and $\gamma > -\beta/2$, so that $\beta + \gamma > 0$ and $|\gamma/(\beta+\gamma)| < 1$. In addition, suppose Γ has a unique and interior NE $\boldsymbol{y}^\star(\boldsymbol{\alpha}, \beta, \gamma, f, G) := (y_1^\star(\boldsymbol{\alpha}, \beta, \gamma, f, G), \ldots, y_n^\star(\boldsymbol{\alpha}, \beta, \gamma, f, G))$, which is given by (1.7). Hereinafter, some or all of the arguments of $\boldsymbol{y}^\star(\boldsymbol{\alpha}, \beta, \gamma, f, G)$ and of its components may be omitted.

First, note that, for all $i \in \mathcal{I}$, $y_i^\star = y_i^\star(\boldsymbol{\alpha}, \beta, \gamma, f, G) = f^{-1}(y_i^\star(\boldsymbol{\alpha}, \beta, \gamma, \mathrm{id}_{\mathcal{Y}}, G))$. Consequently,

$$\langle \mathbf{1}_n, \boldsymbol{y}^\star \rangle = \sum_{i=1}^n y_i^\star = \sum_{i=1}^n f^{-1}\big(y_i^\star(\boldsymbol{\alpha}, \beta, \gamma, \mathrm{id}_{\mathcal{Y}}, G)\big).$$

Second, note that, according to Lemma 1.1,

$$\forall i \in \mathcal{I} \quad \partial f^{-1}\big(f(y_i^\star)\big) = \frac{1}{\partial f\big(f^{-1}\big(f(y_i^\star)\big)\big)} = \frac{1}{\partial f(y_i^\star)} > 0.$$

Let

$$\boldsymbol{w}(f, \boldsymbol{y}^\star) := \left(\frac{1}{\partial f(y_1^\star)}, \ldots, \frac{1}{\partial f(y_n^\star)}\right) \in \mathbb{R}^n_{++}. \tag{D.135}$$

First, I prove (1.48). Let $j \in \mathcal{I}$. We find

$$\begin{aligned}
\frac{\partial \langle \mathbf{1}_n, \boldsymbol{y}^\star \rangle}{\partial \alpha_j} &= \sum_{i=1}^n \frac{\partial f^{-1}\big(y_i^\star(\boldsymbol{\alpha}, \beta, \gamma, \mathrm{id}_{\mathcal{Y}}, G)\big)}{\partial \alpha_j} \\
&= \sum_{i=1}^n \partial f^{-1}\big(y_i^\star(\boldsymbol{\alpha}, \beta, \gamma, \mathrm{id}_{\mathcal{Y}}, G)\big) \frac{\partial y_i^\star(\boldsymbol{\alpha}, \beta, \gamma, \mathrm{id}_{\mathcal{Y}}, G)}{\partial \alpha_j} \\
&= \sum_{i=1}^n \partial f^{-1}\big(f(y_i^\star)\big) \left[\frac{\partial \boldsymbol{y}^\star(\boldsymbol{\alpha}, \beta, \gamma, \mathrm{id}_{\mathcal{Y}}, G)}{\partial \boldsymbol{\alpha}}\right]_{i,j} \\
&= \sum_{i=1}^n [\boldsymbol{w}(f, \boldsymbol{y}^\star)]_i [\boldsymbol{J}(\beta, \gamma, G)]_{i,j} \\
&= \sum_{i=1}^n \big[\boldsymbol{J}(\beta, \gamma, G)^\top\big]_{j,i} [\boldsymbol{w}(f, \boldsymbol{y}^\star)]_i \\
&= \big[\boldsymbol{J}(\beta, \gamma, G)^\top \boldsymbol{w}(f, \boldsymbol{y}^\star)\big]_j,
\end{aligned}$$

where the fourth equality is according to (1.25) and (D.135).

Second, I prove (1.49). We find

$$\begin{aligned}
\frac{\partial \langle \mathbf{1}_n, \boldsymbol{y}^\star \rangle}{\partial \beta} &= \sum_{i=1}^n \partial f^{-1}\big(f(y_i^\star)\big) \left[\frac{\partial \boldsymbol{y}^\star(\boldsymbol{\alpha}, \beta, \gamma, \mathrm{id}_{\mathcal{Y}}, G)}{\partial \beta}\right]_i \\
&= \sum_{i=1}^n [\boldsymbol{w}(f, \boldsymbol{y}^\star)]_i [-\boldsymbol{J}(\beta, \gamma, G) \boldsymbol{f}(\boldsymbol{y}^\star)]_i
\end{aligned}$$

$$= -\langle \boldsymbol{w}(f, \boldsymbol{y}^\star), \boldsymbol{J}(\beta, \gamma, G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle,$$

where the second equality is according to (1.26) and (D.135).

Third, I prove (1.50). We find

$$\begin{aligned}\frac{\partial\langle \mathbf{1}_n, \boldsymbol{y}^\star\rangle}{\partial\gamma} &= \sum_{i=1}^{n} \partial f^{-1}(f(y_i^\star))\left[\frac{\partial \boldsymbol{y}^\star(\boldsymbol{\alpha}, \beta, \gamma, \mathrm{id}_{\mathcal{Y}}, G)}{\partial\gamma}\right]_i \\ &= \sum_{i=1}^{n}[\boldsymbol{w}(f, \boldsymbol{y}^\star)]_i\big[\boldsymbol{J}(\beta, \gamma, G)\big(\bar{\boldsymbol{A}}(G) - \boldsymbol{I}_n\big)\boldsymbol{f}(\boldsymbol{y}^\star)\big]_i \\ &= \big\langle \boldsymbol{w}(f, \boldsymbol{y}^\star), \boldsymbol{J}(\beta, \gamma, G)\big(\bar{\boldsymbol{A}}(G) - \boldsymbol{I}_n\big)\boldsymbol{f}(\boldsymbol{y}^\star)\big\rangle,\end{aligned}$$

where the second equality is according to (1.27) and (D.135).

Results 1.53.1, 1.53.2, and 1.53.3 are straightforward to show giving consideration to $\boldsymbol{J}(\beta, \gamma, G) = (1/\beta)\boldsymbol{I}_n$ if $\gamma = 0$ (Result 1.23.2) $1/(\beta+\gamma)\boldsymbol{I}_n \leq_c \boldsymbol{J}(\beta, \gamma, G)$ if $\gamma > 0$ (Result 1.23.3). ■

Proof of Proposition 1.55

Let $\Gamma := (\mathcal{I}, G, \mathcal{Y}, \{(\alpha_i, \beta, \gamma)\}_{i\in\mathcal{I}}, f)$ be a generic NALA game, and let $\boldsymbol{\alpha}$ be defined as in Section 1.3.2. Suppose $\beta > 0$ and $\gamma > -\beta/2$, so that $\beta + \gamma > 0$ and $|\gamma/(\beta+\gamma)| < 1$. In addition, suppose Γ has a unique and interior NE $\boldsymbol{y}^\star$, which is given by (1.7).

I begin with establishing four auxiliary results. First, note that (1.7) is equivalent to

$$(\beta+\gamma)\boldsymbol{f}(\boldsymbol{y}^\star) = \boldsymbol{\alpha} + \gamma\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star). \tag{D.136}$$

Second, note that $\bar{\boldsymbol{A}}(G)$ and $\boldsymbol{J}(\beta, \gamma, G)$ commute, that is,

$$\bar{\boldsymbol{A}}(G)\boldsymbol{J}(\beta, \gamma, G) = \boldsymbol{J}(\beta, \gamma, G)\bar{\boldsymbol{A}}(G). \tag{D.137}$$

Indeed, we have

$$\begin{aligned}\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1} &= \boldsymbol{I}_n + \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1} \\ &= \boldsymbol{I}_n + \frac{\gamma}{\beta+\gamma}\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1}\bar{\boldsymbol{A}}(G),\end{aligned}$$

which implies that

$$(\beta+\gamma)\boldsymbol{J}(\beta, \gamma, G) = \boldsymbol{I}_n + \gamma\bar{\boldsymbol{A}}(G)\boldsymbol{J}(\beta, \gamma, G) = \boldsymbol{I}_n + \gamma\boldsymbol{J}(\beta, \gamma, G)\bar{\boldsymbol{A}}(G), \tag{D.138}$$

from which (D.137) follows. Third, note that

$$\frac{\partial \boldsymbol{J}(\beta, \gamma, G)}{\partial\beta} = -\boldsymbol{J}(\beta, \gamma, G)^2. \tag{D.139}$$

Indeed, we have

$$\begin{aligned}
\frac{\partial \boldsymbol{J}(\beta,\gamma,G)}{\partial \beta} &= \frac{\partial}{\partial \beta}\left(\frac{1}{\beta+\gamma}\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1}\right) \\
&= -\frac{1}{(\beta+\gamma)^2}\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1} - \frac{1}{\beta+\gamma}\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1} \\
&\quad \times \left(\frac{\partial}{\partial \beta}\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)\right)\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1} \\
&= -\frac{1}{\beta+\gamma}\boldsymbol{J}(\beta,\gamma,G) - \frac{\gamma}{\beta+\gamma}\boldsymbol{J}(\beta,\gamma,G)\bar{\boldsymbol{A}}(G)\boldsymbol{J}(\beta,\gamma,G) \\
&= -\frac{1}{\beta+\gamma}\left(\boldsymbol{I}_n + \gamma \boldsymbol{J}(\beta,\gamma,G)\bar{\boldsymbol{A}}(G)\right)\boldsymbol{J}(\beta,\gamma,G) \\
&= -\frac{1}{\beta+\gamma}(\beta+\gamma)\boldsymbol{J}(\beta,\gamma,G)\boldsymbol{J}(\beta,\gamma,G) \\
&= -\boldsymbol{J}(\beta,\gamma,G)^2,
\end{aligned}$$

where the fifth equality is according to (D.138). Fourth, note that

$$\frac{\partial \boldsymbol{J}(\beta,\gamma,G)}{\partial \gamma} = -\left(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\right)\boldsymbol{J}(\beta,\gamma,G)^2. \tag{D.140}$$

Indeed, we have

$$\begin{aligned}
\frac{\partial \boldsymbol{J}(\beta,\gamma,G)}{\partial \gamma} &= \frac{\partial}{\partial \gamma}\left(\frac{1}{\beta+\gamma}\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1}\right) \\
&= -\frac{1}{(\beta+\gamma)^2}\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1} - \frac{1}{\beta+\gamma}\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1} \\
&\quad \times \left(\frac{\partial}{\partial \gamma}\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)\right)\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\bar{\boldsymbol{A}}(G)\right)^{-1} \\
&= -\frac{1}{\beta+\gamma}\boldsymbol{J}(\beta,\gamma,G) + \frac{\beta}{\beta+\gamma}\boldsymbol{J}(\beta,\gamma,G)\bar{\boldsymbol{A}}(G)\boldsymbol{J}(\beta,\gamma,G) \\
&= -\frac{1}{\beta+\gamma}\left(\boldsymbol{I}_n - \beta \boldsymbol{J}(\beta,\gamma,G)\bar{\boldsymbol{A}}(G)\right)\boldsymbol{J}(\beta,\gamma,G) \\
&= -\frac{1}{\beta+\gamma}\left((\beta+\gamma)\boldsymbol{J}(\beta,\gamma,G) - (\beta+\gamma)\boldsymbol{J}(\beta,\gamma,G)\bar{\boldsymbol{A}}(G)\right)\boldsymbol{J}(\beta,\gamma,G) \\
&= -\left(\boldsymbol{J}(\beta,\gamma,G) - \bar{\boldsymbol{A}}(G)\boldsymbol{J}(\beta,\gamma,G)\right)\boldsymbol{J}(\beta,\gamma,G) \\
&= -\left(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\right)\boldsymbol{J}(\beta,\gamma,G)^2,
\end{aligned}$$

where the fifth equality is according to (D.138) and the sixth equality is according to (D.137).

Next, I prove (1.51). We have

$$w(\boldsymbol{y}^\star) = \langle \boldsymbol{\alpha}, \boldsymbol{f}(\boldsymbol{y}^\star)\rangle - \frac{\beta}{2}\langle \boldsymbol{f}(\boldsymbol{y}^\star), \boldsymbol{f}(\boldsymbol{y}^\star)\rangle$$

$$
\begin{aligned}
&\quad - \frac{\gamma}{2}\langle \boldsymbol{f}(\boldsymbol{y}^\star) - \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \boldsymbol{f}(\boldsymbol{y}^\star) - \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle \\
&= \frac{1}{2}\langle \boldsymbol{f}(\boldsymbol{y}^\star), \boldsymbol{\alpha}\rangle + \frac{1}{2}\langle \boldsymbol{f}(\boldsymbol{y}^\star), \gamma\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle \\
&\quad + \frac{1}{2}\langle \boldsymbol{\alpha}, \boldsymbol{f}(\boldsymbol{y}^\star)\rangle + \frac{1}{2}\langle \gamma\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \boldsymbol{f}(\boldsymbol{y}^\star)\rangle \\
&\quad - \frac{\beta+\gamma}{2}\langle \boldsymbol{f}(\boldsymbol{y}^\star), \boldsymbol{f}(\boldsymbol{y}^\star)\rangle - \frac{\gamma}{2}\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle \\
&= \frac{1}{2}\langle \boldsymbol{f}(\boldsymbol{y}^\star), \boldsymbol{\alpha} + \gamma\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle + \frac{1}{2}\langle \boldsymbol{\alpha} + \gamma\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \boldsymbol{f}(\boldsymbol{y}^\star)\rangle \\
&\quad - \frac{\beta+\gamma}{2}\langle \boldsymbol{f}(\boldsymbol{y}^\star), \boldsymbol{f}(\boldsymbol{y}^\star)\rangle - \frac{\gamma}{2}\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle \\
&= \frac{\beta+\gamma}{2}\langle \boldsymbol{f}(\boldsymbol{y}^\star), \boldsymbol{f}(\boldsymbol{y}^\star)\rangle - \frac{\gamma}{2}\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle \qquad \text{(D.141)} \\
&= \frac{\beta+\gamma}{2}\|\boldsymbol{f}(\boldsymbol{y}^\star)\|_2^2 - \frac{\gamma}{2}\|\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\|_2^2.
\end{aligned}
$$

The first equality is according to (1.4) and the fourth equality follows from (D.136).

In order to prove (1.52), (1.53), and (1.54), I compute the partial derivatives of $\langle \boldsymbol{f}(\boldsymbol{y}^\star), \boldsymbol{f}(\boldsymbol{y}^\star)\rangle$ and $\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle$ with respect to $\boldsymbol{\alpha}$, β, and γ. The partial derivatives of $\langle \boldsymbol{f}(\boldsymbol{y}^\star), \boldsymbol{f}(\boldsymbol{y}^\star)\rangle$ and $\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle$ with respect to $\boldsymbol{\alpha}$ are given by

$$
\frac{\partial\langle \boldsymbol{f}(\boldsymbol{y}^\star), \boldsymbol{f}(\boldsymbol{y}^\star)\rangle}{\partial\boldsymbol{\alpha}} = 2\boldsymbol{\alpha}^\top \boldsymbol{J}(\beta,\gamma,G)^\top \boldsymbol{J}(\beta,\gamma,G) \qquad \text{(D.142)}
$$

and

$$
\frac{\partial\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle}{\partial\boldsymbol{\alpha}} = 2\boldsymbol{\alpha}^\top \boldsymbol{J}(\beta,\gamma,G)^\top \bar{\boldsymbol{A}}(G)^\top \bar{\boldsymbol{A}}(G)\boldsymbol{J}(\beta,\gamma,G). \qquad \text{(D.143)}
$$

The partial derivative of $\langle \boldsymbol{f}(\boldsymbol{y}^\star), \boldsymbol{f}(\boldsymbol{y}^\star)\rangle$ with respect to β is given by

$$
\begin{aligned}
\frac{\partial\langle \boldsymbol{f}(\boldsymbol{y}^\star), \boldsymbol{f}(\boldsymbol{y}^\star)\rangle}{\partial\beta} &= \left\langle \frac{\partial \boldsymbol{f}(\boldsymbol{y}^\star)}{\partial\beta}, \boldsymbol{f}(\boldsymbol{y}^\star)\right\rangle + \left\langle \boldsymbol{f}(\boldsymbol{y}^\star), \frac{\partial \boldsymbol{f}(\boldsymbol{y}^\star)}{\partial\beta}\right\rangle \\
&= 2\left\langle \boldsymbol{f}(\boldsymbol{y}^\star), \frac{\partial \boldsymbol{f}(\boldsymbol{y}^\star)}{\partial\beta}\right\rangle \\
&= 2\left\langle \boldsymbol{f}(\boldsymbol{y}^\star), \frac{\partial \boldsymbol{J}(\beta,\gamma,G)}{\partial\beta}\boldsymbol{\alpha}\right\rangle \\
&= 2\left\langle \boldsymbol{f}(\boldsymbol{y}^\star), -\boldsymbol{J}(\beta,\gamma,G)^2\boldsymbol{\alpha}\right\rangle \\
&= -2\langle \boldsymbol{f}(\boldsymbol{y}^\star), \boldsymbol{J}(\beta,\gamma,G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle, \qquad \text{(D.144)}
\end{aligned}
$$

where the fourth equality is according to (D.139). The partial derivative with respect to β of $\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle$ is given by

$$
\begin{aligned}
&\frac{\partial\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle}{\partial\beta} \\
&= \left\langle \frac{\partial \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)}{\partial\beta}, \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\right\rangle + \left\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \frac{\partial \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)}{\partial\beta}\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= 2\left\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \bar{\boldsymbol{A}}(G)\frac{\partial \boldsymbol{J}(\beta,\gamma,G)}{\partial \beta}\boldsymbol{\alpha}\right\rangle \\
&= -2\left\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \bar{\boldsymbol{A}}(G)\boldsymbol{J}(\beta,\gamma,G)^2\boldsymbol{\alpha}\right\rangle \\
&= -2\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \bar{\boldsymbol{A}}(G)\boldsymbol{J}(\beta,\gamma,G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle \\
&= -2\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \boldsymbol{J}(\beta,\gamma,G)\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle, \qquad \text{(D.145)}
\end{aligned}
$$

where the third equality is according to (D.139) and the fifth equality is according to (D.137). The partial derivative of $\langle \boldsymbol{f}(\boldsymbol{y}^\star), \boldsymbol{f}(\boldsymbol{y}^\star)\rangle$ with respect to γ is given by

$$
\begin{aligned}
\frac{\partial\langle \boldsymbol{f}(\boldsymbol{y}^\star), \boldsymbol{f}(\boldsymbol{y}^\star)\rangle}{\partial\gamma} &= 2\left\langle \boldsymbol{f}(\boldsymbol{y}^\star), \frac{\partial \boldsymbol{J}(\beta,\gamma,G)}{\partial\gamma}\boldsymbol{\alpha}\right\rangle \\
&= 2\left\langle \boldsymbol{f}(\boldsymbol{y}^\star), -\big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)\boldsymbol{J}(\beta,\gamma,G)^2\boldsymbol{\alpha}\right\rangle \\
&= -2\big\langle \boldsymbol{f}(\boldsymbol{y}^\star), \big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)\boldsymbol{J}(\beta,\gamma,G)\boldsymbol{f}(\boldsymbol{y}^\star)\big\rangle, \qquad \text{(D.146)}
\end{aligned}
$$

where the second equality is according to (D.140). The partial derivative with respect to γ of $\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle$ is given by

$$
\begin{aligned}
&\frac{\partial\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle}{\partial\gamma} \\
&= 2\left\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \bar{\boldsymbol{A}}(G)\frac{\partial \boldsymbol{J}(\beta,\gamma,G)}{\partial\gamma}\boldsymbol{\alpha}\right\rangle \\
&= -2\left\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \bar{\boldsymbol{A}}(G)\big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)\boldsymbol{J}(\beta,\gamma,G)^2\boldsymbol{\alpha}\right\rangle \\
&= -2\big\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \bar{\boldsymbol{A}}(G)\big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)\boldsymbol{J}(\beta,\gamma,G)\boldsymbol{f}(\boldsymbol{y}^\star)\big\rangle \\
&= -2\big\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)\boldsymbol{J}(\beta,\gamma,G)\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\big\rangle, \qquad \text{(D.147)}
\end{aligned}
$$

where the second equality is according to (D.140) and the fourth equality is according to (D.137). Next, I prove (1.52), (1.53), and (1.54). First, we find

$$
\begin{aligned}
\frac{\partial w(\boldsymbol{y}^\star)}{\partial\boldsymbol{\alpha}} &= \frac{\beta+\gamma}{2}\frac{\partial\langle \boldsymbol{f}(\boldsymbol{y}^\star), \boldsymbol{f}(\boldsymbol{y}^\star)\rangle}{\partial\boldsymbol{\alpha}} - \frac{\gamma}{2}\frac{\partial\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle}{\partial\boldsymbol{\alpha}} \\
&= (\beta+\gamma)\boldsymbol{\alpha}^\top\boldsymbol{J}(\beta,\gamma,G)^\top\boldsymbol{J}(\beta,\gamma,G) \\
&\quad - \gamma\boldsymbol{\alpha}^\top\boldsymbol{J}(\beta,\gamma,G)^\top\bar{\boldsymbol{A}}(G)^\top\bar{\boldsymbol{A}}(G)\boldsymbol{J}(\beta,\gamma,G) \\
&= \boldsymbol{\alpha}^\top\boldsymbol{J}(\beta,\gamma,G)^\top\big((\beta+\gamma)\boldsymbol{I}_n - \gamma\bar{\boldsymbol{A}}(G)^\top\bar{\boldsymbol{A}}(G)\big)\boldsymbol{J}(\beta,\gamma,G),
\end{aligned}
$$

where the second equality follows from (D.142) and (D.143). Second, we find

$$
\begin{aligned}
\frac{\partial w(\boldsymbol{y}^\star)}{\partial\beta} &= \frac{1}{2}\langle \boldsymbol{f}(\boldsymbol{y}^\star), \boldsymbol{f}(\boldsymbol{y}^\star)\rangle + \frac{\beta+\gamma}{2}\frac{\partial\langle \boldsymbol{f}(\boldsymbol{y}^\star), \boldsymbol{f}(\boldsymbol{y}^\star)\rangle}{\partial\beta} \\
&\quad - \frac{\gamma}{2}\frac{\partial\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle}{\partial\beta} \\
&= \frac{1}{2}\langle \boldsymbol{f}(\boldsymbol{y}^\star), \boldsymbol{f}(\boldsymbol{y}^\star)\rangle - \langle \boldsymbol{f}(\boldsymbol{y}^\star), (\beta+\gamma)\boldsymbol{J}(\beta,\gamma,G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+\gamma\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star),\boldsymbol{J}(\beta,\gamma,G)\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle\\
&=\frac{1}{2}\langle \boldsymbol{f}(\boldsymbol{y}^\star),\boldsymbol{f}(\boldsymbol{y}^\star)\rangle-\big\langle \boldsymbol{f}(\boldsymbol{y}^\star),\big(\boldsymbol{I}_n+\gamma\boldsymbol{J}(\beta,\gamma,G)\bar{\boldsymbol{A}}(G)\big)\boldsymbol{f}(\boldsymbol{y}^\star)\big\rangle\\
&\quad+\gamma\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star),\boldsymbol{J}(\beta,\gamma,G)\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle\\
&=-\frac{1}{2}\langle \boldsymbol{f}(\boldsymbol{y}^\star),\boldsymbol{f}(\boldsymbol{y}^\star)\rangle-\gamma\langle \boldsymbol{f}(\boldsymbol{y}^\star)-\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star),\boldsymbol{J}(\beta,\gamma,G)\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle\\
&=-\frac{1}{2}\|\boldsymbol{f}(\boldsymbol{y}^\star)\|_2^2-\gamma\big\langle\big(\boldsymbol{I}_n-\bar{\boldsymbol{A}}(G)\big)\boldsymbol{f}(\boldsymbol{y}^\star),\boldsymbol{J}(\beta,\gamma,G)\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\big\rangle,
\end{aligned}
$$

where the second equality follows from (D.144) and (D.145) and the third equality follows from (D.138). Third, we find

$$
\begin{aligned}
\frac{\partial w(\boldsymbol{y}^\star)}{\partial\gamma}&=\frac{1}{2}\langle \boldsymbol{f}(\boldsymbol{y}^\star),\boldsymbol{f}(\boldsymbol{y}^\star)\rangle-\frac{1}{2}\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star),\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle\\
&\quad+\frac{\beta+\gamma}{2}\frac{\partial\langle \boldsymbol{f}(\boldsymbol{y}^\star),\boldsymbol{f}(\boldsymbol{y}^\star)\rangle}{\partial\gamma}-\frac{\gamma}{2}\frac{\partial\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star),\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle}{\partial\gamma}\\
&=\frac{1}{2}\langle \boldsymbol{f}(\boldsymbol{y}^\star),\boldsymbol{f}(\boldsymbol{y}^\star)\rangle-\frac{1}{2}\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star),\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle\\
&\quad-(\beta+\gamma)\big\langle \boldsymbol{f}(\boldsymbol{y}^\star),\big(\boldsymbol{I}_n-\bar{\boldsymbol{A}}(G)\big)\boldsymbol{J}(\beta,\gamma,G)\boldsymbol{f}(\boldsymbol{y}^\star)\big\rangle\\
&\quad+\gamma\big\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star),\big(\boldsymbol{I}_n-\bar{\boldsymbol{A}}(G)\big)\boldsymbol{J}(\beta,\gamma,G)\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\big\rangle\\
&=\frac{1}{2}\langle \boldsymbol{f}(\boldsymbol{y}^\star),\boldsymbol{f}(\boldsymbol{y}^\star)\rangle-\frac{1}{2}\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star),\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle\\
&\quad-\langle \boldsymbol{f}(\boldsymbol{y}^\star),(\beta+\gamma)\boldsymbol{J}(\beta,\gamma,G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle\\
&\quad+(\beta+\gamma)\langle \boldsymbol{f}(\boldsymbol{y}^\star),\bar{\boldsymbol{A}}(G)\boldsymbol{J}(\beta,\gamma,G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle\\
&\quad+\gamma\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star),\boldsymbol{J}(\beta,\gamma,G)\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle\\
&\quad-\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star),\gamma\bar{\boldsymbol{A}}(G)\boldsymbol{J}(\beta,\gamma,G)\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle\\
&=\frac{1}{2}\langle \boldsymbol{f}(\boldsymbol{y}^\star),\boldsymbol{f}(\boldsymbol{y}^\star)\rangle-\frac{1}{2}\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star),\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle\\
&\quad-\big\langle \boldsymbol{f}(\boldsymbol{y}^\star),\big(\boldsymbol{I}_n+\gamma\boldsymbol{J}(\beta,\gamma,G)\bar{\boldsymbol{A}}(G)\big)\boldsymbol{f}(\boldsymbol{y}^\star\big\rangle\\
&\quad+(\beta+\gamma)\langle \boldsymbol{f}(\boldsymbol{y}^\star),\boldsymbol{J}(\beta,\gamma,G)\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle\\
&\quad+\gamma\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star),\boldsymbol{J}(\beta,\gamma,G)\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle\\
&\quad-\big\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star),\big((\beta+\gamma)\boldsymbol{J}(\beta,\gamma,G)-\boldsymbol{I}_n\big)\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\big\rangle\\
&=-\frac{1}{2}\langle \boldsymbol{f}(\boldsymbol{y}^\star),\boldsymbol{f}(\boldsymbol{y}^\star)\rangle+\frac{1}{2}\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star),\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle\\
&\quad+\beta\langle \boldsymbol{f}(\boldsymbol{y}^\star),\boldsymbol{J}(\beta,\gamma,G)\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle\\
&\quad-\beta\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star),\boldsymbol{J}(\beta,\gamma,G)\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle \qquad\text{(D.148)}\\
&=-\frac{1}{2}\|\boldsymbol{f}(\boldsymbol{y}^\star)\|_2^2+\frac{1}{2}\|\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\|_2^2\\
&\quad+\beta\big\langle\big(\boldsymbol{I}_n-\bar{\boldsymbol{A}}(G)\big)\boldsymbol{f}(\boldsymbol{y}^\star),\boldsymbol{J}(\beta,\gamma,G)\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\big\rangle,
\end{aligned}
$$

where the second equality follows from (D.146) and (D.147) and the forth equality follows from (D.137) and (D.138).

Finally, I prove Results 1.55.1 to 1.55.3. First, I prove Results 1.55.1 and 1.55.2. If $\gamma = 0$, then $\boldsymbol{J}(\beta, \gamma, G) = (1/\beta)\boldsymbol{I}_n$ (Result 1.23.2) and $\boldsymbol{y}^\star = (1/\beta)\boldsymbol{\alpha}$ (Result 1.31.2 and Remark 1.36), so that $\partial w(\boldsymbol{y}^\star)/\partial \boldsymbol{\alpha} = (1/\beta)\boldsymbol{\alpha}^\top$ according to (1.52) and $\partial w(\boldsymbol{y}^\star)/\partial \beta = -1/(2\beta^2)\|\boldsymbol{\alpha}\|_2^2$ according to (1.53). If $\boldsymbol{\alpha} \neq \boldsymbol{0}_n$, $\gamma > 0$, and $\boldsymbol{0}_n \leq_c \boldsymbol{f}(\boldsymbol{y}^\star) \leq_c (1/\beta)\boldsymbol{\alpha}$, then $\partial w(\boldsymbol{y}^\star)/\partial \beta < 0$ according to (1.53) because $\boldsymbol{\alpha} \neq \boldsymbol{0}_n$ is equivalent to $\boldsymbol{f}(\boldsymbol{y}^\star) \neq \boldsymbol{0}_n$ and $\boldsymbol{f}(\boldsymbol{y}^\star) \leq_c (1/\beta)\boldsymbol{\alpha}$ is equivalent to $\bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star) \leq_c \boldsymbol{f}(\boldsymbol{y}^\star)$ if $\gamma > 0$ (Result 1.33.2 and Remark 1.36). Second, I prove Result 1.55.3. We have

$$\begin{aligned}
\left.\frac{\partial w(\boldsymbol{y}^\star)}{\partial \gamma}\right|_{\gamma=0} &= -\frac{1}{2}\langle \boldsymbol{f}(\boldsymbol{y}^\star), \boldsymbol{f}(\boldsymbol{y}^\star)\rangle + \frac{1}{2}\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle \\
&\quad + \langle \boldsymbol{f}(\boldsymbol{y}^\star), \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle - \langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle \\
&= -\frac{1}{2}\langle \boldsymbol{f}(\boldsymbol{y}^\star), \boldsymbol{f}(\boldsymbol{y}^\star)\rangle + \frac{1}{2}\langle \boldsymbol{f}(\boldsymbol{y}^\star), \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle \\
&\quad + \frac{1}{2}\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \boldsymbol{f}(\boldsymbol{y}^\star)\rangle - \frac{1}{2}\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle \\
&= -\frac{1}{2}\langle \boldsymbol{f}(\boldsymbol{y}^\star) - \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star), \boldsymbol{f}(\boldsymbol{y}^\star) - \bar{\boldsymbol{A}}(G)\boldsymbol{f}(\boldsymbol{y}^\star)\rangle \\
&= -\frac{1}{2}\big\|\big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)\boldsymbol{f}(\boldsymbol{y}^\star)\big\|_2^2 \\
&= -\frac{1}{2\beta^2}\|\bar{\boldsymbol{A}}(G)\boldsymbol{\alpha} - \boldsymbol{\alpha}\|_2^2,
\end{aligned}$$

where the first equality follows from (D.148) because $\boldsymbol{J}(\beta, \gamma, G) = (1/\beta)\boldsymbol{I}_n$ if $\gamma = 0$ (Result 1.23.2) and the last equality follows from the fact that $\boldsymbol{y}^\star = (1/\beta)\boldsymbol{\alpha}$ if $\gamma = 0$ (Result 1.31.2 and Remark 1.36). ■

Example 1.56: Proofs of (1.55) and (1.56)

Consider the setup of Example 1.56. If $\gamma = 0$, then $\partial w(\boldsymbol{y}^\star)/\partial \boldsymbol{\alpha} = (1/\beta)\boldsymbol{\alpha}^\top$ (Result 1.55.1), which is equivalent to (1.55). Suppose $\gamma \neq 0$ in what follows. We have

$$\bar{\boldsymbol{A}}(G) = \boldsymbol{1}_n \boldsymbol{e}_1^\top. \tag{D.149}$$

The inverse of the matrix $\boldsymbol{I}_n - (\gamma/(\beta+\gamma))\boldsymbol{1}_n\boldsymbol{e}_1^\top$ is given by Sherman and Morrison's (1949) formula (see, for example, Bartlett 1951, p. 107) because $(\gamma/(\beta+\gamma))\boldsymbol{1}_n\boldsymbol{e}_1^\top$ has rank 1:

$$\left(\boldsymbol{I}_n - \frac{\gamma}{\beta+\gamma}\boldsymbol{1}_n\boldsymbol{e}_1^\top\right)^{-1} = \boldsymbol{I}_n + \frac{\dfrac{\gamma}{\beta+\gamma}\boldsymbol{1}_n\boldsymbol{e}_1^\top}{1 - \dfrac{\gamma}{\beta+\gamma}\langle \boldsymbol{1}_n, \boldsymbol{e}_1\rangle} = \boldsymbol{I}_n + \frac{\gamma}{\beta}\boldsymbol{1}_n\boldsymbol{e}_1^\top. \tag{D.150}$$

Using (1.52), (D.149), and (D.150), straightforward algebra yields

$$\frac{\partial w(\boldsymbol{y}^\star)}{\partial \boldsymbol{\alpha}} = \frac{1}{\beta+\gamma}\boldsymbol{\alpha}^\top + \frac{\gamma}{\beta(\beta+\gamma)}\langle \boldsymbol{\alpha}, \boldsymbol{1}_n - n\boldsymbol{e}_1\rangle \boldsymbol{e}_1^\top + \frac{\gamma}{\beta(\beta+\gamma)}\langle \boldsymbol{\alpha}, \boldsymbol{e}_1\rangle \boldsymbol{1}_n^\top$$

$$= \frac{1}{\beta+\gamma}\boldsymbol{\alpha}^{\mathsf{T}} + \frac{\gamma}{\beta(\beta+\gamma)}\Big(\sum_{i\in\mathcal{I}}\alpha_i - n\alpha_1\Big)\boldsymbol{e}_1^{\mathsf{T}} + \frac{\alpha_1\gamma}{\beta(\beta+\gamma)}\mathbf{1}_n^{\mathsf{T}},$$

which is equivalent to (1.55). If $\alpha_1 = \bar{\alpha} + \Delta\alpha$ and $\alpha_2 = \cdots = \alpha_n = \bar{\alpha}$ for some $(\bar{\alpha}, \Delta\alpha) \in \mathbb{R}_{++} \times \mathbb{R}$ with $\bar{\alpha} + \Delta\alpha > 0$ and $\beta = 1$, then

$$\frac{\partial w(\boldsymbol{y}^\star)}{\partial \alpha_1} = \bar{\alpha} - \frac{(n-2)\gamma - 1}{1+\gamma}\Delta\alpha,$$

from which (1.56) follows, provided that $(n-2)\gamma - 1 > 0$. ∎

Example 1.57: Proofs of (1.57) and (1.58)

Consider the setup of Example 1.57. Using (D.149) and (D.150), we find

$$\boldsymbol{y}^\star = \frac{1}{\beta+\gamma}\boldsymbol{\alpha} + \frac{\alpha_1\gamma}{\beta(\beta+\gamma)}\mathbf{1}_n, \quad \bar{\boldsymbol{A}}(G)\boldsymbol{y}^\star = \frac{\alpha_1}{\beta}\mathbf{1}_n,$$

and

$$\boldsymbol{J}(\beta,\gamma,G)\bar{\boldsymbol{A}}(G)\boldsymbol{y}^\star = \frac{\alpha_1}{\beta^2}\mathbf{1}_n,$$

from which

$$\|\boldsymbol{y}^\star\|_2^2 = \frac{n\alpha_1^2\gamma^2}{\beta^2(\beta+\gamma)^2} + \frac{2\alpha_1\gamma}{\beta(\beta+\gamma)^2}\sum_{i\in\mathcal{I}}\alpha_i + \frac{1}{(\beta+\gamma)^2}\sum_{i\in\mathcal{I}}\alpha_i^2 \tag{D.151}$$

and

$$R(\boldsymbol{\alpha},\beta,\gamma,f,G) = \frac{\alpha_1}{\beta^2(\beta+\gamma)}\Big(\sum_{i\in\mathcal{I}}\alpha_i - n\alpha_1\Big) \tag{D.152}$$

follow. Using (1.53), (D.151), and (D.152), straightforward algebra yields (1.57). If $\alpha_1 = 1 + \Delta\alpha$ for some $\Delta\alpha \in (-1, +\infty)$ and $\alpha_2 = \cdots = \alpha_n = 1$, then

$$\left.\frac{\partial w(\boldsymbol{y}^\star)}{\partial \beta}\right|_{\beta=1} = \frac{(n-2)(2+\gamma)\gamma - 1}{2(1+\gamma)^2}(\Delta\alpha)^2 - \Delta\alpha - \frac{n}{2},$$

from which (1.58) follows. If $n = 10$, $\alpha_1 = 1 + \Delta\alpha$ with $\Delta\alpha \in (-1, +\infty)$, $\alpha_2 = \cdots = \alpha_n = 1$, and $\gamma = 5/6$, then

$$\left.\frac{\partial w(\boldsymbol{y}^\star)}{\partial \beta}\right|_{\beta=1} = \frac{322}{121}(\Delta\alpha)^2 - \Delta\alpha - 5.$$

The zero set of the polynomial $(322/121)(\Delta\alpha)^2 - \Delta\alpha - 5$ is $\{-55/46, 11/7\}$, with $\{-55/46, 11/7\} \cap (-1, +\infty) = \{11/7\}$. ∎

Proof of Proposition 1.58

Let $\Gamma := (\mathcal{I}, G, \mathcal{Y}, \{(\alpha_i, \beta, \gamma)\}_{i \in \mathcal{I}}, f)$ be a generic NALA game, and let $\boldsymbol{\alpha}$ be defined as in Section 1.3.2. Suppose $\beta > 0$ and $\gamma > -\beta/2$ with $\gamma \neq 0$, so that $\beta + \gamma > 0$ and $0 < |\gamma/(\beta+\gamma)| < 1$. In addition, suppose Γ has a unique and interior NE $\boldsymbol{y}^\star(\gamma)$, which is given by (1.7). We find

$$\begin{aligned}
&w\big(\boldsymbol{y}^\star(\gamma)\big) - w\big(\boldsymbol{y}^\star(0)\big) \\
&\quad = \frac{\beta+\gamma}{2}\big\langle \boldsymbol{f}\big(\boldsymbol{y}^\star(\gamma)\big), \boldsymbol{f}\big(\boldsymbol{y}^\star(\gamma)\big)\big\rangle \\
&\qquad - \frac{\gamma}{2}\big\langle \bar{\boldsymbol{A}}(G)\boldsymbol{f}\big(\boldsymbol{y}^\star(\gamma)\big), \bar{\boldsymbol{A}}(G)\boldsymbol{f}\big(\boldsymbol{y}^\star(\gamma)\big)\big\rangle - \frac{1}{2\beta}\langle \boldsymbol{\alpha}, \boldsymbol{\alpha}\rangle \\
&\quad = \frac{\beta+\gamma}{2}\big\langle \boldsymbol{f}\big(\boldsymbol{y}^\star(\gamma)\big), \boldsymbol{f}\big(\boldsymbol{y}^\star(\gamma)\big)\big\rangle \\
&\qquad - \frac{1}{2\gamma}\big\langle (\beta+\gamma)\boldsymbol{f}\big(\boldsymbol{y}^\star(\gamma)\big) - \boldsymbol{\alpha}, (\beta+\gamma)\boldsymbol{f}\big(\boldsymbol{y}^\star(\gamma)\big) - \boldsymbol{\alpha}\big\rangle - \frac{1}{2\beta}\langle \boldsymbol{\alpha}, \boldsymbol{\alpha}\rangle \\
&\quad = -\frac{\beta(\beta+\gamma)}{2\gamma}\big\langle \boldsymbol{f}\big(\boldsymbol{y}^\star(\gamma)\big), \boldsymbol{f}\big(\boldsymbol{y}^\star(\gamma)\big)\big\rangle \\
&\qquad + \frac{\beta+\gamma}{2\gamma}\big\langle \boldsymbol{f}\big(\boldsymbol{y}^\star(\gamma)\big), \boldsymbol{\alpha}\big\rangle + \frac{\beta+\gamma}{2\gamma}\big\langle \boldsymbol{\alpha}, \boldsymbol{f}\big(\boldsymbol{y}^\star(\gamma)\big)\big\rangle - \frac{\beta+\gamma}{2\beta\gamma}\langle \boldsymbol{\alpha}, \boldsymbol{\alpha}\rangle \\
&\quad = -\frac{\beta(\beta+\gamma)}{2\gamma}\left\langle \boldsymbol{f}\big(\boldsymbol{y}^\star(\gamma)\big), \boldsymbol{f}\big(\boldsymbol{y}^\star(\gamma)\big) - \frac{1}{\beta}\boldsymbol{\alpha}\right\rangle + \frac{\beta+\gamma}{2\gamma}\left\langle \boldsymbol{\alpha}, \boldsymbol{f}\big(\boldsymbol{y}^\star(\gamma)\big) - \frac{1}{\beta}\boldsymbol{\alpha}\right\rangle \\
&\quad = -\frac{\beta(\beta+\gamma)}{2\gamma}\left\langle \boldsymbol{f}\big(\boldsymbol{y}^\star(\gamma)\big) - \frac{1}{\beta}\boldsymbol{\alpha}, \boldsymbol{f}\big(\boldsymbol{y}^\star(\gamma)\big) - \frac{1}{\beta}\boldsymbol{\alpha}\right\rangle \\
&\quad = -\frac{\beta(\beta+\gamma)}{2\gamma}\left\|\boldsymbol{f}\big(\boldsymbol{y}^\star(\gamma)\big) - \frac{1}{\beta}\boldsymbol{\alpha}\right\|_2^2.
\end{aligned}$$

The first equality is according to (1.51). The second equality follows from (D.136). The remaining equalities are straightforward to show. ∎

Proof of Corollary 1.59

The statement follows from Proposition 1.58. ∎

Proof of Proposition 1.64

Note that $\gamma > 0$ by assumption. Let $D \in \mathcal{D}$, and let $i \in \mathcal{I}$. If $\deg_D^+(i) = 0$, then

$$u_i^\star(D) = \alpha(\chi, D)(i) f\big(y_i^\star(D)\big) - \frac{\beta}{2} f\big(y_i^\star(D)\big)^2 = \frac{\alpha(\chi, D)(i)^2}{2\beta}$$

because $f(y_i^\star(D)) = \alpha(\chi, D)(i)/\beta$. If $\deg_D^+(i) > 0$, then

$$u_i^\star(D) = \alpha(\chi, D)(i) f\big(y_i^\star(D)\big) - \frac{\beta}{2} f\big(y_i^\star(D)\big)^2$$

$$
\begin{aligned}
&-\frac{\gamma}{2}\Big(f\big(y_i^\star(D)\big)-\sum_{j\in\mathcal{I}}\bar{a}_{i,j}(D)f\big(y_j^\star(D)\big)\Big)^2\\
&=\alpha(\chi,D)(i)f\big(y_i^\star(D)\big)-\frac{\beta}{2}f\big(y_i^\star(D)\big)^2-\frac{1}{2\gamma}\Big(\alpha(\chi,D)(i)-\beta f\big(y_i^\star(D)\big)\Big)^2\\
&=\frac{\alpha(\chi,D)(i)^2}{2\beta}-\frac{\beta+\gamma}{2\beta\gamma}\Big(\alpha(\chi,D)(i)-\beta f\big(y_i^\star(D)\big)\Big)^2,
\end{aligned}
$$

where the second equality follows from the first-order condition (see (D.4)),

$$
f\big(y_i^\star(D)\big)=\frac{1}{\beta+\gamma}\alpha(\chi,D)(i)+\frac{\gamma}{\beta+\gamma}\sum_{j\in\mathcal{I}}\bar{a}_{i,j}(D)f\big(y_j^\star(D)\big).
$$

If the mapping $\alpha(\chi,\cdot)\colon\mathcal{D}\to\mathbb{R}^{\mathcal{I}}$ is constant, then

$$
\alpha(\chi,D)(i)=\alpha(\chi,D_i)(i)=\alpha(\chi,(\mathcal{I},\varnothing))(i),
$$

which implies that

$$
u_i^\star(D_i)-u_i^\star(D)=\frac{\beta+\gamma}{2\beta\gamma}\Big(\alpha(\chi,D)(i)-\beta f\big(y_i^\star(D)\big)\Big)^2\geq 0
$$

because $f(y_i^\star(D_i))=\alpha(\chi,D_i)(i)/\beta$ and $\gamma>0$. ■

Proof of Proposition 1.65

The proof is omitted because it is similar to the proof of Proposition 1.13. ■

Proof of Proposition 1.66

The proof is omitted because it is similar to the proof of Proposition 1.16. ■

Proof of Proposition 1.67

The proof is omitted because it is similar to the proof of Proposition 1.14. ■

Proof of Proposition 1.69

The proof is omitted because it is similar to the proof of Proposition 1.13. ■

Proof of Proposition 1.70

The proof is omitted because it is similar to the proof of Proposition 1.16. ■

Proof of Proposition 1.71

The proof is omitted because it is similar to the proof of Proposition 1.14. ■

Proof of Lemma 1.74

Let $\boldsymbol{H}$ and $\boldsymbol{K}$ be real symmetric matrices of the same order. Suppose $\boldsymbol{H}$ is positive definite and $\boldsymbol{K}$ is positive semidefinite but not positive definite. Since $\boldsymbol{H}$ is symmetric and positive definite, it has a unique positive definite square root $\boldsymbol{H}^{1/2}$, which is symmetric. We have

$$\sigma(\boldsymbol{HK}) = \sigma\big(\boldsymbol{H}^{1/2}\boldsymbol{H}^{1/2}\boldsymbol{K}\big) = \sigma\big(\boldsymbol{H}^{1/2}\boldsymbol{K}\boldsymbol{H}^{1/2}\big) \subset \mathbb{R}.$$

For the second equality see, for example, Horn and Johnson (2012, Theorem 1.3.22). The set inclusion follows from the fact that symmetric matrices have real spectra. Finally, the inequality $\min \sigma(\boldsymbol{HK}) \geq 0$ follows from Johnson (1977, Lemma 1). In particular, the inertia of $\boldsymbol{HK}$ and $\boldsymbol{K}$ are the same because $\boldsymbol{H}$ is positive definite, where the negative index of inertia of $\boldsymbol{K}$, that is, the number of negative eigenvalues of $\boldsymbol{K}$, is zero because $\boldsymbol{K}$ is positive semidefinite, from which it follows that $\boldsymbol{HK}$ has no negative eigenvalues, so that $\min \sigma(\boldsymbol{HK}) \geq 0$. ■

Proofs of Lemmata 1.76, 1.77, and 1.78

In what follows, I write $\bar{\boldsymbol{A}}$ for $\bar{\boldsymbol{A}}(G)$. Consider the setup of Section 1.4.2.2. The matrix $\bar{\boldsymbol{A}}$ is block diagonal with R blocks. For all $r \in \{1, \ldots, R\}$, let $\bar{\boldsymbol{A}}_r$ denote the rth block of $\bar{\boldsymbol{A}}$ (that is, $\bar{\boldsymbol{A}}_r$ is the row-normalized adjacency matrix of $\mathrm{sl}(G_r)$ with respect to the unique order isomorphism $h\colon \mathcal{I}_r \to \{1, \ldots, n_r\}$), which is, by assumption, of order $n_r > 1$. The matrix $q(\bar{\boldsymbol{A}})$ is block diagonal. Let $r \in \{1, \ldots, R\}$. The rth block of $q(\bar{\boldsymbol{A}})$ is equal to $q(\bar{\boldsymbol{A}}_r)$, that is, $\bar{\boldsymbol{A}}_r - \boldsymbol{I}_{n_r}$. The matrix $q(\bar{\boldsymbol{A}}_r)$ has rank at least one. To see this, suppose, for the sake of contradiction, the rank of $q(\bar{\boldsymbol{A}}_r)$ is zero. It follows that $q(\bar{\boldsymbol{A}}_r) = \boldsymbol{O}_{n_r}$ or, equivalently, $\bar{\boldsymbol{A}}_r = \boldsymbol{I}_{n_r}$. Note that $\bar{\boldsymbol{A}}_r = \boldsymbol{I}_{n_r}$ if and only if all players in G_r are isolated. As G_r is a weakly connected component of G and all players in G_r are isolated, we must have $n_r = 1$, which contradicts the assumption that $n_r > 1$. This concludes the proof that the rank of $q(\bar{\boldsymbol{A}}_r)$ is at least one. The matrix $q(\bar{\boldsymbol{A}}_r)$ has rank at most $n_r - 1$ because $q(\bar{\boldsymbol{A}}_r)\boldsymbol{1}_{n_r} = \boldsymbol{0}_{n_r}$.

Proof of Lemma 1.76

Proof of Result 1.76.1 For all $r \in \{1, \ldots, R\}$, $q(\bar{\boldsymbol{A}}_r)$ has rank at most $n_r - 1$. As the rank of a block diagonal matrix is equal to the sum of the ranks of its blocks, $q(\bar{\boldsymbol{A}})$ has rank at most $\sum_{r=1}^{R}(n_r - 1) = n - R$. ■

Proof of Result 1.76.2 This property is obvious. ■

Proof of Result 1.76.3 We have

$$q(\bar{\boldsymbol{A}})\boldsymbol{\iota} = \begin{pmatrix} q(\bar{\boldsymbol{A}}_1)\mathbf{1}_{n_1} & \mathbf{0}_{n_1} & \cdots & \mathbf{0}_{n_1} \\ \mathbf{0}_{n_2} & q(\bar{\boldsymbol{A}}_2)\mathbf{1}_{n_2} & \cdots & \mathbf{0}_{n_2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{n_R} & \mathbf{0}_{n_R} & \cdots & q(\bar{\boldsymbol{A}}_R)\mathbf{1}_{n_R} \end{pmatrix} = \mathbf{0}_n\mathbf{0}_R^\top,$$

from which $q(\bar{\boldsymbol{A}})\boldsymbol{\iota}\boldsymbol{\eta} = \mathbf{0}_n$ follows. ∎

Proof of Result 1.76.4 Premultiplying (respectively, postmultiplying) both sides of the identity

$$\boldsymbol{I}_n = \boldsymbol{I}_n - \gamma_0 q(\bar{\boldsymbol{A}}) + \gamma_0 q(\bar{\boldsymbol{A}})$$

by $(\boldsymbol{I}_n - \gamma_0 q(\bar{\boldsymbol{A}}))^{-1}$ gives $(\boldsymbol{I}_n - \gamma_0 q(\bar{\boldsymbol{A}}))^{-1} = \boldsymbol{I}_n + \gamma_0(\boldsymbol{I}_n - \gamma_0 q(\bar{\boldsymbol{A}}))^{-1}q(\bar{\boldsymbol{A}})$ (respectively, $(\boldsymbol{I}_n - \gamma_0 q(\bar{\boldsymbol{A}}))^{-1} = \boldsymbol{I}_n + \gamma_0 q(\bar{\boldsymbol{A}})(\boldsymbol{I}_n - \gamma_0 q(\bar{\boldsymbol{A}}))^{-1}$. Therefore,

$$\left(\boldsymbol{I}_n - \gamma_0 q(\bar{\boldsymbol{A}})\right)^{-1} - \boldsymbol{I}_n = \gamma_0\left(\boldsymbol{I}_n - \gamma_0 q(\bar{\boldsymbol{A}})\right)^{-1}q(\bar{\boldsymbol{A}}) = \gamma_0 q(\bar{\boldsymbol{A}})\left(\boldsymbol{I}_n - \gamma_0 q(\bar{\boldsymbol{A}})\right)^{-1},$$

from which it follows that $q(\bar{\boldsymbol{A}})$ and $(\boldsymbol{I}_n - \gamma_0 q(\bar{\boldsymbol{A}}))^{-1}$ commute. ∎

Proof of Lemma 1.77

For all $r \in \{1, \ldots, R\}$, let s_r denote the nullity of $q(\bar{\boldsymbol{A}}_r)$, that is, the rank of $q(\bar{\boldsymbol{A}}_r)$ is equal to $n_r - s_r$, where $0 < s_r < n_r$ because $0 < n_r - s_r < n_r$. It follows that the rank of $q(\bar{\boldsymbol{A}})$ is equal to $\sum_{r=1}^{R}(n_r - s_r) = n - S$, where $S := \sum_{r=1}^{R} s_r$. Let $r \in \{1, \ldots, R\}$. The matrix $q(\bar{\boldsymbol{A}}_r)q(\bar{\boldsymbol{A}}_r)^\top$ is symmetric and positive semidefinite but not positive definite with rank $n_r - s_r$. (The notation introduced hereinafter is based on the assumption that $s_r > 1$ and $n_r - s_r > 1$. All results are, however, true if $s_r = 1$ or $n_r - s_r = 1$, with the obvious changes in notation.) Consequently, there exists a spectral decomposition of $q(\bar{\boldsymbol{A}}_r)q(\bar{\boldsymbol{A}}_r)^\top$ that is given by

$$q(\bar{\boldsymbol{A}}_r)q(\bar{\boldsymbol{A}}_r)^\top = (\boldsymbol{U}_r : \boldsymbol{V}_r)\begin{pmatrix} \boldsymbol{\Lambda}_r & \mathbf{0}_{n_r - s_r}\mathbf{0}_{s_r}^\top \\ \mathbf{0}_{s_r}\mathbf{0}_{n_r - s_r}^\top & \boldsymbol{O}_{s_r} \end{pmatrix}(\boldsymbol{U}_r : \boldsymbol{V}_r)^\top = \boldsymbol{U}_r\boldsymbol{\Lambda}_r\boldsymbol{U}_r^\top,$$

where $(\boldsymbol{U}_r : \boldsymbol{V}_r)$ is an orthogonal matrix of order n_r and $\boldsymbol{\Lambda}_r$ is a positive definite diagonal matrix of order $n_r - s_r$. The $n_r \times (n_r - s_r)$ matrix $\boldsymbol{U}_r$ and the $n_r \times s_r$ matrix $\boldsymbol{V}_r$ satisfy $\boldsymbol{U}_r^\top\boldsymbol{U}_r = \boldsymbol{I}_{n_r - s_r}$, $\boldsymbol{V}_r^\top\boldsymbol{V}_r = \boldsymbol{I}_{s_r}$, $\boldsymbol{U}_r^\top\boldsymbol{V}_r = \mathbf{0}_{n_r - s_r}\mathbf{0}_{s_r}^\top$, and $\boldsymbol{U}_r\boldsymbol{U}_r^\top + \boldsymbol{V}_r\boldsymbol{V}_r^\top = \boldsymbol{I}_{n_r}$. Let the $(n_r - s_r) \times n_r$ matrix $\boldsymbol{Q}_r$ be defined by

$$\boldsymbol{Q}_r := \boldsymbol{\Lambda}_r^{-1/2}\boldsymbol{U}_r^\top q(\bar{\boldsymbol{A}}_r),$$

where $\boldsymbol{\Lambda}_r^{-1/2}$ denotes the inverse of the unique principal square root $\boldsymbol{\Lambda}_r^{1/2}$ of $\boldsymbol{\Lambda}_r$. Note that $\boldsymbol{Q}_r\mathbf{1}_{n_r} = \mathbf{0}_{n_r - s_r}$, $\boldsymbol{Q}_r\boldsymbol{Q}_r^\top = \boldsymbol{I}_{n_r - s_r}$, which implies that $\boldsymbol{Q}_r$ has full row rank, $\boldsymbol{Q}_r^\top\boldsymbol{Q}_r = q(\bar{\boldsymbol{A}}_r)^\top\boldsymbol{U}_r\boldsymbol{\Lambda}_r^{-1}\boldsymbol{U}_r^\top q(\bar{\boldsymbol{A}}_r)$, and $q(\bar{\boldsymbol{A}}_r)\boldsymbol{Q}_r^\top\boldsymbol{Q}_r q(\bar{\boldsymbol{A}}_r)^\top = q(\bar{\boldsymbol{A}}_r)q(\bar{\boldsymbol{A}}_r)^\top$.

First, I show that

$$q(\bar{\boldsymbol{A}}_r)\boldsymbol{Q}_r^\top\boldsymbol{Q}_r = q(\bar{\boldsymbol{A}}_r), \tag{D.153}$$

from which it follows that $\boldsymbol{Q}_r\bar{\boldsymbol{A}}_r\boldsymbol{Q}_r^\top\boldsymbol{Q}_r = \boldsymbol{Q}_r\bar{\boldsymbol{A}}_r$. We have

$$q(\bar{\boldsymbol{A}}_r)\boldsymbol{Q}_r^\top\boldsymbol{Q}_r = \boldsymbol{U}_r\boldsymbol{\Lambda}_r\boldsymbol{U}_r^\top\boldsymbol{U}_r\boldsymbol{\Lambda}_r^{-1}\boldsymbol{U}_r^\top q(\bar{\boldsymbol{A}}_r) = \boldsymbol{U}_r\boldsymbol{U}_r^\top q(\bar{\boldsymbol{A}}_r),$$

where $\boldsymbol{U}_r\boldsymbol{U}_r^\top = \boldsymbol{U}_r(\boldsymbol{U}_r^\top\boldsymbol{U}_r)^{-1}\boldsymbol{U}_r^\top$ is the idempotent matrix that represents the projection mapping from $\mathbb{R}^{n_r}$ to the column space of $\boldsymbol{U}_r$ with respect to the standard basis for $\mathbb{R}^{n_r}$. First, note that the columns of $q(\bar{\boldsymbol{A}}_r)q(\bar{\boldsymbol{A}}_r)^\top$ lie in the column space of $\boldsymbol{U}_r$ because $\boldsymbol{U}_r\boldsymbol{U}_r^\top q(\bar{\boldsymbol{A}}_r)q(\bar{\boldsymbol{A}}_r)^\top = q(\bar{\boldsymbol{A}}_r)q(\bar{\boldsymbol{A}}_r)^\top$. Second, note that $q(\bar{\boldsymbol{A}}_r)$ and $q(\bar{\boldsymbol{A}}_r)q(\bar{\boldsymbol{A}}_r)^\top$ have the same column space (see, for example, Puntanen, Styan, and Isotalo 2011, Theorem 1). It follows that the columns of $q(\bar{\boldsymbol{A}}_r)$ lie in the column space of $\boldsymbol{U}_r$, that is, $\boldsymbol{U}_r\boldsymbol{U}_r^\top q(\bar{\boldsymbol{A}}_r) = q(\bar{\boldsymbol{A}}_r)$. This concludes the proof of (D.153).

Second, I show that $\boldsymbol{Q}_r(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r))^{-1} = \boldsymbol{Q}_r(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r))^{-1}\boldsymbol{Q}_r^\top\boldsymbol{Q}_r$. We find

$$\begin{aligned}
\boldsymbol{Q}_r\big(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r)\big)^{-1}\boldsymbol{Q}_r^\top\boldsymbol{Q}_r &= \boldsymbol{Q}_r\Big(\boldsymbol{I}_{n_r} + \gamma_0\big(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r)\big)^{-1}q(\bar{\boldsymbol{A}}_r)\Big)\boldsymbol{Q}_r^\top\boldsymbol{Q}_r \\
&= \boldsymbol{Q}_r + \gamma_0\boldsymbol{Q}_r\big(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r)\big)^{-1}q(\bar{\boldsymbol{A}}_r) \\
&= \boldsymbol{Q}_r\Big(\boldsymbol{I}_{n_r} + \gamma_0\big(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r)\big)^{-1}q(\bar{\boldsymbol{A}}_r)\Big) \\
&= \boldsymbol{Q}_r\big(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r)\big)^{-1},
\end{aligned}$$

where the first and the forth equality follow from

$$\big(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r)\big)^{-1} = \boldsymbol{I}_{n_r} + \gamma_0\big(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r)\big)^{-1}q(\bar{\boldsymbol{A}}_r) \tag{D.154}$$

and the second equality follows from $\boldsymbol{Q}_r\boldsymbol{Q}_r^\top = \boldsymbol{I}_{n_r-s_r}$ and (D.153).

Third, I show that the matrix $\boldsymbol{I}_{n_r-s_r} - \gamma_0\boldsymbol{Q}_r q(\bar{\boldsymbol{A}}_r)\boldsymbol{Q}_r^\top$ is nonsingular with inverse $\boldsymbol{Q}_r(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r))^{-1}\boldsymbol{Q}_r^\top$. We find

$$\begin{aligned}
&\big(\boldsymbol{I}_{n_r-s_r} - \gamma_0\boldsymbol{Q}_r q(\bar{\boldsymbol{A}}_r)\boldsymbol{Q}_r^\top\big)\boldsymbol{Q}_r\big(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r)\big)^{-1}\boldsymbol{Q}_r^\top \\
&\quad= \boldsymbol{Q}_r\big(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r)\big)^{-1}\boldsymbol{Q}_r^\top - \gamma_0\boldsymbol{Q}_r q(\bar{\boldsymbol{A}}_r)\boldsymbol{Q}_r^\top\boldsymbol{Q}_r\big(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r)\big)^{-1}\boldsymbol{Q}_r^\top \\
&\quad= \boldsymbol{Q}_r\big(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r)\big)^{-1}\boldsymbol{Q}_r^\top - \gamma_0\boldsymbol{Q}_r q(\bar{\boldsymbol{A}}_r)\big(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r)\big)^{-1}\boldsymbol{Q}_r^\top \\
&\quad= \boldsymbol{Q}_r\big(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r)\big)\big(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r)\big)^{-1}\boldsymbol{Q}_r^\top \\
&\quad= \boldsymbol{Q}_r\boldsymbol{Q}_r^\top \\
&\quad= \boldsymbol{I}_{n_r-s_r},
\end{aligned}$$

where the second equality follows from (D.153), and

$$\begin{aligned}
&\boldsymbol{Q}_r\big(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r)\big)^{-1}\boldsymbol{Q}_r^\top\big(\boldsymbol{I}_{n_r-s_r} - \gamma_0\boldsymbol{Q}_r q(\bar{\boldsymbol{A}}_r)\boldsymbol{Q}_r^\top\big) \\
&\quad= \boldsymbol{Q}_r\big(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r)\big)^{-1}\boldsymbol{Q}_r^\top - \gamma_0\boldsymbol{Q}_r\big(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r)\big)^{-1}\boldsymbol{Q}_r^\top\boldsymbol{Q}_r q(\bar{\boldsymbol{A}}_r)\boldsymbol{Q}_r^\top \\
&\quad= \boldsymbol{Q}_r\big(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r)\big)^{-1}\boldsymbol{Q}_r^\top \\
&\qquad\quad - \gamma_0\boldsymbol{Q}_r\Big(\boldsymbol{I}_{n_r} + \gamma_0\big(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r)\big)^{-1}q(\bar{\boldsymbol{A}}_r)\Big)\boldsymbol{Q}_r^\top\boldsymbol{Q}_r q(\bar{\boldsymbol{A}}_r)\boldsymbol{Q}_r^\top \\
&\quad= \boldsymbol{Q}_r\big(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r)\big)^{-1}\boldsymbol{Q}_r^\top
\end{aligned}$$

$$
\begin{aligned}
&- \gamma_0 \boldsymbol{Q}_r q(\bar{\boldsymbol{A}}_r)\boldsymbol{Q}_r^\top - \gamma_0^2 \boldsymbol{Q}_r \big(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r)\big)^{-1} q(\bar{\boldsymbol{A}}_r)\boldsymbol{Q}_r^\top \boldsymbol{Q}_r q(\bar{\boldsymbol{A}}_r)\boldsymbol{Q}_r^\top \\
&= \boldsymbol{Q}_r \big(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r)\big)^{-1} \boldsymbol{Q}_r^\top \\
&\quad - \gamma_0 \boldsymbol{Q}_r q(\bar{\boldsymbol{A}}_r)\boldsymbol{Q}_r^\top - \gamma_0^2 \boldsymbol{Q}_r \big(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r)\big)^{-1} q(\bar{\boldsymbol{A}}_r) q(\bar{\boldsymbol{A}}_r)\boldsymbol{Q}_r^\top \\
&= \boldsymbol{Q}_r \big(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r)\big)^{-1} \Big(\boldsymbol{I}_{n_r} - \gamma_0 \big(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r)\big) q(\bar{\boldsymbol{A}}_r) - \gamma_0^2 q(\bar{\boldsymbol{A}}_r) q(\bar{\boldsymbol{A}}_r)\Big) \boldsymbol{Q}_r^\top \\
&= \boldsymbol{Q}_r \boldsymbol{Q}_r^\top \\
&= \boldsymbol{I}_{n_r - s_r},
\end{aligned}
$$

where the second equality follows from (D.154) and the forth equality from (D.153).

Combining the two results $\boldsymbol{Q}_r(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r))^{-1} = \boldsymbol{Q}_r(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r))^{-1}\boldsymbol{Q}_r^\top \boldsymbol{Q}_r$ and $\boldsymbol{Q}_r(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r))^{-1}\boldsymbol{Q}_r^\top = (\boldsymbol{I}_{n_r - s_r} - \gamma_0 \boldsymbol{Q}_r q(\bar{\boldsymbol{A}}_r)\boldsymbol{Q}_r^\top)^{-1}$ gives

$$
\boldsymbol{Q}_r \big(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r)\big)^{-1} = \big(\boldsymbol{I}_{n_r - s_r} - \gamma_0 \boldsymbol{Q}_r q(\bar{\boldsymbol{A}}_r)\boldsymbol{Q}_r^\top\big)^{-1} \boldsymbol{Q}_r.
$$

To sum up, for all $r \in \{1, \dots, R\}$, the $(n_r - s_r) \times n_r$ matrix $\boldsymbol{Q}_r$ has the following properties:

(1) $\boldsymbol{Q}_r$ has full row rank;

(2) $\boldsymbol{Q}_r \boldsymbol{Q}_r^\top = \boldsymbol{I}_{n_r - s_r}$;

(3) $q(\bar{\boldsymbol{A}}_r) q(\bar{\boldsymbol{A}}_r)^\top = q(\bar{\boldsymbol{A}}_r)\boldsymbol{Q}_r^\top \boldsymbol{Q}_r q(\bar{\boldsymbol{A}}_r)^\top$;

(4) $\boldsymbol{Q}_r \boldsymbol{1}_{n_r} = \boldsymbol{0}_{n_r - s_r}$;

(5) $\boldsymbol{Q}_r \bar{\boldsymbol{A}}_r = \boldsymbol{Q}_r \bar{\boldsymbol{A}}_r \boldsymbol{Q}_r^\top \boldsymbol{Q}_r$; and

(6) $\boldsymbol{Q}_r(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r))^{-1} = (\boldsymbol{I}_{n_r - s_r} - \gamma_0 \boldsymbol{Q}_r q(\bar{\boldsymbol{A}}_r)\boldsymbol{Q}_r^\top)^{-1} \boldsymbol{Q}_r$.

Finally, define the $(n - S) \times n$ matrix $\boldsymbol{Q}_\ell$ by

$$
\boldsymbol{Q}_\ell := \begin{pmatrix}
\boldsymbol{Q}_1 & \boldsymbol{0}_{n_1 - s_1}\boldsymbol{0}_{n_2}^\top & \cdots & \boldsymbol{0}_{n_1 - s_1}\boldsymbol{0}_{n_R}^\top \\
\boldsymbol{0}_{n_2 - s_2}\boldsymbol{0}_{n_1}^\top & \boldsymbol{Q}_2 & \cdots & \boldsymbol{0}_{n_2 - s_2}\boldsymbol{0}_{n_R}^\top \\
\vdots & \vdots & \ddots & \vdots \\
\boldsymbol{0}_{n_R - s_R}\boldsymbol{0}_{n_1}^\top & \boldsymbol{0}_{n_R - s_R}\boldsymbol{0}_{n_2}^\top & \cdots & \boldsymbol{Q}_R
\end{pmatrix}.
$$

It is straightforward to show that Results 1.77.1 to 1.77.7 follow from the definition of $\boldsymbol{Q}_\ell$ and Results 1 to 6. ■

Proof of Lemma 1.78

Let $c \neq 0$. Let $\boldsymbol{P}(c)$ be the block diagonal matrix of order n that has the same block structure as $\bar{\boldsymbol{A}}$, where for all $r \in \{1, \dots, R\}$, the rth block of $\boldsymbol{P}(c)$ is denoted by $\boldsymbol{P}_r(c)$ and satisfies

$$
\boldsymbol{P}_r(c) = c\left(\boldsymbol{I}_{n_r} - \frac{1}{n_r}\boldsymbol{1}_{n_r}\boldsymbol{1}_{n_r}^\top\right).
$$

Let $r \in \{1, \ldots, R\}$. The matrix $\boldsymbol{P}_r(1)$ is idempotent and symmetric. As $\boldsymbol{P}_r(1)$ is idempotent, its spectrum is a subset of $\{0, 1\}$ and its rank is equal to its trace, which is equal to $n_r - 1$. It follows that $\boldsymbol{P}_r(1)$ is positive semidefinite but not positive definite. As $\boldsymbol{P}_r(1)$ is idempotent with rank $n_r - 1$, the algebraic multiplicity of the eigenvalue 1 is $n_r - 1$ and that of the eigenvalue 0 is 1. Note that $\boldsymbol{P}_r(c) = c\boldsymbol{P}_r(1)$, from which it follows that the matrix $\boldsymbol{P}_r(c)$ has the following properties: (i) it is symmetric; (ii) it is positive (respectively, negative) semidefinite but not positive (respectively, negative) definite if $c > 0$ (respectively, $c < 0$); (iii) its rank is equal to $n_r - 1$; and (iv) its spectrum is equal to $\{0, c\}$, where the algebraic multiplicity of the eigenvalue c is $n_r - 1$ and that of the eigenvalue 0 is 1. (The notation introduced hereinafter is based on the assumption that $n_r - 1 > 1$. All results are, however, true if $n_r - 1 = 1$, with the obvious changes in notation.) Consequently, there exists a spectral decomposition of $\boldsymbol{P}_r(c)$ that is given by

$$\boldsymbol{P}_r(c) = (\boldsymbol{U}_r : \boldsymbol{v}_r) \begin{pmatrix} c\boldsymbol{I}_{n_r-1} & \boldsymbol{0}_{n_r-1} \\ \boldsymbol{0}_{n_r-1}^\top & 0 \end{pmatrix} (\boldsymbol{U}_r : \boldsymbol{v}_r)^\top = c\boldsymbol{U}_r\boldsymbol{U}_r^\top,$$

where $(\boldsymbol{U}_r : \boldsymbol{v}_r)$ is an orthogonal matrix of order n_r. The $n_r \times (n_r - 1)$ matrix $\boldsymbol{U}_r$ and the $n_r \times 1$ vector $\boldsymbol{v}_r$ (the normalized eigenvector of $\boldsymbol{P}_r$ corresponding to the eigenvalue 0) satisfy $\boldsymbol{U}_r^\top\boldsymbol{U}_r = \boldsymbol{I}_{n_r-1}$, $\boldsymbol{v}_r = (1/\sqrt{n_r})\boldsymbol{1}_{n_r}$, $\boldsymbol{v}_r^\top\boldsymbol{v}_r = 1$, $\boldsymbol{U}_r^\top\boldsymbol{v}_r = \boldsymbol{0}_{n_r-1}$, and $\boldsymbol{U}_r\boldsymbol{U}_r^\top + \boldsymbol{v}_r\boldsymbol{v}_r^\top = \boldsymbol{I}_{n_r}$. Let the $(n_r - 1) \times n_r$ matrix $\boldsymbol{Q}_r$ be defined by

$$\boldsymbol{Q}_r := \sqrt{c}\boldsymbol{U}_r^\top.$$

Note that $\boldsymbol{Q}_r\boldsymbol{1}_{n_r} = \boldsymbol{0}_{n_r-1}$ and $\boldsymbol{Q}_r\boldsymbol{Q}_r^\top = c\boldsymbol{I}_{n_r-1}$, which implies that $\boldsymbol{Q}_r$ has full row rank. Also note that

$$q(\bar{\boldsymbol{A}}_r)\boldsymbol{Q}_r^\top\boldsymbol{Q}_r = cq(\bar{\boldsymbol{A}}_r), \tag{D.155}$$

from which it follows that $\boldsymbol{Q}_r\bar{\boldsymbol{A}}_r\boldsymbol{Q}_r^\top\boldsymbol{Q}_r = c\boldsymbol{Q}_r\bar{\boldsymbol{A}}_r$. Indeed, $\boldsymbol{U}_r\boldsymbol{U}_r^\top + \boldsymbol{v}_r\boldsymbol{v}_r^\top = \boldsymbol{I}_{n_r}$ and $q(\bar{\boldsymbol{A}}_r)\boldsymbol{1}_{n_r} = \boldsymbol{0}_{n_r}$ imply that

$$q(\bar{\boldsymbol{A}}_r)\boldsymbol{Q}_r^\top\boldsymbol{Q}_r = cq(\bar{\boldsymbol{A}}_r)\left(\boldsymbol{I}_{n_r} - \boldsymbol{v}_r\boldsymbol{v}_r^\top\right) = cq(\bar{\boldsymbol{A}}_r)\left(\boldsymbol{I}_{n_r} - \frac{1}{n_r}\boldsymbol{1}_{n_r}\boldsymbol{1}_{n_r}^\top\right) = cq(\bar{\boldsymbol{A}}_r).$$

Analogous to the proof of Result 6 of $\boldsymbol{Q}_r$ in the proof of Lemma 1.77, (D.154) and (D.155) imply that $\boldsymbol{Q}_r(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r))^{-1} = (\boldsymbol{I}_{n_r-1} - (\gamma_0/c)\boldsymbol{Q}_r q(\bar{\boldsymbol{A}}_r)\boldsymbol{Q}_r^\top)^{-1}\boldsymbol{Q}_r$. To sum up, for all $r \in \{1, \ldots, R\}$, the $(n_r - 1) \times n_r$ matrix $\boldsymbol{Q}_r$ has the following properties:

(1) $\boldsymbol{Q}_r$ has full row rank;

(2) $\boldsymbol{Q}_r\boldsymbol{Q}_r^\top = c\boldsymbol{I}_{n_r-1}$;

(3) $\boldsymbol{Q}_r^\top\boldsymbol{Q}_r = c(\boldsymbol{I}_{n_r} - (1/n_r)\boldsymbol{1}_{n_r}\boldsymbol{1}_{n_r}^\top)$;

(4) $\boldsymbol{Q}_r\boldsymbol{1}_{n_r} = \boldsymbol{0}_{n_r-1}$;

(5) $c\boldsymbol{Q}_r\bar{\boldsymbol{A}}_r = \boldsymbol{Q}_r\bar{\boldsymbol{A}}_r\boldsymbol{Q}_r^\top\boldsymbol{Q}_r$; and

(6) $\boldsymbol{Q}_r(\boldsymbol{I}_{n_r} - \gamma_0 q(\bar{\boldsymbol{A}}_r))^{-1} = (\boldsymbol{I}_{n_r-1} - (\gamma_0/c)\boldsymbol{Q}_r q(\bar{\boldsymbol{A}}_r)\boldsymbol{Q}_r^{\top})^{-1}\boldsymbol{Q}_r$.

Finally, define the $(n-R) \times n$ matrix $\boldsymbol{Q}_g$ by

$$\boldsymbol{Q}_g := \begin{pmatrix} \boldsymbol{Q}_1 & \boldsymbol{0}_{n_1-1}\boldsymbol{0}_{n_2}^{\top} & \cdots & \boldsymbol{0}_{n_1-1}\boldsymbol{0}_{n_R}^{\top} \\ \boldsymbol{0}_{n_2-1}\boldsymbol{0}_{n_1}^{\top} & \boldsymbol{Q}_2 & \cdots & \boldsymbol{0}_{n_2-1}\boldsymbol{0}_{n_R}^{\top} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0}_{n_R-1}\boldsymbol{0}_{n_1}^{\top} & \boldsymbol{0}_{n_R-1}\boldsymbol{0}_{n_2}^{\top} & \cdots & \boldsymbol{Q}_R \end{pmatrix}.$$

It is straightforward to show that Results 1.78.1 to 1.78.7 follow from the definition of $\boldsymbol{Q}_g$ and Results 1 to 6. ■

Proof of Proposition 1.79

In what follows, I write $\bar{\boldsymbol{A}}$ for $\bar{\boldsymbol{A}}(G)$ and $\bar{\boldsymbol{C}}$ for $\bar{\boldsymbol{C}}(H)$. Let $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ be a pair of parameter points in Θ^2. Suppose $\boldsymbol{X}\boldsymbol{\phi}_1 + \bar{\boldsymbol{C}}\boldsymbol{X}\boldsymbol{\psi}_1 \in \ker(q(\bar{\boldsymbol{A}}))$, that is, $q(\bar{\boldsymbol{A}})(\boldsymbol{X}\boldsymbol{\phi}_1 + \bar{\boldsymbol{C}}\boldsymbol{X}\boldsymbol{\psi}_1) = \boldsymbol{0}_n$. Note that

$$\big(\boldsymbol{I}_n - \gamma_1 q(\bar{\boldsymbol{A}})\big)^{-1} = \boldsymbol{I}_n + \gamma_1\big(\boldsymbol{I}_n - \gamma_1 q(\bar{\boldsymbol{A}})\big)^{-1} q(\bar{\boldsymbol{A}}).$$

Using this result, we find

$$\begin{aligned} \mathbb{E}\big(\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_1)) \mid \mathfrak{F}\big) &= \big(\boldsymbol{I}_n - \gamma_1 q(\bar{\boldsymbol{A}})\big)^{-1}(\boldsymbol{X}\boldsymbol{\phi}_1 + \bar{\boldsymbol{C}}\boldsymbol{X}\boldsymbol{\psi}_1) \\ &= \boldsymbol{X}\boldsymbol{\phi}_1 + \bar{\boldsymbol{C}}\boldsymbol{X}\boldsymbol{\psi}_1 + \gamma_1\big(\boldsymbol{I}_n - \gamma_1 q(\bar{\boldsymbol{A}})\big)^{-1} q(\bar{\boldsymbol{A}})(\boldsymbol{X}\boldsymbol{\phi}_1 + \bar{\boldsymbol{C}}\boldsymbol{X}\boldsymbol{\psi}_1) \\ &= \boldsymbol{X}\boldsymbol{\phi}_1 + \bar{\boldsymbol{C}}\boldsymbol{X}\boldsymbol{\psi}_1. \end{aligned}$$

Suppose

$$\mathbb{E}\big(\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_1)) \mid \mathfrak{F}\big) = \mathbb{E}\big(\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_2)) \mid \mathfrak{F}\big) \tag{D.156}$$

is true. We find that (D.156) is equivalent to

$$\boldsymbol{X}\boldsymbol{\phi}_1 + \bar{\boldsymbol{C}}\boldsymbol{X}\boldsymbol{\psi}_1 = \big(\boldsymbol{I}_n - \gamma_2 q(\bar{\boldsymbol{A}})\big)^{-1}(\boldsymbol{X}\boldsymbol{\phi}_2 + \bar{\boldsymbol{C}}\boldsymbol{X}\boldsymbol{\psi}_2),$$

which is equivalent to

$$\big(\boldsymbol{I}_n - \gamma_2 q(\bar{\boldsymbol{A}})\big)(\boldsymbol{X}\boldsymbol{\phi}_1 + \bar{\boldsymbol{C}}\boldsymbol{X}\boldsymbol{\psi}_1) = \boldsymbol{X}\boldsymbol{\phi}_2 + \bar{\boldsymbol{C}}\boldsymbol{X}\boldsymbol{\psi}_2,$$

which in turn is equivalent to

$$\boldsymbol{X}(\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2) + \bar{\boldsymbol{C}}\boldsymbol{X}(\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2) = \boldsymbol{0}_n.$$

It follows that γ is not identified by $\mathcal{P}(\Theta)$ through the mean. Finally, note that $q(\bar{\boldsymbol{A}})$ is singular because $1 \in \sigma(\bar{\boldsymbol{A}})$. ■

Proof of Proposition 1.79'

The proof is omitted because it is similar to the proof of Proposition 1.79. ■

Proof of Corollary 1.80

The first statement of the corollary follows immediately from Proposition 1.79. The second statement follows from the first by noting that $\ker(q(\bar{\boldsymbol{A}}(G)))$ is a vector space over $\mathbb{R}$ containing $\mathbf{1}_n$, from which it follows that for all $c \in \mathbb{R}$ and for all $\boldsymbol{v} \in \mathbb{R}^n$, $\boldsymbol{v} \in \ker(q(\bar{\boldsymbol{A}}(G)))$ if and only if $(\boldsymbol{v} - c\mathbf{1}_n) \in \ker(q(\bar{\boldsymbol{A}}(G)))$. ■

Proof of Corollary 1.80'

The proof is omitted because it is similar to the proof of Corollary 1.80. ■

Proof of Corollary 1.81

The statement of the corollary follows immediately from Proposition 1.79. ■

Proof of Corollary 1.82

In what follows, I write $\bar{\boldsymbol{A}}$ for $\bar{\boldsymbol{A}}(G)$ and $\bar{\boldsymbol{C}}$ for $\bar{\boldsymbol{C}}(H)$. Suppose $\bar{\boldsymbol{A}} = \bar{\boldsymbol{C}}$, $\boldsymbol{\psi} = \zeta\boldsymbol{\phi}$, and for all subparameter points ζ_0 in $\zeta(\Theta)$, $-1 \notin \sigma(\zeta_0\bar{\boldsymbol{A}})$. Let $\boldsymbol{\theta}_1$ be a parameter point in Θ. First, note that $\boldsymbol{I}_n + \zeta_1\bar{\boldsymbol{A}}$ is nonsingular because $-1 \notin \sigma(\zeta_1\bar{\boldsymbol{A}})$. Second, note that $q(\bar{\boldsymbol{A}})$ and $\boldsymbol{I}_n + \zeta_1\bar{\boldsymbol{A}}$ commute. Using the preceding two results, the statement of the corollary follows from Proposition 1.79 because

$$\begin{aligned} q(\bar{\boldsymbol{A}})(\boldsymbol{X}\boldsymbol{\phi}_1 + \bar{\boldsymbol{C}}\boldsymbol{X}\boldsymbol{\psi}_1) = \mathbf{0}_n \quad &\Leftrightarrow \quad q(\bar{\boldsymbol{A}})(\boldsymbol{I}_n + \zeta_1\bar{\boldsymbol{A}})\boldsymbol{X}\boldsymbol{\phi}_1 = \mathbf{0}_n \\ &\Leftrightarrow \quad (\boldsymbol{I}_n + \zeta_1\bar{\boldsymbol{A}})q(\bar{\boldsymbol{A}})\boldsymbol{X}\boldsymbol{\phi}_1 = \mathbf{0}_n \\ &\Leftrightarrow \quad q(\bar{\boldsymbol{A}})\boldsymbol{X}\boldsymbol{\phi}_1 = \mathbf{0}_n. \end{aligned}$$ ■

Proof of Lemma 1.83

In what follows, I write $\bar{\boldsymbol{A}}$ for $\bar{\boldsymbol{A}}(G)$ and $\bar{\boldsymbol{C}}$ for $\bar{\boldsymbol{C}}(H)$. Results 1.83.1 and 1.83.2 follow directly from the definitions of $\bar{\boldsymbol{A}}$ and $\bar{\boldsymbol{C}}$. As to the proof of Result 1.83.3, suppose $\bar{\boldsymbol{A}} = \bar{\boldsymbol{C}}$. Recall that $\bar{\boldsymbol{C}}$ has only zeros on its main diagonal. The equality of $\bar{\boldsymbol{A}}$ and $\bar{\boldsymbol{C}}$ implies that $\bar{\boldsymbol{A}}$ has only zeros on its main diagonal, which is equivalent to $\mathcal{I}_0^+(G) = \emptyset$. The definitions of $\bar{\boldsymbol{A}}$ and $\bar{\boldsymbol{C}}$ and the assumption that the former matrix is equal to the latter imply that $\mathbf{0}_n = (\bar{\boldsymbol{A}} - \bar{\boldsymbol{C}})\mathbf{1}_n = \bar{\boldsymbol{A}}\mathbf{1}_n - \bar{\boldsymbol{A}}(H)\mathbf{1}_n + \operatorname{diag}(\boldsymbol{\iota}_0^+(H))\mathbf{1}_n = \mathbf{1}_n - \mathbf{1}_n + \boldsymbol{\iota}_0^+(H) = \boldsymbol{\iota}_0^+(H)$, where $\boldsymbol{\iota}_0^+(H) = \mathbf{0}_n$ if and only if $\mathcal{I}_0^+(H) = \emptyset$. ■

Proof of Proposition 1.84

In what follows, I write $\bar{\boldsymbol{A}}$ for $\bar{\boldsymbol{A}}(G)$ and $\bar{\boldsymbol{C}}$ for $\bar{\boldsymbol{C}}(H)$. Let $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ be a pair of parameter points in Θ^2. Note that, for all $(\boldsymbol{v}_1, \boldsymbol{v}_2) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\left(\boldsymbol{I}_n - \gamma_1 q(\bar{\boldsymbol{A}})\right)^{-1}\boldsymbol{v}_1 = \left(\boldsymbol{I}_n - \gamma_2 q(\bar{\boldsymbol{A}})\right)^{-1}\boldsymbol{v}_2$$

$$\Leftrightarrow \qquad \big(\boldsymbol{I}_n - \gamma_2 q(\bar{\boldsymbol{A}})\big)\boldsymbol{v}_1 = \big(\boldsymbol{I}_n - \gamma_1 q(\bar{\boldsymbol{A}})\big)\boldsymbol{v}_2,$$

because $\boldsymbol{I}_n - \gamma_1 q(\bar{\boldsymbol{A}})$ and $\big(\boldsymbol{I}_n - \gamma_2 q(\bar{\boldsymbol{A}})\big)^{-1}$ commute (Lemma B.4).

Proof of Result 1.84.1 Suppose $\bar{\boldsymbol{A}} \neq \bar{\boldsymbol{C}}$, $\mathcal{I}_0^+(H) \neq \emptyset$, the kernel condition (1.75) is not satisfied, and the matrix (1.76) has full column rank. We must show that $(\gamma_1, \boldsymbol{\phi}_1, \boldsymbol{\psi}_1) = (\gamma_2, \boldsymbol{\phi}_2, \boldsymbol{\psi}_2)$ is necessary for

$$\mathbb{E}\big(\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_1)\big) \,\big|\, \mathfrak{F}\big) = \mathbb{E}\big(\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_2)\big) \,\big|\, \mathfrak{F}\big). \tag{D.157}$$

Suppose (D.157) is true. We find that (D.157) is equivalent to

$$\big(\boldsymbol{I}_n - \gamma_1 q(\bar{\boldsymbol{A}})\big)^{-1}\big(\boldsymbol{X}\boldsymbol{\phi}_1 + \bar{\boldsymbol{C}}\boldsymbol{X}\boldsymbol{\psi}_1\big) = \big(\boldsymbol{I}_n - \gamma_2 q(\bar{\boldsymbol{A}})\big)^{-1}\big(\boldsymbol{X}\boldsymbol{\phi}_2 + \bar{\boldsymbol{C}}\boldsymbol{X}\boldsymbol{\psi}_2\big),$$

which is equivalent to

$$\big(\boldsymbol{I}_n - \gamma_2 q(\bar{\boldsymbol{A}})\big)\big(\boldsymbol{X}\boldsymbol{\phi}_1 + \bar{\boldsymbol{C}}\boldsymbol{X}\boldsymbol{\psi}_1\big) = \big(\boldsymbol{I}_n - \gamma_1 q(\bar{\boldsymbol{A}})\big)\big(\boldsymbol{X}\boldsymbol{\phi}_2 + \bar{\boldsymbol{C}}\boldsymbol{X}\boldsymbol{\psi}_2\big),$$

which in turn is equivalent to

$$\begin{aligned}\boldsymbol{0}_n = {} & \boldsymbol{X}\big((1+\gamma_2)\boldsymbol{\phi}_1 - (1+\gamma_1)\boldsymbol{\phi}_2\big) \\ & + \bar{\boldsymbol{A}}\boldsymbol{X}\big(\gamma_1\boldsymbol{\phi}_2 - \gamma_2\boldsymbol{\phi}_1\big) \\ & + \bar{\boldsymbol{C}}\boldsymbol{X}\big((1+\gamma_2)\boldsymbol{\psi}_1 - (1+\gamma_1)\boldsymbol{\psi}_2\big) \\ & + \bar{\boldsymbol{A}}\bar{\boldsymbol{C}}\boldsymbol{X}\big(\gamma_1\boldsymbol{\psi}_2 - \gamma_2\boldsymbol{\psi}_1\big),\end{aligned}$$

which in turn is equivalent to

$$\begin{aligned}\boldsymbol{0}_n = {} & \boldsymbol{1}_n\big(\phi_1 + \psi_1 - \phi_2 - \psi_2\big) \\ & + \boldsymbol{\iota}_0^+(H)\big((1+\gamma_1)\psi_2 - (1+\gamma_2)\psi_1\big) \\ & + \bar{\boldsymbol{A}}\boldsymbol{\iota}_0^+(H)\big(\gamma_2\psi_1 - \gamma_1\psi_2\big) \\ & + \boldsymbol{X}_2\big((1+\gamma_2)\boldsymbol{\phi}_{1,-1} - (1+\gamma_1)\boldsymbol{\phi}_{2,-1}\big) \\ & + \bar{\boldsymbol{A}}\boldsymbol{X}_2\big(\gamma_1\boldsymbol{\phi}_{2,-1} - \gamma_2\boldsymbol{\phi}_{1,-1}\big) \\ & + \bar{\boldsymbol{C}}\boldsymbol{X}_2\big((1+\gamma_2)\boldsymbol{\psi}_{1,-1} - (1+\gamma_1)\boldsymbol{\psi}_{2,-1}\big) \\ & + \bar{\boldsymbol{A}}\bar{\boldsymbol{C}}\boldsymbol{X}_2\big(\gamma_1\boldsymbol{\psi}_{2,-1} - \gamma_2\boldsymbol{\psi}_{1,-1}\big)\end{aligned}$$

because $\boldsymbol{X} = (\boldsymbol{1}_n : \boldsymbol{X}_2)$ (Assumption $\mathcal{P}$-$\boldsymbol{X}$), $\bar{\boldsymbol{A}}\boldsymbol{1}_n = \boldsymbol{1}_n$, $\bar{\boldsymbol{C}}\boldsymbol{1}_n = \boldsymbol{1}_n - \boldsymbol{\iota}_0^+(H)$, and $\bar{\boldsymbol{A}}\bar{\boldsymbol{C}}\boldsymbol{1}_n = \boldsymbol{1}_n - \bar{\boldsymbol{A}}\boldsymbol{\iota}_0^+(H)$. We have

$$\phi_1 + \psi_1 - \phi_2 - \psi_2 = 0 \tag{D.158}$$
$$(1+\gamma_1)\psi_2 - (1+\gamma_2)\psi_1 = 0 \tag{D.159}$$
$$\gamma_2\psi_1 - \gamma_1\psi_2 = 0 \tag{D.160}$$
$$(1+\gamma_2)\boldsymbol{\phi}_{1,-1} - (1+\gamma_1)\boldsymbol{\phi}_{2,-1} = \boldsymbol{0}_{K-1} \tag{D.161}$$
$$\gamma_1\boldsymbol{\phi}_{2,-1} - \gamma_2\boldsymbol{\phi}_{1,-1} = \boldsymbol{0}_{K-1} \tag{D.162}$$
$$(1+\gamma_2)\boldsymbol{\psi}_{1,-1} - (1+\gamma_1)\boldsymbol{\psi}_{2,-1} = \boldsymbol{0}_{K-1} \tag{D.163}$$

$$\gamma_1 \boldsymbol{\psi}_{2,-1} - \gamma_2 \boldsymbol{\psi}_{1,-1} = \mathbf{0}_{K-1} \tag{D.164}$$

because matrix (1.76) has full column rank. The system of equations (D.158) to (D.164) has a unique solution. Adding equations (D.159) and (D.160) gives $\psi_1 = \psi_2$. This equality together with equation (D.158) gives $\phi_1 = \phi_2$. Adding equations (D.161) and (D.162) gives $\boldsymbol{\phi}_{1,-1} = \boldsymbol{\phi}_{2,-1}$. Adding equations (D.163) and (D.164) gives $\boldsymbol{\psi}_{1,-1} = \boldsymbol{\psi}_{2,-1}$. Using the equalities $\phi_1 = \phi_2$, $\boldsymbol{\phi}_{1,-1} = \boldsymbol{\phi}_{2,-1}$, $\psi_1 = \psi_2$, and $\boldsymbol{\psi}_{1,-1} = \boldsymbol{\psi}_{2,-1}$, equations (D.159) and (D.160) are equivalent to $(\gamma_1 - \gamma_2)\psi_1 = 0$, equations (D.161) and (D.162) are equivalent to $(\gamma_1 - \gamma_2)\boldsymbol{\phi}_{1,-1} = \mathbf{0}_{K-1}$, and equations (D.163) and (D.164) are equivalent to $(\gamma_1 - \gamma_2)\boldsymbol{\psi}_{1,-1} = \mathbf{0}_{K-1}$. We must have $\psi_1 \neq 0$, $\boldsymbol{\phi}_{1,-1} \neq \mathbf{0}_{K-1}$, or $\boldsymbol{\psi}_{1,-1} \neq \mathbf{0}_{K-1}$ because the kernel condition (1.75) is not satisfied (Corollary 1.80). It follows that $\gamma_1 = \gamma_2$. To sum up, I have established that $(\gamma_1, \phi_1, \boldsymbol{\phi}_{1,-1}, \psi_1, \boldsymbol{\psi}_{1,-1}) = (\gamma_2, \phi_2, \boldsymbol{\phi}_{2,-1}, \psi_2, \boldsymbol{\psi}_{2,-1})$.

The proof that $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}$, $\bar{\boldsymbol{C}}$, $\bar{\boldsymbol{A}}\bar{\boldsymbol{C}}$ are linearly independent if γ, $\boldsymbol{\phi}$, and $\boldsymbol{\psi}$ are identified through the mean by $\mathcal{P}(\Theta)$ is omitted. ∎

Proof of Result 1.84.2 Suppose $\bar{\boldsymbol{A}} \neq \bar{\boldsymbol{C}}$, $\mathcal{I}_0^+(H) = \emptyset$, the kernel condition (1.75) is not satisfied, and the matrix (1.77) has full column rank. We must show that $(\gamma_1, \boldsymbol{\phi}_{1,-1}, \boldsymbol{\psi}_{1,-1}) = (\gamma_2, \boldsymbol{\phi}_{2,-1}, \boldsymbol{\psi}_{2,-1})$ is necessary for

$$\mathbb{E}\big(\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_1)\big) \,\big|\, \mathfrak{F}\big) = \mathbb{E}\big(\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_2)\big) \,\big|\, \mathfrak{F}\big). \tag{D.165}$$

Suppose (D.165) is true. We find that (D.165) is equivalent to

$$\begin{aligned}
\mathbf{0}_n = {} & \mathbf{1}_n(\phi_1 + \psi_1 - \phi_2 - \psi_2) \\
& + \boldsymbol{X}_2\big((1+\gamma_2)\boldsymbol{\phi}_{1,-1} - (1+\gamma_1)\boldsymbol{\phi}_{2,-1}\big) \\
& + \bar{\boldsymbol{A}}\boldsymbol{X}_2\big(\gamma_1\boldsymbol{\phi}_{2,-1} - \gamma_2\boldsymbol{\phi}_{1,-1}\big) \\
& + \bar{\boldsymbol{C}}\boldsymbol{X}_2\big((1+\gamma_2)\boldsymbol{\psi}_{1,-1} - (1+\gamma_1)\boldsymbol{\psi}_{2,-1}\big) \\
& + \bar{\boldsymbol{A}}\bar{\boldsymbol{C}}\boldsymbol{X}_2\big(\gamma_1\boldsymbol{\psi}_{2,-1} - \gamma_2\boldsymbol{\psi}_{1,-1}\big).
\end{aligned}$$

We have

$$\phi_1 + \psi_1 - \phi_2 - \psi_2 = 0$$

$$(1+\gamma_2)\boldsymbol{\phi}_{1,-1} - (1+\gamma_1)\boldsymbol{\phi}_{2,-1} = \mathbf{0}_{K-1} \tag{D.166}$$

$$\gamma_1 \boldsymbol{\phi}_{2,-1} - \gamma_2 \boldsymbol{\phi}_{1,-1} = \mathbf{0}_{K-1} \tag{D.167}$$

$$(1+\gamma_2)\boldsymbol{\psi}_{1,-1} - (1+\gamma_1)\boldsymbol{\psi}_{2,-1} = \mathbf{0}_{K-1} \tag{D.168}$$

$$\gamma_1 \boldsymbol{\psi}_{2,-1} - \gamma_2 \boldsymbol{\psi}_{1,-1} = \mathbf{0}_{K-1} \tag{D.169}$$

because matrix (1.77) has full column rank. The system of equations (D.166) to (D.169) has a unique solution. Adding equations (D.166) and (D.167) gives $\boldsymbol{\phi}_{1,-1} = \boldsymbol{\phi}_{2,-1}$. Adding equations (D.168) and (D.169) gives $\boldsymbol{\psi}_{1,-1} = \boldsymbol{\psi}_{2,-1}$. Using the equalities $\boldsymbol{\phi}_{1,-1} = \boldsymbol{\phi}_{2,-1}$ and $\boldsymbol{\psi}_{1,-1} = \boldsymbol{\psi}_{2,-1}$, equations (D.166) and (D.167) are equivalent to $(\gamma_1 - \gamma_2)\boldsymbol{\phi}_{1,-1} = \mathbf{0}_{K-1}$ and equations (D.168) and (D.169) are equivalent to $(\gamma_1 - \gamma_2)\boldsymbol{\psi}_{1,-1} = \mathbf{0}_{K-1}$. We must have $\boldsymbol{\phi}_{1,-1} \neq \mathbf{0}_{K-1}$ or $\boldsymbol{\psi}_{1,-1} \neq \mathbf{0}_{K-1}$ because the kernel condition (1.75) is not satisfied (Corollary 1.80). It follows that $\gamma_1 = \gamma_2$. To sum up, I have established that $(\gamma_1, \boldsymbol{\phi}_{1,-1}, \boldsymbol{\psi}_{1,-1}) = (\gamma_2, \boldsymbol{\phi}_{2,-1}, \boldsymbol{\psi}_{2,-1})$.

The proof that $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}$, $\bar{\boldsymbol{C}}$, $\bar{\boldsymbol{A}}\bar{\boldsymbol{C}}$ are linearly independent if γ, $\boldsymbol{\phi}_{-1}$, and $\boldsymbol{\psi}_{-1}$ are identified through the mean by $\mathcal{P}(\Theta)$ is omitted. ■

Proof of Proposition 1.84'

The proof is omitted because it is similar to the proof of Proposition 1.84. ■

Proof of Proposition 1.86

In what follows, I write $\bar{\boldsymbol{A}}$ for $\bar{\boldsymbol{A}}(G)$ and $\bar{\boldsymbol{C}}$ for $\bar{\boldsymbol{C}}(H)$. Suppose $\bar{\boldsymbol{A}} = \bar{\boldsymbol{C}}$, the kernel condition (1.75) is not satisfied, $\gamma\boldsymbol{\phi}_{-1} + (1+\gamma)\boldsymbol{\psi}_{-1} \neq \mathbf{0}_{K-1}$, and the matrix (1.78) has full column rank. Let $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ be a pair of parameter points in Θ^2. Note that $\mathcal{I}_0^+(H) = \emptyset$ because $\bar{\boldsymbol{A}} = \bar{\boldsymbol{C}}$ (Result 1.83.3). We must show that $(\gamma_1, \boldsymbol{\phi}_{1,-1}, \boldsymbol{\psi}_{1,-1}) = (\gamma_2, \boldsymbol{\phi}_{2,-1}, \boldsymbol{\psi}_{2,-1})$ is necessary for

$$\mathbb{E}\big(\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_1)\big) \,\big|\, \mathfrak{F}\big) = \mathbb{E}\big(\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_2)\big) \,\big|\, \mathfrak{F}\big). \tag{D.170}$$

Suppose (D.170) is true. We find that (D.170) is equivalent to

$$\begin{aligned}
\mathbf{0}_n = \mathbf{1}_n&(\phi_1 + \psi_1 - \phi_2 - \psi_2) \\
&+ \boldsymbol{X}_2\big((1+\gamma_2)\boldsymbol{\phi}_{1,-1} - (1+\gamma_1)\boldsymbol{\phi}_{2,-1}\big) \\
&+ \bar{\boldsymbol{A}}\boldsymbol{X}_2\big(\gamma_1\boldsymbol{\phi}_{2,-1} + (1+\gamma_2)\boldsymbol{\psi}_{1,-1} - \gamma_2\boldsymbol{\phi}_{1,-1} - (1+\gamma_1)\boldsymbol{\psi}_{2,-1}\big) \\
&+ \bar{\boldsymbol{A}}^2\boldsymbol{X}_2\big(\gamma_1\boldsymbol{\psi}_{2,-1} - \gamma_2\boldsymbol{\psi}_{1,-1}\big).
\end{aligned}$$

We have

$$\phi_1 + \psi_1 - \phi_2 - \psi_2 = 0$$

$$(1+\gamma_2)\boldsymbol{\phi}_{1,-1} - (1+\gamma_1)\boldsymbol{\phi}_{2,-1} = \mathbf{0}_{K-1} \tag{D.171}$$

$$\gamma_1\boldsymbol{\phi}_{2,-1} + (1+\gamma_2)\boldsymbol{\psi}_{1,-1} - \gamma_2\boldsymbol{\phi}_{1,-1} - (1+\gamma_1)\boldsymbol{\psi}_{2,-1} = \mathbf{0}_{K-1} \tag{D.172}$$

$$\gamma_1\boldsymbol{\psi}_{2,-1} - \gamma_2\boldsymbol{\psi}_{1,-1} = \mathbf{0}_{K-1} \tag{D.173}$$

because matrix (1.78) has full column rank. The system of equations (D.171), (D.172), and (D.173) has a unique solution because $\gamma\boldsymbol{\phi}_{-1} + (1+\gamma)\boldsymbol{\psi}_{-1} \neq \mathbf{0}_{K-1}$ implies that $\gamma_1\boldsymbol{\phi}_{1,-1} + (1+\gamma_1)\boldsymbol{\psi}_{1,-1} \neq \mathbf{0}_{K-1}$ and $\gamma_2\boldsymbol{\phi}_{2,-1} + (1+\gamma_2)\boldsymbol{\psi}_{2,-1} \neq \mathbf{0}_{K-1}$. To show this, I consider two cases:

(1) Suppose $\gamma_1 = 0$. The inequality $\gamma_1\boldsymbol{\phi}_{1,-1} + (1+\gamma_1)\boldsymbol{\psi}_{1,-1} \neq \mathbf{0}_{K-1}$ is equivalent to $\boldsymbol{\psi}_{1,-1} \neq \mathbf{0}_{K-1}$. Using this result, equation (D.173) implies that $\gamma_2 = 0$. Using $\gamma_1 = \gamma_2 = 0$, equation (D.171) gives $\boldsymbol{\phi}_{1,-1} = \boldsymbol{\phi}_{2,-1}$ and equation (D.172) gives $\boldsymbol{\psi}_{1,-1} = \boldsymbol{\psi}_{2,-1}$.

(2) Suppose $\gamma_1 \neq 0$. We must have $\gamma_2 \neq 0$. Indeed, if $\gamma_2 = 0$, then $\boldsymbol{\psi}_{2,-1} \neq \mathbf{0}_{K-1}$ according to the inequality $\gamma_2\boldsymbol{\phi}_{2,-1} + (1+\gamma_2)\boldsymbol{\psi}_{2,-1} \neq \mathbf{0}_{K-1}$ and $\boldsymbol{\psi}_{2,-1} = \mathbf{0}_{K-1}$ according to equation (D.173), a contradiction. Multiplying both sides of

equation (D.172) by $1+\gamma_1$ gives $\gamma_1(1+\gamma_1)\boldsymbol{\phi}_{2,-1} + (1+\gamma_1)(1+\gamma_2)\boldsymbol{\psi}_{1,-1} - (1+\gamma_1)\gamma_2\boldsymbol{\phi}_{1,-1} - (1+\gamma_1)^2\boldsymbol{\psi}_{2,-1} = \mathbf{0}_{K-1}$, which is equivalent to

$$(\gamma_1-\gamma_2)\big(\gamma_1\boldsymbol{\phi}_{1,-1} + (1+\gamma_1)\boldsymbol{\psi}_{1,-1}\big) = \mathbf{0}_{K-1} \tag{D.174}$$

because $\gamma_1(1+\gamma_2)\boldsymbol{\phi}_{1,-1} = \gamma_1(1+\gamma_1)\boldsymbol{\phi}_{2,-1}$ according to equation (D.171) and $\boldsymbol{\psi}_{2,-1} = (\gamma_2/\gamma_1)\boldsymbol{\psi}_{1,-1}$ according to equation (D.173). Equation (D.174) and the inequality $\gamma_1\boldsymbol{\phi}_{1,-1} + (1+\gamma_1)\boldsymbol{\psi}_{1,-1} \neq \mathbf{0}_{K-1}$ imply that $\gamma_1 = \gamma_2$. Using this result, equation (D.173) gives $\boldsymbol{\psi}_{1,-1} = \boldsymbol{\psi}_{2,-1}$. Adding equations (D.171), (D.172), and (D.173) gives $\boldsymbol{\phi}_{1,-1} - \boldsymbol{\phi}_{2,-1} = \boldsymbol{\psi}_{2,-1} - \boldsymbol{\psi}_{1,-1}$, from which $\boldsymbol{\phi}_{1,-1} = \boldsymbol{\phi}_{2,-1}$ follows because $\boldsymbol{\psi}_{1,-1} = \boldsymbol{\psi}_{2,-1}$.

To sum up, I have established that $(\gamma_1, \boldsymbol{\phi}_{1,-1}, \boldsymbol{\psi}_{1,-1}) = (\gamma_2, \boldsymbol{\phi}_{2,-1}, \boldsymbol{\psi}_{2,-1})$. Finally, note that the system of equations (D.171), (D.172), and (D.173) has infinitely many solutions without the restrictions $\gamma_1\boldsymbol{\phi}_{1,-1} + (1+\gamma_1)\boldsymbol{\psi}_{1,-1} \neq \mathbf{0}_{K-1}$ and $\gamma_2\boldsymbol{\phi}_{2,-1} + (1+\gamma_2)\boldsymbol{\psi}_{2,-1} \neq \mathbf{0}_{K-1}$, for example, for all $(c_1, c_2, \boldsymbol{p}) \in \gamma(\Theta)^2 \times \mathbb{R}^{K-1}$ with $c_1 \neq 0$,

$$\big(\gamma_1, \gamma_2, \boldsymbol{\phi}_{1,-1}, \boldsymbol{\phi}_{2,-1}, \boldsymbol{\psi}_{1,-1}, \boldsymbol{\psi}_{2,-1}\big) = \big(0, c_2, \boldsymbol{p}, (1+c_2)\boldsymbol{p}, \mathbf{0}_{K-1}, -c_2\boldsymbol{p}\big)$$

and

$$\big(\gamma_1, \gamma_2, \boldsymbol{\phi}_{1,-1}, \boldsymbol{\phi}_{2,-1}, \boldsymbol{\psi}_{1,-1}, \boldsymbol{\psi}_{2,-1}\big) = \left(c_1, c_2, -\frac{1+c_1}{c_1}\boldsymbol{p}, -\frac{1+c_2}{c_1}\boldsymbol{p}, \boldsymbol{p}, \frac{c_2}{c_1}\boldsymbol{p}\right)$$

are solutions.

The proof that $\boldsymbol{I}_n, \bar{\boldsymbol{A}}, \bar{\boldsymbol{A}}^2$ are linearly independent if γ, $\boldsymbol{\phi}_{-1}$, and $\boldsymbol{\psi}_{-1}$ are identified through the mean by $\mathcal{P}(\Theta)$ is omitted. ■

Proof of Proposition 1.86'

The proof is omitted because it is similar to the proof of Proposition 1.86. ■

Proof of Lemma 1.87

In what follows, I write $\bar{\boldsymbol{A}}$ for $\bar{\boldsymbol{A}}(G)$. Suppose (x, y, z) is an intransitive triple in G. It follows that $\deg_G^+(x) > 0$, $\deg_G^+(y) > 0$, $[\bar{\boldsymbol{A}}]_{x,x} = 0$, $[\bar{\boldsymbol{A}}]_{x,y} = 1/\deg_G^+(x) > 0$, $[\bar{\boldsymbol{A}}]_{y,z} = 1/\deg_G^+(y) > 0$, and $[\bar{\boldsymbol{A}}]_{x,z} = 0$. Suppose, for the sake of contradiction, $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}$, $\bar{\boldsymbol{A}}^2$ are linearly dependent. There exists a triple $(c_0, c_1, c_2) \neq (0,0,0)$ such that

$$c_0\boldsymbol{I}_n + c_1\bar{\boldsymbol{A}} + c_2\bar{\boldsymbol{A}}^2 = \boldsymbol{O}_n. \tag{D.175}$$

Note that $c_2 \neq 0$. In order to prove this, suppose, for the sake of contradiction, $c_2 = 0$. It follows from (D.175) and $[\bar{\boldsymbol{A}}]_{x,x} = 0$ that $c_0 + c_2[\bar{\boldsymbol{A}}^2]_{x,x} = 0$, from which $c_0 = 0$ follows because $c_2 = 0$. It follows from (D.175) and $c_0 = c_2 = 0$ that $c_1\bar{\boldsymbol{A}} = \boldsymbol{O}_n$, which implies that $c_1 = 0$ because $\bar{\boldsymbol{A}}$ is nonnegative and different from $\boldsymbol{O}_n$. Consequently, $(c_0, c_1, c_2) = (0,0,0)$, which contradicts $(c_0, c_1, c_2) \neq (0,0,0)$. This concludes the proof that $c_2 \neq 0$. We find $[\bar{\boldsymbol{A}}^2]_{x,z} = \sum_{k=1}^n [\bar{\boldsymbol{A}}]_{x,k}[\bar{\boldsymbol{A}}]_{k,z} \geq [\bar{\boldsymbol{A}}]_{x,y}[\bar{\boldsymbol{A}}]_{y,z} >$

0 because $\bar{A}$ is nonnegative, $[\bar{A}]_{x,y} > 0$, and $[\bar{A}]_{y,z} > 0$. It follows from (D.175) and $[\bar{A}]_{x,z} = 0$ that $c_2[\bar{A}^2]_{x,z} = 0$, which is equivalent to $[\bar{A}^2]_{x,z} = 0$ because $c_2 \neq 0$, which contradicts $[\bar{A}^2]_{x,z} > 0$. This concludes the proof that $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}$, $\bar{\boldsymbol{A}}^2$ are linearly independent. ■

Proof of Proposition 1.89

In what follows, I write $\bar{\boldsymbol{A}}$ for $\bar{\boldsymbol{A}}(G)$ and $\bar{\boldsymbol{C}}$ for $\bar{\boldsymbol{C}}(H)$. Suppose the kernel condition (1.75) is not satisfied, $G = H$, $\boldsymbol{0}_n \neq \boldsymbol{\iota}_0^+(G) \in \text{c-sp}(\text{diag}(\boldsymbol{\iota}_0^+(G))\boldsymbol{X}_2)$, $\gamma\boldsymbol{\phi}_{-1} + (1+\gamma)\boldsymbol{\psi}_{-1} \neq \boldsymbol{0}_{K-1}$, and the matrix (1.79) has full column rank. Let $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ be a pair of parameter points in Θ^2. Note that $\bar{\boldsymbol{A}}\boldsymbol{X}_2 - \bar{\boldsymbol{C}}\boldsymbol{X}_2 = \text{diag}(\boldsymbol{\iota}_0^+(G))\boldsymbol{X}_2$. Note also that $\boldsymbol{\iota}_0^+(G) \in \text{c-sp}(\text{diag}(\boldsymbol{\iota}_0^+(G))\boldsymbol{X}_2)$ if and only if there exists a $\boldsymbol{\lambda} \in \mathbb{R}^{K-1}$ such that $\text{diag}(\boldsymbol{\iota}_0^+(G))\boldsymbol{X}_2\boldsymbol{\lambda} = \boldsymbol{\iota}_0^+(G)$. It follows that $\boldsymbol{\iota}_0^+(G) = \bar{\boldsymbol{A}}\boldsymbol{X}_2\boldsymbol{\lambda} - \bar{\boldsymbol{C}}\boldsymbol{X}_2\boldsymbol{\lambda}$ for some $\boldsymbol{\lambda} \in \mathbb{R}^{K-1} \setminus \{\boldsymbol{0}_{K-1}\}$, where $\boldsymbol{\lambda} \neq \boldsymbol{0}_{K-1}$ because $\boldsymbol{\iota}_0^+(G) \neq \boldsymbol{0}_n$. We must show that $(\gamma_1, \boldsymbol{\phi}_1, \boldsymbol{\psi}_1) = (\gamma_2, \boldsymbol{\phi}_2, \boldsymbol{\psi}_2)$ is necessary for

$$\mathbb{E}\big(\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_1)\big) \,\big|\, \mathfrak{F}\big) = \mathbb{E}\big(\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_2)\big) \,\big|\, \mathfrak{F}\big). \tag{D.176}$$

Suppose (D.176) is true. We find that (D.176) is equivalent to

$$\begin{aligned}
\boldsymbol{0}_n = \boldsymbol{1}_n\big(&\phi_1 + \psi_1 - \phi_2 - \psi_2\big)\\
&+ \boldsymbol{\iota}_0^+(G)\big((1+\gamma_1)\psi_2 - (1+\gamma_2)\psi_1\big)\\
&+ \bar{\boldsymbol{A}}\boldsymbol{\iota}_0^+(G)\big(\gamma_2\psi_1 - \gamma_1\psi_2\big)\\
&+ \boldsymbol{X}_2\big((1+\gamma_2)\boldsymbol{\phi}_{1,-1} - (1+\gamma_1)\boldsymbol{\phi}_{2,-1}\big)\\
&+ \bar{\boldsymbol{A}}\boldsymbol{X}_2\big(\gamma_1\boldsymbol{\phi}_{2,-1} - \gamma_2\boldsymbol{\phi}_{1,-1}\big)\\
&+ \bar{\boldsymbol{C}}\boldsymbol{X}_2\big((1+\gamma_2)\boldsymbol{\psi}_{1,-1} - (1+\gamma_1)\boldsymbol{\psi}_{2,-1}\big)\\
&+ \bar{\boldsymbol{A}}\bar{\boldsymbol{C}}\boldsymbol{X}_2\big(\gamma_1\boldsymbol{\psi}_{2,-1} - \gamma_2\boldsymbol{\psi}_{1,-1}\big),
\end{aligned}$$

which in turn is equivalent to

$$\begin{aligned}
\boldsymbol{0}_n = \boldsymbol{1}_n\big(&\phi_1 + \psi_1 - \phi_2 - \psi_2\big)\\
&+ \boldsymbol{X}_2\big((1+\gamma_2)\boldsymbol{\phi}_{1,-1} - (1+\gamma_1)\boldsymbol{\phi}_{2,-1}\big)\\
&+ \bar{\boldsymbol{A}}\boldsymbol{X}_2\big(\gamma_1\boldsymbol{\phi}_{2,-1} - \gamma_2\boldsymbol{\phi}_{1,-1} + (1+\gamma_1)\psi_2\boldsymbol{\lambda} - (1+\gamma_2)\psi_1\boldsymbol{\lambda}\big)\\
&+ \bar{\boldsymbol{C}}\boldsymbol{X}_2\big((1+\gamma_2)\boldsymbol{\psi}_{1,-1} - (1+\gamma_1)\boldsymbol{\psi}_{2,-1} - (1+\gamma_1)\psi_2\boldsymbol{\lambda} + (1+\gamma_2)\psi_1\boldsymbol{\lambda}\big)\\
&+ \bar{\boldsymbol{A}}^2\boldsymbol{X}_2\big(\gamma_2\psi_1\boldsymbol{\lambda} - \gamma_1\psi_2\boldsymbol{\lambda}\big)\\
&+ \bar{\boldsymbol{A}}\bar{\boldsymbol{C}}\boldsymbol{X}_2\big(\gamma_1\boldsymbol{\psi}_{2,-1} - \gamma_2\boldsymbol{\psi}_{1,-1} - \gamma_2\psi_1\boldsymbol{\lambda} + \gamma_1\psi_2\boldsymbol{\lambda}\big)
\end{aligned}$$

because $\boldsymbol{\iota}_0^+(G) = \bar{\boldsymbol{A}}\boldsymbol{X}_2\boldsymbol{\lambda} - \bar{\boldsymbol{C}}\boldsymbol{X}_2\boldsymbol{\lambda}$. We have

$$\phi_1 + \psi_1 - \phi_2 - \psi_2 = 0 \tag{D.177}$$

$$(1+\gamma_2)\boldsymbol{\phi}_{1,-1} - (1+\gamma_1)\boldsymbol{\phi}_{2,-1} = \boldsymbol{0}_{K-1} \tag{D.178}$$

$$\gamma_1\boldsymbol{\phi}_{2,-1} - \gamma_2\boldsymbol{\phi}_{1,-1} + (1+\gamma_1)\psi_2\boldsymbol{\lambda} - (1+\gamma_2)\psi_1\boldsymbol{\lambda} = \boldsymbol{0}_{K-1} \tag{D.179}$$

$$(1+\gamma_2)\boldsymbol{\psi}_{1,-1} - (1+\gamma_1)\boldsymbol{\psi}_{2,-1} - (1+\gamma_1)\psi_2\boldsymbol{\lambda} + (1+\gamma_2)\psi_1\boldsymbol{\lambda} = \boldsymbol{0}_{K-1} \tag{D.180}$$

$$\gamma_2\psi_1\boldsymbol{\lambda} - \gamma_1\psi_2\boldsymbol{\lambda} = \mathbf{0}_{K-1} \tag{D.181}$$
$$\gamma_1\boldsymbol{\psi}_{2,-1} - \gamma_2\boldsymbol{\psi}_{1,-1} - \gamma_2\psi_1\boldsymbol{\lambda} + \gamma_1\psi_2\boldsymbol{\lambda} = \mathbf{0}_{K-1} \tag{D.182}$$

because matrix (1.79) has full column rank. Adding equations (D.179) and (D.180) gives

$$\gamma_1\boldsymbol{\phi}_{2,-1} + (1+\gamma_2)\boldsymbol{\psi}_{1,-1} - \gamma_2\boldsymbol{\phi}_{1,-1} - (1+\gamma_1)\boldsymbol{\psi}_{2,-1} = \mathbf{0}_{K-1}, \tag{D.183}$$

and adding equations (D.181) and (D.182) gives

$$\gamma_1\boldsymbol{\psi}_{2,-1} - \gamma_2\boldsymbol{\psi}_{1,-1} = \mathbf{0}_{K-1}. \tag{D.184}$$

The system of equations (D.178), (D.183), and (D.184) has a unique solution, that is, $(\gamma_1, \boldsymbol{\phi}_{1,-1}, \boldsymbol{\psi}_{1,-1}) = (\gamma_2, \boldsymbol{\phi}_{2,-1}, \boldsymbol{\psi}_{2,-1})$, because it is equivalent to the system of equations (D.171), (D.172), and (D.173) (see the proof of Proposition 1.86). Note that the system of equations (D.178), (D.183), and (D.184) has infinitely many solutions without the restrictions $\gamma_1\boldsymbol{\phi}_{1,-1} + (1+\gamma_1)\boldsymbol{\psi}_{1,-1} \neq \mathbf{0}_{K-1}$ and $\gamma_2\boldsymbol{\phi}_{2,-1} + (1+\gamma_2)\boldsymbol{\psi}_{2,-1} \neq \mathbf{0}_{K-1}$ (see the proof of Proposition 1.86). Equation (D.180), $\boldsymbol{\lambda} \neq \mathbf{0}_{K-1}$, $\gamma_1 = \gamma_2$, $1+\gamma_1 > 0$ (Assumption $\mathcal{P}$-γ), and $\boldsymbol{\psi}_{1,-1} = \boldsymbol{\psi}_{2,-1}$ imply that $\psi_1 = \psi_2$. This result and (D.177) imply that $\phi_1 = \phi_2$. To sum up, I have established that $(\gamma_1, \phi_1, \boldsymbol{\phi}_{1,-1}, \psi_1, \boldsymbol{\psi}_{1,-1}) = (\gamma_2, \phi_2, \boldsymbol{\phi}_{2,-1}, \psi_2, \boldsymbol{\psi}_{2,-1})$.

The proof that $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}$, $\bar{\boldsymbol{C}}$, $\bar{\boldsymbol{A}}^2$, $\bar{\boldsymbol{A}}\bar{\boldsymbol{C}}$ are linearly independent if γ, $\boldsymbol{\phi}$, and $\boldsymbol{\psi}$ are identified through the mean by $\mathcal{P}(\Theta)$ is omitted. ∎

Proof of Proposition 1.89'

The proof is omitted because it is similar to the proof of Proposition 1.89. ∎

Proof of Proposition 1.90

In what follows, I write $\bar{\boldsymbol{A}}$ for $\bar{\boldsymbol{A}}(G)$ and $\bar{\boldsymbol{C}}$ for $\bar{\boldsymbol{C}}(H)$. Suppose γ is identified by $\mathcal{P}(\Theta)$. Let $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ be a pair of parameter points in Θ^2.

Proof of Result 1.90.1 Suppose $\mathcal{I}_0^+(H) \neq \varnothing$ and the matrix $(\mathbf{1}_n : \boldsymbol{\iota}_0^+(H) : \boldsymbol{X}_2 : \bar{\boldsymbol{C}}\boldsymbol{X}_2)$ has full column rank. We must show that $(\boldsymbol{\phi}_1, \boldsymbol{\psi}_1) = (\boldsymbol{\phi}_2, \boldsymbol{\psi}_2)$ is necessary for

$$\mathbb{E}\big(\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_1)\big) \,\big|\, \mathfrak{F}\big) = \mathbb{E}\big(\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_2)\big) \,\big|\, \mathfrak{F}\big). \tag{D.185}$$

Suppose (D.185) is true. Note that $\gamma_1 = \gamma_2$ because γ is identified by $\mathcal{P}(\Theta)$. We find that (D.185) is equivalent to

$$\big(\boldsymbol{I}_n - \gamma_1 q(\bar{\boldsymbol{A}})\big)^{-1}(\boldsymbol{X}\boldsymbol{\phi}_1 + \bar{\boldsymbol{C}}\boldsymbol{X}\boldsymbol{\psi}_1) = \big(\boldsymbol{I}_n - \gamma_2 q(\bar{\boldsymbol{A}})\big)^{-1}(\boldsymbol{X}\boldsymbol{\phi}_2 + \bar{\boldsymbol{C}}\boldsymbol{X}\boldsymbol{\psi}_2),$$

which is equivalent to

$$\boldsymbol{X}(\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2) + \bar{\boldsymbol{C}}\boldsymbol{X}(\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2) = \mathbf{0}_n$$

because $\gamma_1 = \gamma_2$, which in turn is equivalent to

$$\begin{aligned}\mathbf{0}_n = \mathbf{1}_n(\phi_1 + \psi_1 - \phi_2 - \psi_2)\\ + \boldsymbol{\iota}_0^+(H)(\psi_2 - \psi_1)\\ + \boldsymbol{X}_2(\boldsymbol{\phi}_{1,-1} - \boldsymbol{\phi}_{2,-1})\\ + \bar{\boldsymbol{C}}\boldsymbol{X}_2(\boldsymbol{\psi}_{1,-1} - \boldsymbol{\psi}_{2,-1})\end{aligned}$$

because $\boldsymbol{X} = (\mathbf{1}_n : \boldsymbol{X}_2)$ (Assumption $\mathcal{P}$-$\boldsymbol{X}$) and $\bar{\boldsymbol{C}}\mathbf{1}_n = \mathbf{1}_n - \boldsymbol{\iota}_0^+(H)$. We have

$$\phi_1 + \psi_1 - \phi_2 - \psi_2 = 0 \tag{D.186}$$
$$\psi_2 - \psi_1 = 0 \tag{D.187}$$
$$\boldsymbol{\phi}_{1,-1} - \boldsymbol{\phi}_{2,-1} = \mathbf{0}_{K-1} \tag{D.188}$$
$$\boldsymbol{\psi}_{1,-1} - \boldsymbol{\psi}_{2,-1} = \mathbf{0}_{K-1} \tag{D.189}$$

because the matrix $(\mathbf{1}_n : \boldsymbol{\iota}_0^+(H) : \boldsymbol{X}_2 : \bar{\boldsymbol{C}}\boldsymbol{X}_2)$ has full column rank. The system of equations (D.186) to (D.189) has a unique solution, that is, $(\phi_1, \boldsymbol{\phi}_{1,-1}, \psi_1, \boldsymbol{\psi}_{1,-1}) = (\phi_2, \boldsymbol{\phi}_{2,-1}, \psi_2, \boldsymbol{\psi}_{2,-1})$. ∎

Proof of Result 1.90.2 Suppose $\mathcal{I}_0^+(H) = \emptyset$ and the matrix $(\mathbf{1}_n : \boldsymbol{X}_2 : \bar{\boldsymbol{C}}\boldsymbol{X}_2)$ has full column rank. We must show that $(\boldsymbol{\phi}_{1,-1}, \boldsymbol{\psi}_{1,-1}) = (\boldsymbol{\phi}_{2,-1}, \boldsymbol{\psi}_{2,-1})$ is necessary for

$$\mathbb{E}(\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_1)) \mid \mathfrak{F}) = \mathbb{E}(\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_2)) \mid \mathfrak{F}). \tag{D.190}$$

Suppose (D.190) is true. Note that $\gamma_1 = \gamma_2$ because γ is identified by $\mathcal{P}(\Theta)$. We find that (D.190) is equivalent to

$$\mathbf{1}_n(\phi_1 + \psi_1 - \phi_2 - \psi_2) + \boldsymbol{X}_2(\boldsymbol{\phi}_{1,-1} - \boldsymbol{\phi}_{2,-1}) + \bar{\boldsymbol{C}}\boldsymbol{X}_2(\boldsymbol{\psi}_{1,-1} - \boldsymbol{\psi}_{2,-1}) = \mathbf{0}_n$$

because $\gamma_1 = \gamma_2$, $\boldsymbol{X} = (\mathbf{1}_n : \boldsymbol{X}_2)$ (Assumption $\mathcal{P}$-$\boldsymbol{X}$), and $\bar{\boldsymbol{C}}\mathbf{1}_n = \mathbf{1}_n$. We have

$$\phi_1 + \psi_1 - \phi_2 - \psi_2 = 0$$
$$\boldsymbol{\phi}_{1,-1} - \boldsymbol{\phi}_{2,-1} = \mathbf{0}_{K-1} \tag{D.191}$$
$$\boldsymbol{\psi}_{1,-1} - \boldsymbol{\psi}_{2,-1} = \mathbf{0}_{K-1} \tag{D.192}$$

because the matrix $(\mathbf{1}_n : \boldsymbol{X}_2 : \bar{\boldsymbol{C}}\boldsymbol{X}_2)$ has full column rank. The system of equations (D.191) and (D.192) has a unique solution, that is, $(\boldsymbol{\phi}_{1,-1}, \boldsymbol{\psi}_{1,-1}) = (\boldsymbol{\phi}_{2,-1}, \boldsymbol{\psi}_{2,-1})$. ∎

Proof of Proposition 1.90'

The proof is omitted because it is similar to the proof of Proposition 1.90. ∎

Proof of Proposition 1.91

In what follows, I write $\bar{A}$ for $\bar{A}(G)$. Suppose $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top$, $\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\top$ are linearly independent. Let $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ be a pair of parameter points in Θ_0^2. We must show that $(\gamma_1, \varsigma_1) = (\gamma_2, \varsigma_2)$ is necessary for

$$\operatorname{var}(\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_1)) \mid \mathfrak{F}) = \operatorname{var}(\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_2)) \mid \mathfrak{F}). \tag{D.193}$$

Suppose (D.193) is true. Using Lemma B.4, we find that (D.193) is equivalent to

$$\varsigma_1^2 \big(\boldsymbol{I}_n - \gamma_2 q(\bar{\boldsymbol{A}})\big)\big(\boldsymbol{I}_n - \gamma_2 q(\bar{\boldsymbol{A}})^\top\big) = \varsigma_2^2 \big(\boldsymbol{I}_n - \gamma_1 q(\bar{\boldsymbol{A}})\big)\big(\boldsymbol{I}_n - \gamma_1 q(\bar{\boldsymbol{A}})^\top\big),$$

which in turn is equivalent to

$$\begin{aligned}\boldsymbol{O}_n = {} & \big((1+\gamma_2)^2\varsigma_1^2 - (1+\gamma_1)^2\varsigma_2^2\big)\boldsymbol{I}_n \\ & + \big(\gamma_1(1+\gamma_1)\varsigma_2^2 - \gamma_2(1+\gamma_2)\varsigma_1^2\big)\big(\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top\big) \\ & + \big(\gamma_2^2\varsigma_1^2 - \gamma_1^2\varsigma_2^2\big)\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\top.\end{aligned}$$

We have

$$(1+\gamma_2)^2\varsigma_1^2 - (1+\gamma_1)^2\varsigma_2^2 = 0 \tag{D.194}$$
$$\gamma_1(1+\gamma_1)\varsigma_2^2 - \gamma_2(1+\gamma_2)\varsigma_1^2 = 0 \tag{D.195}$$
$$\gamma_2^2\varsigma_1^2 - \gamma_1^2\varsigma_2^2 = 0 \tag{D.196}$$

because $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top$, $\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\top$ are linearly independent. The system of equations (D.194), (D.195), and (D.196) has a unique solution. Adding equations (D.195) and (D.196) gives

$$\gamma_1\varsigma_2^2 - \gamma_2\varsigma_1^2 = 0. \tag{D.197}$$

Adding equations (D.194), (D.195), and (D.197) gives $\varsigma_1^2 = \varsigma_2^2$, which is equivalent to $\varsigma_1 = \varsigma_2$ because $\varsigma_1 > 0$ and $\varsigma_2 > 0$. This equality together with equation (D.197) gives $(\gamma_1 - \gamma_2)\varsigma_1^2 = 0$, from which $\gamma_1 = \gamma_2$ follows because $\varsigma_1 > 0$. To sum up, I have established that $(\gamma_1, \varsigma_1) = (\gamma_2, \varsigma_2)$.

The proof that $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top$, $\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\top$ are linearly independent if γ and ς are identified through the variance by $\mathcal{P}(\Theta_0)$ is omitted. ■

Proof of Lemma 1.92

In what follows, I write $\bar{\boldsymbol{A}}$ for $\bar{\boldsymbol{A}}(G)$. Note that, for all $i \in \mathcal{I}$,

$$[\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top]_{i,i} = \begin{cases} 2 & \text{if } \deg_G^+(i) = 0, \\ 0 & \text{if } \deg_G^+(i) > 0, \end{cases}$$

and

$$[\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\top]_{i,i} = \begin{cases} 1 & \text{if } \deg_G^+(i) = 0, \\ \dfrac{1}{\deg_G^+(i)} & \text{if } \deg_G^+(i) > 0, \end{cases}$$

and

$$[\bar{\boldsymbol{A}}^\top\bar{\boldsymbol{A}}]_{i,i} = \begin{cases} \mathbb{1}_{\{0\}}\big(\deg_G^+(i)\big) & \text{if } \deg_G^-(i) = 0, \\ \mathbb{1}_{\{0\}}\big(\deg_G^+(i)\big) + \displaystyle\sum_{j \in \mathcal{N}_G^-(i)} \frac{1}{\deg_G^+(j)^2} & \text{if } \deg_G^-(i) > 0. \end{cases}$$

The statement of the lemma is equivalent to the following equivalence: $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top$, $\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\top$ are linearly dependent if and only if $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top$, $\bar{\boldsymbol{A}}^\top\bar{\boldsymbol{A}}$ are linearly dependent. I prove only one implication, namely, $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top$, $\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\top$ are linearly dependent if $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top$, $\bar{\boldsymbol{A}}^\top\bar{\boldsymbol{A}}$ are linearly dependent; the proof of the converse is similar. Suppose $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top$, $\bar{\boldsymbol{A}}^\top\bar{\boldsymbol{A}}$ are linearly dependent. I show that $\bar{\boldsymbol{A}}$ is normal, that is, $\bar{\boldsymbol{A}}^\top\bar{\boldsymbol{A}} = \bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\top$, from which it follows that $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top$, $\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\top$ are linearly dependent. There exists a triple $(c_0, c_1, c_2) \neq (0,0,0)$ such that

$$c_0\boldsymbol{I}_n + c_1\big(\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top\big) + c_2\bar{\boldsymbol{A}}^\top\bar{\boldsymbol{A}} = \boldsymbol{O}_n. \tag{D.198}$$

There exists an $i \in \mathcal{I}$ with $\deg_G^-(i) > 0$ because G is not empty. Let $a := [\bar{\boldsymbol{A}}^\top\bar{\boldsymbol{A}}]_{i,i}$. Note that $a > 0$ because $\deg_G^-(i) > 0$ is equivalent to $\mathcal{N}_G^-(i) \neq \emptyset$, which implies that there exists a $j \in \mathcal{N}_G^-(i)$ with $\deg_G^+(j) > 0$. We must have $\deg_G^+(i) > 0$. To see this, suppose, for the sake of contradiction, $\deg_G^+(i) = 0$. It follows from (D.198) that $c_0[\boldsymbol{I}_n]_{i,i} + c_1[\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top]_{i,i} + c_2[\bar{\boldsymbol{A}}^\top\bar{\boldsymbol{A}}]_{i,i} = 0$, which is equivalent to $c_0 + 2c_1 + ac_2 = 0$, and $c_0\boldsymbol{1}_n^\top\boldsymbol{1}_n + c_1\boldsymbol{1}_n^\top(\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top)\boldsymbol{1}_n + c_2\boldsymbol{1}_n^\top\bar{\boldsymbol{A}}^\top\bar{\boldsymbol{A}}\boldsymbol{1}_n = 0$, which is equivalent to $c_0 + 2c_1 + c_2 = 0$. The two equalities $c_0 + 2c_1 + ac_2 = 0$ and $c_0 + 2c_1 + c_2 = 0$ imply that $ac_2 = c_2$, which is equivalent to $c_2 = 0$ because $a > 0$. It follows from $c_0 + 2c_1 = 0$ that $c_0 = 0$ if and only if $c_1 = 0$. I show that $c_0 \neq 0$, that is, $c_1 \neq 0$. Suppose, for the sake of contradiction, $c_0 = 0$, which implies that $c_1 = 0$. Consequently, $(c_0, c_1, c_2) = (0,0,0)$, which contradicts $(c_0, c_1, c_2) \neq (0,0,0)$. This concludes the proof that $c_0 \neq 0$. It follows from (D.198) and $c_2 = 0$ that $c_0\boldsymbol{I}_n + c_1(\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top) = \boldsymbol{O}_n$, which is equivalent to $\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top = 2\boldsymbol{I}_n$ because $c_0 + 2c_1 = 0$ and $c_0 \neq 0$ and $c_1 \neq 0$, which in turn is equivalent to $\bar{\boldsymbol{A}} = \boldsymbol{I}_n$ because $\bar{\boldsymbol{A}}$ is nonnegative, which contradicts the assumption that G is not empty. This concludes the proof that $\deg_G^+(i) > 0$. It follows from $c_0[\boldsymbol{I}_n]_{i,i} + c_1[\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top]_{i,i} + c_2[\bar{\boldsymbol{A}}^\top\bar{\boldsymbol{A}}]_{i,i} = 0$ and $\deg_G^+(i) > 0$ that

$$-c_2 = a^{-1}c_0. \tag{D.199}$$

It follows from (D.199) and $a^{-1} > 0$ that $c_0 = 0$ if and only if $c_2 = 0$. I show that $c_0 \neq 0$, that is, $c_2 \neq 0$. Suppose, for the sake of contradiction, $c_0 = 0$, which implies that $c_2 = 0$. It follows from (D.198) that $c_1(\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top) = \boldsymbol{O}_n$, which implies that $c_1 = 0$ because $\bar{\boldsymbol{A}}$ is nonnegative and different from $\boldsymbol{O}_n$. Consequently, $(c_0, c_1, c_2) = (0,0,0)$, which contradicts $(c_0, c_1, c_2) \neq (0,0,0)$. This concludes the proof that $c_0 \neq 0$. Let

$$c_3 := \frac{c_1^2 - c_0c_2}{c_2^2} = \frac{c_1^2 + a^{-1}c_0^2}{c_2^2} > 0 \quad \text{and} \quad \boldsymbol{N} := \bar{\boldsymbol{A}} + \frac{c_1}{c_2}\boldsymbol{I}_n.$$

Note that (D.198) is equivalent to $\boldsymbol{N}\boldsymbol{N}^\top = c_3\boldsymbol{I}_n$. I show that $\boldsymbol{N}$ is nonsingular and normal. First, I show that $\boldsymbol{N}$ is nonsingular. We have $\det(\boldsymbol{N}) \neq 0$ because $0 \neq c_3^n = \det(c_3\boldsymbol{I}_n) = \det(\boldsymbol{N}\boldsymbol{N}^\top) = \det(\boldsymbol{N})^2$. Second, I show that $\boldsymbol{N}$ is normal. We have

$$\boldsymbol{N}^\top\boldsymbol{N} = \boldsymbol{N}^{-1}(\boldsymbol{N}\boldsymbol{N}^\top)\boldsymbol{N} = \boldsymbol{N}^{-1}(c_3\boldsymbol{I}_n)\boldsymbol{N} = c_3\boldsymbol{I}_n = \boldsymbol{N}\boldsymbol{N}^\top,$$

that is, $\boldsymbol{N}$ is normal. Finally, we find

$$\boldsymbol{O}_n = \boldsymbol{N}^\top\boldsymbol{N} - \boldsymbol{N}\boldsymbol{N}^\top$$

$$
\begin{aligned}
&= \left(\bar{\boldsymbol{A}} + \frac{c_1}{c_2}\boldsymbol{I}_n\right)^\intercal \left(\bar{\boldsymbol{A}} + \frac{c_1}{c_2}\boldsymbol{I}_n\right) - \left(\bar{\boldsymbol{A}} + \frac{c_1}{c_2}\boldsymbol{I}_n\right)\left(\bar{\boldsymbol{A}} + \frac{c_1}{c_2}\boldsymbol{I}_n\right)^\intercal \\
&= \bar{\boldsymbol{A}}^\intercal \bar{\boldsymbol{A}} - \bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\intercal,
\end{aligned}
$$

that is, $\bar{\boldsymbol{A}}$ is normal. ∎

Proof of Proposition 1.95

In what follows, I write $\mathcal{I}^\star$ for $\mathcal{I}^\star(G)$ and $\bar{\boldsymbol{A}}$ for $\bar{\boldsymbol{A}}(G)$. Note that, for all $(i,j) \in \mathcal{I}^2$,

$$
[\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\intercal]_{i,j} = \begin{cases}
2\delta_{i,j} & \text{if } \deg_G^+(i) = 0 \text{ and } \deg_G^+(j) = 0, \\
\delta_{i,j} + \dfrac{\mathbb{1}_{\mathcal{N}_G^+(i)}(j)}{\deg_G^+(i)} & \text{if } \deg_G^+(i) > 0 \text{ and } \deg_G^+(j) = 0, \\
\delta_{i,j} + \dfrac{\mathbb{1}_{\mathcal{N}_G^+(j)}(i)}{\deg_G^+(j)} & \text{if } \deg_G^+(i) = 0 \text{ and } \deg_G^+(j) > 0, \\
\dfrac{\mathbb{1}_{\mathcal{N}_G^+(i)}(j)}{\deg_G^+(i)} + \dfrac{\mathbb{1}_{\mathcal{N}_G^+(j)}(i)}{\deg_G^+(j)} & \text{if } \deg_G^+(i) > 0 \text{ and } \deg_G^+(j) > 0,
\end{cases}
$$

and

$$
[\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\intercal]_{i,j} = \begin{cases}
\delta_{i,j} & \text{if } \deg_G^+(i) = 0 \text{ and } \deg_G^+(j) = 0, \\
\dfrac{\mathbb{1}_{\mathcal{N}_G^+(i)}(j)}{\deg_G^+(i)} & \text{if } \deg_G^+(i) > 0 \text{ and } \deg_G^+(j) = 0, \\
\dfrac{\mathbb{1}_{\mathcal{N}_G^+(j)}(i)}{\deg_G^+(j)} & \text{if } \deg_G^+(i) = 0 \text{ and } \deg_G^+(j) > 0, \\
\dfrac{\left|\mathcal{N}_G^+(i) \cap \mathcal{N}_G^+(j)\right|}{\deg_G^+(i)\deg_G^+(j)} & \text{if } \deg_G^+(i) > 0 \text{ and } \deg_G^+(j) > 0,
\end{cases}
$$

and

$$
[\bar{\boldsymbol{A}}^\intercal \bar{\boldsymbol{A}}]_{i,j} = \begin{cases}
\delta_{i,j}\,\mathbb{1}_{\{0\}}\left(\deg_G^+(i)\right) & \text{if } \mathcal{N}_G^-(i) \cap \mathcal{N}_G^-(j) = \emptyset, \\
\delta_{i,j}\,\mathbb{1}_{\{0\}}\left(\deg_G^+(i)\right) + \displaystyle\sum_{k \in \mathcal{N}_G^-(i) \cap \mathcal{N}_G^-(j)} \frac{1}{\deg_G^+(k)^2} & \text{if } \mathcal{N}_G^-(i) \cap \mathcal{N}_G^-(j) \neq \emptyset.
\end{cases}
$$

Let $n^\star := |\mathcal{I}^\star|$. Note that $n^\star > 1$ and $G\langle \mathcal{I}^\star \rangle$ is not empty because G is not empty. Assume without loss of generality that $\mathcal{I}^\star = \{1, \ldots, n^\star\}$. It follows that $\bar{\boldsymbol{A}}$ is block diagonal if $n^\star < n$, specifically,

$$
\bar{\boldsymbol{A}} = \begin{cases}
\begin{pmatrix} [\bar{\boldsymbol{A}}]_{\mathcal{I}^\star,\mathcal{I}^\star} & 0 \\ 0 & 1 \end{pmatrix} & \text{if } n^\star = n-1, \\
\begin{pmatrix} [\bar{\boldsymbol{A}}]_{\mathcal{I}^\star,\mathcal{I}^\star} & \boldsymbol{0}_{n^\star}\boldsymbol{0}_{n-n^\star}^\intercal \\ \boldsymbol{0}_{n-n^\star}\boldsymbol{0}_{n^\star}^\intercal & \boldsymbol{I}_{n-n^\star} \end{pmatrix} & \text{if } n^\star < n-1.
\end{cases} \tag{D.200}
$$

First, I prove that if $G\langle\mathcal{I}^\star\rangle$ is an NRD$(n^\star, d, (d-1)/2, 0)$ for some positive integer d, then $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top$, $\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\top$ are linearly dependent. Suppose $G\langle\mathcal{I}^\star\rangle$ is an NRD$(n^\star, d, (d-1)/2, 0)$ for some positive integer d, which implies that $G\langle\mathcal{I}^\star\rangle$ is d-regular (see Jørgensen 2015, Corollary 11). The adjacency matrix of $G\langle\mathcal{I}^\star\rangle$ with respect to the identity mapping $\mathrm{id}_{\mathcal{I}^\star}$, $\dot{\boldsymbol{A}}(G\langle\mathcal{I}^\star\rangle)$, satisfies (see Proposition 1)

$$\dot{\boldsymbol{A}}(G\langle\mathcal{I}^\star\rangle)\dot{\boldsymbol{A}}(G\langle\mathcal{I}^\star\rangle)^\top = d\boldsymbol{I}_{n^\star} + \frac{d-1}{2}\left(\dot{\boldsymbol{A}}(G\langle\mathcal{I}^\star\rangle) + \dot{\boldsymbol{A}}(G\langle\mathcal{I}^\star\rangle)^\top\right). \tag{D.201}$$

Note that the adjacency matrix of G with respect to the identity mapping $\mathrm{id}_{\mathcal{I}}$, $\dot{\boldsymbol{A}}(G)$, satisfies $[\dot{\boldsymbol{A}}(G)]_{\mathcal{I}^\star,\mathcal{I}^\star} = \dot{\boldsymbol{A}}(G\langle\mathcal{I}^\star\rangle)$. Note also that $[\bar{\boldsymbol{A}}]_{\mathcal{I}^\star,\mathcal{I}^\star} = (1/d)[\dot{\boldsymbol{A}}(G)]_{\mathcal{I}^\star,\mathcal{I}^\star}$ because $G\langle\mathcal{I}^\star\rangle$ is d-regular. It follows from (D.201) and $\dot{\boldsymbol{A}}(G\langle\mathcal{I}^\star\rangle) = d[\bar{\boldsymbol{A}}]_{\mathcal{I}^\star,\mathcal{I}^\star}$ that

$$d\boldsymbol{I}_{n^\star} + \frac{d(d-1)}{2}\left([\bar{\boldsymbol{A}}]_{\mathcal{I}^\star,\mathcal{I}^\star} + [\bar{\boldsymbol{A}}]_{\mathcal{I}^\star,\mathcal{I}^\star}^\top\right) - d^2[\bar{\boldsymbol{A}}]_{\mathcal{I}^\star,\mathcal{I}^\star}[\bar{\boldsymbol{A}}]_{\mathcal{I}^\star,\mathcal{I}^\star}^\top = \boldsymbol{O}_{n^\star}. \tag{D.202}$$

This shows that $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top$, $\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\top$ are linearly dependent if $n^\star = n$, that is, $[\bar{\boldsymbol{A}}]_{\mathcal{I}^\star,\mathcal{I}^\star} = \bar{\boldsymbol{A}}$. If $n^\star < n$, then (D.200), (D.202), and $d + (d(d-1)/2)(1+1) - d^2 = 0$ in case $n^\star = n-1$ or $d\boldsymbol{I}_{n-n^\star} + (d(d-1)/2)(\boldsymbol{I}_{n-n^\star} + \boldsymbol{I}_{n-n^\star}) - d^2\boldsymbol{I}_{n-n^\star} = \boldsymbol{O}_{n-n^\star}$ in case $n^\star < n-1$ imply that

$$d\boldsymbol{I}_n + \frac{d(d-1)}{2}\left(\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top\right) - d^2\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\top = \boldsymbol{O}_n,$$

that is, $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top$, $\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\top$ are linearly dependent.

Second, I prove that if $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top$, $\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\top$ are linearly dependent, then there exists a positive integer d—to economize on notation, I reuse the symbol d introduced in the preceding proof—such that

$$\dot{\boldsymbol{A}}(G\langle\mathcal{I}^\star\rangle)\dot{\boldsymbol{A}}(G\langle\mathcal{I}^\star\rangle)^\top = d\boldsymbol{I}_{n^\star} + \frac{d-1}{2}\left(\dot{\boldsymbol{A}}(G\langle\mathcal{I}^\star\rangle) + \dot{\boldsymbol{A}}(G\langle\mathcal{I}^\star\rangle)^\top\right), \tag{D.203}$$

that is, $G\langle\mathcal{I}^\star\rangle$ is an NRD$(n^\star, d, (d-1)/2, 0)$ (see Proposition 1). Suppose $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top$, $\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\top$ are linearly dependent. There exists a triple $(c_0, c_1, c_2) \neq (0,0,0)$ such that

$$c_0\boldsymbol{I}_n + c_1\left(\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top\right) + c_2\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\top = \boldsymbol{O}_n. \tag{D.204}$$

There exists an $x \in \mathcal{I}^\star$ with $\deg_G^+(x) > 0$ because G is not empty.

First, I show that $c_0 \neq 0$ and $c_2 \neq 0$. It follows from (D.204) that $c_0[\boldsymbol{I}_n]_{x,x} + c_1[\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top]_{x,x} + c_2[\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\top]_{x,x} = 0$, which is equivalent to

$$c_0\deg_G^+(x) + c_2 = 0. \tag{D.205}$$

It follows from (D.205) and $\deg_G^+(x) > 0$ that $c_0 = 0$ if and only if $c_2 = 0$. I show that $c_0 \neq 0$, that is, $c_2 \neq 0$. Suppose, for the sake of contradiction, $c_0 = 0$, which implies that $c_2 = 0$. It follows from (D.204) that $c_1(\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top) = \boldsymbol{O}_n$, which implies that $c_1 = 0$ because $\bar{\boldsymbol{A}}$ is nonnegative and different from $\boldsymbol{O}_n$. Consequently, $(c_0, c_1, c_2) = (0,0,0)$, which contradicts $(c_0, c_1, c_2) \neq (0,0,0)$. This concludes the proof that $c_0 \neq 0$ and $c_2 \neq 0$.

Second, I show that $\bar{A}$ is normal, that is, $\bar{A}\bar{A}^\top = \bar{A}^\top\bar{A}$. Let

$$c_3 := \frac{c_1^2 - c_0 c_2}{c_2^2} = \frac{c_1^2 + c_0^2 \deg_G^+(x)}{c_2^2} > 0 \quad \text{and} \quad N := \bar{A} + \frac{c_1}{c_2} I_n.$$

Note that (D.204) is equivalent to $NN^\top = c_3 I_n$. I show that N is nonsingular and normal. First, I show that N is nonsingular. We have $\det(N) \neq 0$ because $0 < c_3^n = \det(c_3 I_n) = \det(NN^\top) = \det(N)^2$. Second, I show that N is normal. We have

$$N^\top N = N^{-1}(NN^\top)N = N^{-1}(c_3 I_n)N = c_3 I_n = NN^\top,$$

that is, N is normal. Finally, we find

$$\begin{aligned} O_n &= N^\top N - NN^\top \\ &= \left(\bar{A} + \frac{c_1}{c_2} I_n\right)^\top \left(\bar{A} + \frac{c_1}{c_2} I_n\right) - \left(\bar{A} + \frac{c_1}{c_2} I_n\right)\left(\bar{A} + \frac{c_1}{c_2} I_n\right)^\top \\ &= \bar{A}^\top \bar{A} - \bar{A}\bar{A}^\top, \end{aligned}$$

that is, $\bar{A}$ is normal.

Third, I show that, for all $i \in \mathcal{I}^\star$, $\deg_G^+(i) > 0$. Suppose, for the sake of contradiction, there exists a $z \in \mathcal{I}^\star$ with $\deg_G^+(z) = 0$. There exists a $y \in \mathcal{I}^\star$ with $z \in \mathcal{N}_G^+(y)$ because $\deg_G^+(z) = 0$ and z lies in some weakly connected component of $G\langle \mathcal{I}^\star \rangle$ of order at least 2. Note that $y \neq z$ and $\deg_G^+(y) > 0$ because $z \in \mathcal{N}_G^+(y)$. It follows from (D.204) that

$$c_0 \deg_G^+(y) + c_2 = 0. \tag{D.206}$$

We have

$$\begin{aligned} & & c_0 [I_n]_{y,z} + c_1 [\bar{A} + \bar{A}^\top]_{y,z} + c_2 [\bar{A}\bar{A}^\top]_{y,z} &= 0 \\ &\Leftrightarrow & c_1 \frac{\mathbb{1}_{\mathcal{N}_G^+(y)}(z)}{\deg_G^+(y)} + c_2 \frac{\mathbb{1}_{\mathcal{N}_G^+(y)}(z)}{\deg_G^+(y)} &= 0 \\ &\Leftrightarrow & c_1 + c_2 &= 0. \end{aligned} \tag{D.207}$$

Since $\bar{A}$ is normal, (D.204) is equivalent to $c_0 I_n + c_1(\bar{A} + \bar{A}^\top) + c_2 \bar{A}^\top \bar{A} = O_n$, which implies that $c_0 \mathbf{1}_n^\top \mathbf{1}_n + c_1 \mathbf{1}_n^\top (\bar{A} + \bar{A}^\top)\mathbf{1}_n + c_2 \mathbf{1}_n^\top \bar{A}^\top \bar{A} \mathbf{1}_n = 0$, which is equivalent to

$$c_0 + 2c_1 + c_2 = 0. \tag{D.208}$$

It follows from (D.207) and (D.208) that $c_0 = c_2$. This equality, $c_0 \neq 0$, and (D.206) give $\deg_G^+(y) = -1$, which contradicts $\deg_G^+(y) > 0$. This concludes the proof that for all $i \in \mathcal{I}^\star$, $\deg_G^+(i) > 0$.

Fourth, I show that, for all $i \in \mathcal{I}^\star$, $\deg_G^-(i) > 0$. Suppose, for the sake of contradiction, there exists a $y \in \mathcal{I}^\star$ with $\deg_G^-(y) = 0$, that is, $\mathcal{N}_G^-(y) = \emptyset$. Since $\bar{A}$ is normal and $\deg_G^+(y) > 0$, (D.204) implies that $c_0[I_n]_{y,y} + c_1[\bar{A} + \bar{A}^\top]_{y,y} + c_2[\bar{A}^\top \bar{A}]_{y,y} = 0$, which is equivalent to $c_0 = 0$, a contradiction to $c_0 \neq 0$. This concludes the proof that for all $i \in \mathcal{I}^\star$, $\deg_G^-(i) > 0$.

Fifth, I show that there exists a positive integer d such that for all $i \in \mathcal{I}^\star$, $\deg_G^-(i) = \deg_G^+(i) = d$, that is, $G\langle\mathcal{I}^\star\rangle$ is d-regular. Let $(y,z) \in (\mathcal{I}^\star)^2$ with $y \neq z$. It follows from (D.204) that $c_0\deg_G^+(y) + c_2 = 0$ and $c_0\deg_G^+(z) + c_2 = 0$ because $\deg_G^+(y) > 0$ and $\deg_G^+(z) > 0$. The inequality $c_0 \neq 0$ and the two equalities $c_0\deg_G^+(y) + c_2 = 0$ and $c_0\deg_G^+(z) + c_2 = 0$ imply that $\deg_G^+(y) = \deg_G^+(z)$. Let $d := \deg_G^+(y) > 0$. Since $\bar{\boldsymbol{A}}$ is normal, we find, for all $i \in \mathcal{I}^\star$,

$$\frac{\deg_G^-(i)}{d^2} = \frac{|\mathcal{N}_G^-(i)|}{d^2} = \sum_{k\in\mathcal{N}_G^-(i)} \frac{1}{\deg_G^+(k)^2} = [\bar{\boldsymbol{A}}^\top\bar{\boldsymbol{A}}]_{i,i} = [\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\top]_{i,i} = \frac{|\mathcal{N}_G^+(i)|}{\deg_G^+(i)^2} = \frac{d}{d^2},$$

that is, $\deg_G^-(i) = d$. This concludes the proof that $G\langle\mathcal{I}^\star\rangle$ is d-regular.

Sixth, I show that (D.203) is true. It follows from $[\bar{\boldsymbol{A}}]_{\mathcal{I}^\star,\mathcal{I}^\star} = (1/d)\dot{\boldsymbol{A}}(G\langle\mathcal{I}^\star\rangle)$ and (D.204) that

$$c_0\boldsymbol{I}_{n^\star} + \frac{c_1}{d}\left(\dot{\boldsymbol{A}}(G\langle\mathcal{I}^\star\rangle) + \dot{\boldsymbol{A}}(G\langle\mathcal{I}^\star\rangle)^\top\right) + \frac{c_2}{d^2}\dot{\boldsymbol{A}}(G\langle\mathcal{I}^\star\rangle)\dot{\boldsymbol{A}}(G\langle\mathcal{I}^\star\rangle)^\top = \boldsymbol{O}_{n^\star},$$

which is equivalent to (D.203) because $c_2 \neq 0$, $c_0 d + c_2 = 0$, and (D.208) imply that

$$-\frac{c_0 d^2}{c_2} = d \quad \text{and} \quad -\frac{c_1 d}{c_2} = \frac{d-1}{2}. \qquad \blacksquare$$

Proof of Corollary 1.96

Results 1.96.1 and 1.96.2 follow from Proposition 1.95. ■

Proof of Proposition 1.97

In what follows, I write $\mathcal{I}^\star$ for $\mathcal{I}^\star(G)$ and $\bar{\boldsymbol{A}}$ for $\bar{\boldsymbol{A}}(G)$. Suppose G is symmetric, which implies that $G\langle\mathcal{I}^\star\rangle$ is symmetric. Note that every weakly connected component of $G\langle\mathcal{I}^\star\rangle$ is of order at least 2.

First, I prove that $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}$, $\bar{\boldsymbol{A}}^2$ are linearly dependent if $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top$, $\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\top$ are linearly dependent. Suppose $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top$, $\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\top$ are linearly dependent. It follows that $G\langle\mathcal{I}^\star\rangle$ is an NRD$(|\mathcal{I}^\star|, d, (d-1)/2, 0)$ for some positive integer d (Proposition 1.95) and $\bar{\boldsymbol{A}}$ satisfies (see the proof of Proposition 1.95)

$$d\boldsymbol{I}_n + \frac{d(d-1)}{2}(\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top) - d^2\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\top = \boldsymbol{O}_n. \tag{D.209}$$

Note that $\bar{\boldsymbol{A}}$ is symmetric, that is, $\bar{\boldsymbol{A}} = \bar{\boldsymbol{A}}^\top$, because $G\langle\mathcal{I}^\star\rangle$ is symmetric and d-regular. It follows from (D.209) and $\bar{\boldsymbol{A}} = \bar{\boldsymbol{A}}^\top$ that $d\boldsymbol{I}_n + d(d-1)\bar{\boldsymbol{A}} - d^2\bar{\boldsymbol{A}}^2 = \boldsymbol{O}_n$, that is, $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}$, $\bar{\boldsymbol{A}}^2$ are linearly dependent.

Second, I prove that $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top$, $\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\top$ are linearly dependent if $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}$, $\bar{\boldsymbol{A}}^2$ are linearly dependent. Suppose $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}$, $\bar{\boldsymbol{A}}^2$ are linearly dependent.

First, I show that every weakly connected component of $G\langle\mathcal{I}^\star\rangle$ is transitive. There do not exist intransitive triples in $G\langle\mathcal{I}^\star\rangle$ because $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}$, $\bar{\boldsymbol{A}}^2$ are linearly dependent (Lemma 1.87). It follows that $G\langle\mathcal{I}^\star\rangle$ is transitive, which in turn implies that all of its weakly connected components are transitive.

Second, I show that all weakly connected components of $G\langle\mathcal{I}^\star\rangle$ are complete. All weakly connected components of $G\langle\mathcal{I}^\star\rangle$ are strongly connected because $G\langle\mathcal{I}^\star\rangle$ is symmetric. Note that a strongly connected digraph D is transitive if and only if D is complete (see Bang-Jensen and Gutin 2009, p. 37). It follows that all weakly connected components of $G\langle\mathcal{I}^\star\rangle$ are complete.

Third, I show that $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}+\bar{\boldsymbol{A}}^\top$, $\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\top$ are linearly dependent. All complete components of $G\langle\mathcal{I}^\star\rangle$ must be of the same order, denoted by m, because $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}$, $\bar{\boldsymbol{A}}^2$ are linearly dependent (see Bramoullé, Djebbari, and Fortin 2009, Section 2.4.1.3). It follows that $G\langle\mathcal{I}^\star\rangle$ is an $\mathrm{NRD}(|\mathcal{I}^\star|, m-1, (m-2)/2, 0)$, from which it follows that $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}+\bar{\boldsymbol{A}}^\top$, $\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\top$ are linearly dependent (Proposition 1.95). ■

Proof of Corollary 1.98

The statement of the corollary follows from Propositions 1.95 and 1.97. ■

Proof of Corollary 1.99

The statement of the corollary follows from Propositions 1.86, 1.91, and 1.97. ■

Proof of Proposition 1.100

In what follows, I write $\bar{\boldsymbol{C}}$ for $\bar{\boldsymbol{C}}(H)$. Suppose $\boldsymbol{I}_n$, $\bar{\boldsymbol{C}}+\bar{\boldsymbol{C}}^\top$, $\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top$ are linearly independent and γ is identified by $\mathcal{P}(\Theta)$. Let $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ be a pair of parameter points in Θ^2. We must show that $(\zeta_1, \varsigma_1) = (\zeta_2, \varsigma_2)$ is necessary for

$$\operatorname{var}\big(\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_1)) \mid \mathfrak{F}\big) = \operatorname{var}\big(\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_2)) \mid \mathfrak{F}\big). \tag{D.210}$$

Suppose (D.210) is true. Note that $\gamma_1 = \gamma_2$ because γ is identified by $\mathcal{P}(\Theta)$. We find that (D.210) is equivalent to

$$\varsigma_1^2\big(\boldsymbol{I}_n + \zeta_1\bar{\boldsymbol{C}}\big)\big(\boldsymbol{I}_n + \zeta_1\bar{\boldsymbol{C}}^\top\big) = \varsigma_2^2\big(\boldsymbol{I}_n + \zeta_2\bar{\boldsymbol{C}}\big)\big(\boldsymbol{I}_n + \zeta_2\bar{\boldsymbol{C}}^\top\big)$$

because $\gamma_1 = \gamma_2$, which in turn is equivalent to

$$\big(\varsigma_1^2 - \varsigma_2^2\big)\boldsymbol{I}_n + \big(\zeta_1\varsigma_1^2 - \zeta_2\varsigma_2^2\big)\big(\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top\big) + \big(\zeta_1^2\varsigma_1^2 - \zeta_2^2\varsigma_2^2\big)\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top = \boldsymbol{O}_n.$$

We have

$$\varsigma_1^2 - \varsigma_2^2 = 0 \tag{D.211}$$
$$\zeta_1\varsigma_1^2 - \zeta_2\varsigma_2^2 = 0 \tag{D.212}$$
$$\zeta_1^2\varsigma_1^2 - \zeta_2^2\varsigma_2^2 = 0 \tag{D.213}$$

because $\boldsymbol{I}_n$, $\bar{\boldsymbol{C}}+\bar{\boldsymbol{C}}^\top$, $\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top$ are linearly independent. The system of equations (D.211), (D.212), and (D.213) has a unique solution. Equation (D.211) gives $\varsigma_1^2 = \varsigma_2^2$, which is equivalent to $\varsigma_1 = \varsigma_2$ because $\varsigma_1 > 0$ and $\varsigma_2 > 0$. This equality together with equation (D.212) gives $(\zeta_1 - \zeta_2)\varsigma_1^2 = 0$, from which $\zeta_1 = \zeta_2$ follows because $\varsigma_1 > 0$. To sum up, I have established that $(\zeta_1, \varsigma_1) = (\zeta_2, \varsigma_2)$.

The proof that $\boldsymbol{I}_n$, $\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top$, $\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top$ are linearly independent if ζ and ς are identified through the variance by $\mathcal{P}(\Theta)$ is omitted. ■

Proof of Lemma 1.101

The proof is omitted because it is similar to the proof of Lemma 1.92. ■

Proof of Proposition 1.102

In what follows, I write $\bar{\boldsymbol{C}}$ for $\bar{\boldsymbol{C}}(H)$. Note that, for all $i \in \mathcal{I}$, $[\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top]_{i,i} = 0$ and

$$[\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top]_{i,i} = \begin{cases} 0 & \text{if } \deg_H^+(i) = 0, \\ \dfrac{1}{\deg_H^+(i)} & \text{if } \deg_H^+(i) > 0. \end{cases}$$

I prove the contrapositive of the conditional statement of the proposition. Suppose $\boldsymbol{I}_n$, $\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top$, $\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top$ are linearly dependent. I show that $\mathcal{I}_0^+(H) = \emptyset$. Suppose, for the sake of contradiction, $\mathcal{I}_0^+(H) \neq \emptyset$. There exists a triple $(c_0, c_1, c_2) \neq (0, 0, 0)$ such that

$$c_0\boldsymbol{I}_n + c_1\left(\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top\right) + c_2\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top = \boldsymbol{O}_n. \tag{D.214}$$

There exist an $x \in \mathcal{I}$ with $\deg_H^+(x) > 0$ (because H is not empty) and a $y \in \mathcal{I}$ with $\deg_H^+(y) = 0$ (because $\mathcal{I}_0^+(H) \neq \emptyset$). It follows from (D.214) that $c_0[\boldsymbol{I}_n]_{x,x} + c_1[\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top]_{x,x} + c_2[\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top]_{x,x} = 0$, which is equivalent to $c_0\deg_H^+(x) + c_2 = 0$, and $c_0[\boldsymbol{I}_n]_{y,y} + c_1[\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top]_{y,y} + c_2[\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top]_{y,y} = 0$, which is equivalent to $c_0 = 0$. The two equalities $c_0\deg_H^+(x) + c_2 = 0$ and $c_0 = 0$ imply that $c_2 = 0$. It follows from (D.214) and $c_0 = c_2 = 0$ that $c_1(\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top) = \boldsymbol{O}_n$, which implies that $c_1 = 0$ because $\bar{\boldsymbol{C}}$ is nonnegative and different from $\boldsymbol{O}_n$. Consequently, $(c_0, c_1, c_2) = (0, 0, 0)$, which contradicts $(c_0, c_1, c_2) \neq (0, 0, 0)$. This concludes the proof that $\mathcal{I}_0^+(H) = \emptyset$. ■

Proof of Proposition 1.103

The following proof is similar to the proof of Proposition 1.95. There are, however, subtle differences that justify a detailed proof.

In what follows, I write $\bar{\boldsymbol{C}}$ for $\bar{\boldsymbol{C}}(H)$. Note that, for all $i \in \mathcal{I}$, $[\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top]_{i,i} = 0$,

$$[\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top]_{i,i} = \begin{cases} 0 & \text{if } \deg_H^+(i) = 0, \\ \dfrac{1}{\deg_H^+(i)} & \text{if } \deg_H^+(i) > 0, \end{cases}$$

and

$$[\bar{\boldsymbol{C}}^\top \bar{\boldsymbol{C}}]_{i,i} = \begin{cases} 0 & \text{if } \mathcal{N}_H^-(i) = \emptyset, \\ \displaystyle\sum_{k \in \mathcal{N}_H^-(i)} \frac{1}{\deg_H^+(k)^2} & \text{if } \mathcal{N}_H^-(i) \neq \emptyset. \end{cases}$$

First, I prove that if H is an NRD$(n, d, (d-1)/2, 0)$ for some positive integer d, then $\boldsymbol{I}_n$, $\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top$, $\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top$ are linearly dependent. Suppose H is an NRD$(n, d, (d-1)/2, 0)$ for some positive integer d, which implies that H is d-regular (see Jørgensen 2015, Corollary 11). The adjacency matrix of H with respect to the identity mapping $\mathrm{id}_{\mathcal{I}}$, $\dot{\boldsymbol{A}}(H)$, satisfies (see Proposition 1)

$$\dot{\boldsymbol{A}}(H)\dot{\boldsymbol{A}}(H)^\top = d\boldsymbol{I}_n + \frac{d-1}{2}\left(\dot{\boldsymbol{A}}(H) + \dot{\boldsymbol{A}}(H)^\top\right). \tag{D.215}$$

Note that $\dot{\boldsymbol{A}}(H) = d\bar{\boldsymbol{C}}$ because H is d-regular. It follows from (D.215) and $\dot{\boldsymbol{A}}(H) = d\bar{\boldsymbol{C}}$ that

$$d\boldsymbol{I}_n + \frac{d(d-1)}{2}\left(\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top\right) - d^2\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top = \boldsymbol{O}_n. \tag{D.216}$$

This concludes the proof that $\boldsymbol{I}_n$, $\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top$, $\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top$ are linearly dependent.

Second, I prove that if $\boldsymbol{I}_n$, $\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top$, $\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top$ are linearly dependent, then there exists a positive integer d—to economize on notation, I reuse the symbol d introduced in the preceding proof—such that

$$\dot{\boldsymbol{A}}(H)\dot{\boldsymbol{A}}(H)^\top = d\boldsymbol{I}_n + \frac{d-1}{2}\left(\dot{\boldsymbol{A}}(H) + \dot{\boldsymbol{A}}(H)^\top\right), \tag{D.217}$$

which is equivalent (see Proposition 1) to H being an NRD$(n, d, (d-1)/2, 0)$. Suppose $\boldsymbol{I}_n$, $\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top$, $\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top$ are linearly dependent. There exists a triple $(c_0, c_1, c_2) \neq (0, 0, 0)$ such that

$$c_0\boldsymbol{I}_n + c_1\left(\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top\right) + c_2\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top = \boldsymbol{O}_n. \tag{D.218}$$

Let $x \in \mathcal{I}$. Note that $\deg_H^+(x) > 0$ because $\mathcal{I}_0^+(H) = \emptyset$ (Proposition 1.102).

First, I show that $c_0 \neq 0$ and $c_2 \neq 0$. It follows from (D.218) that $c_0[\boldsymbol{I}_n]_{x,x} + c_1[\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top]_{x,x} + c_2[\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top]_{x,x} = 0$, which is equivalent to

$$c_0\deg_H^+(x) + c_2 = 0. \tag{D.219}$$

It follows from (D.219) and $\deg_H^+(x) > 0$ that $c_0 = 0$ if and only if $c_2 = 0$. I show that $c_0 \neq 0$, that is, $c_2 \neq 0$. Suppose, for the sake of contradiction, $c_0 = 0$, which implies that $c_2 = 0$. It follows from (D.218) that $c_1(\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top) = \boldsymbol{O}_n$, which implies that $c_1 = 0$ because $\bar{\boldsymbol{C}}$ is nonnegative and different from $\boldsymbol{O}_n$. Consequently, $(c_0, c_1, c_2) = (0, 0, 0)$, which contradicts $(c_0, c_1, c_2) \neq (0, 0, 0)$. This concludes the proof that $c_0 \neq 0$ and $c_2 \neq 0$.

Second, I show that $\bar{\boldsymbol{C}}$ is normal, that is, $\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top = \bar{\boldsymbol{C}}^\top\bar{\boldsymbol{C}}$. Let

$$c_3 := \frac{c_1^2 - c_0c_2}{c_2^2} = \frac{c_1^2 + c_0^2\deg_H^+(x)}{c_2^2} > 0 \quad \text{and} \quad \boldsymbol{N} := \bar{\boldsymbol{C}} + \frac{c_1}{c_2}\boldsymbol{I}_n.$$

Note that (D.218) is equivalent to $\boldsymbol{N}\boldsymbol{N}^\top = c_3\boldsymbol{I}_n$. I show that $\boldsymbol{N}$ is nonsingular and normal. First, I show that $\boldsymbol{N}$ is nonsingular. We have $\det(\boldsymbol{N}) \neq 0$ because $0 < c_3^n = \det(c_3\boldsymbol{I}_n) = \det(\boldsymbol{N}\boldsymbol{N}^\top) = \det(\boldsymbol{N})^2$. Second, I show that $\boldsymbol{N}$ is normal. We have

$$\boldsymbol{N}^\top\boldsymbol{N} = \boldsymbol{N}^{-1}(\boldsymbol{N}\boldsymbol{N}^\top)\boldsymbol{N} = \boldsymbol{N}^{-1}(c_3\boldsymbol{I}_n)\boldsymbol{N} = c_3\boldsymbol{I}_n = \boldsymbol{N}\boldsymbol{N}^\top,$$

that is, $\boldsymbol{N}$ is normal. Finally, we find

$$\begin{aligned}\boldsymbol{O}_n &= \boldsymbol{N}^\top\boldsymbol{N} - \boldsymbol{N}\boldsymbol{N}^\top \\ &= \left(\bar{\boldsymbol{C}} + \frac{c_1}{c_2}\boldsymbol{I}_n\right)^\top\left(\bar{\boldsymbol{C}} + \frac{c_1}{c_2}\boldsymbol{I}_n\right) - \left(\bar{\boldsymbol{C}} + \frac{c_1}{c_2}\boldsymbol{I}_n\right)\left(\bar{\boldsymbol{C}} + \frac{c_1}{c_2}\boldsymbol{I}_n\right)^\top \\ &= \bar{\boldsymbol{C}}^\top\bar{\boldsymbol{C}} - \bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top,\end{aligned}$$

that is, $\bar{\boldsymbol{C}}$ is normal.

Third, I show that, for all $i \in \mathcal{I}$, $\deg_H^+(i) > 0$, that is, $\mathcal{I}_0^+(H) = \emptyset$. Suppose, for the sake of contradiction, there exists a $y \in \mathcal{I}$ with $\deg_H^+(y) = 0$. It follows from (D.218) that $c_0[\boldsymbol{I}_n]_{y,y} + c_1[\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top]_{y,y} + c_2[\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top]_{y,y} = 0$, which is equivalent to $c_0 = 0$, a contradiction to $c_0 \neq 0$. This concludes the proof that for all $i \in \mathcal{I}$, $\deg_H^+(i) > 0$.

Fourth, I show that, for all $i \in \mathcal{I}$, $\deg_H^-(i) > 0$. Suppose, for the sake of contradiction, there exists a $y \in \mathcal{I}$ with $\deg_H^-(y) = 0$, that is, $\mathcal{N}_H^-(y) = \emptyset$. Since $\bar{\boldsymbol{C}}$ is normal and $\deg_H^+(y) > 0$, (D.218) implies that $c_0[\boldsymbol{I}_n]_{y,y} + c_1[\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top]_{y,y} + c_2[\bar{\boldsymbol{C}}^\top\bar{\boldsymbol{C}}]_{y,y} = 0$, which is equivalent to $c_0 = 0$, a contradiction to $c_0 \neq 0$. This concludes the proof that for all $i \in \mathcal{I}$, $\deg_H^-(i) > 0$.

Fifth, I show that there exists a positive integer d such that for all $i \in \mathcal{I}$, $\deg_H^-(i) = \deg_H^+(i) = d$, that is, H is d-regular. Let $(y,z) \in \mathcal{I}^2$ with $y \neq z$. It follows from (D.218) that $c_0\deg_H^+(y) + c_2 = 0$ and $c_0\deg_H^+(z) + c_2 = 0$ because $\deg_H^+(y) > 0$ and $\deg_H^+(z) > 0$. The inequality $c_0 \neq 0$ and the two equalities $c_0\deg_H^+(y) + c_2 = 0$ and $c_0\deg_H^+(z) + c_2 = 0$ imply that $\deg_H^+(y) = \deg_H^+(z)$. Let $d := \deg_H^+(y) > 0$. Since $\bar{\boldsymbol{C}}$ is normal, we find, for all $i \in \mathcal{I}$,

$$\frac{\deg_H^-(i)}{d^2} = \frac{|\mathcal{N}_H^-(i)|}{d^2} = \sum_{k \in \mathcal{N}_H^-(i)} \frac{1}{\deg_H^+(k)^2} = [\bar{\boldsymbol{C}}^\top\bar{\boldsymbol{C}}]_{i,i} = [\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top]_{i,i} = \frac{1}{\deg_H^+(i)} = \frac{1}{d},$$

that is, $\deg_H^-(i) = d$. This concludes the proof that H is d-regular.

Sixth, I show that (D.217) is true. Note that $\bar{\boldsymbol{C}}\boldsymbol{1}_n = \boldsymbol{1}_n$ because $\mathcal{I}_0^+(H) = \emptyset$. Since $\bar{\boldsymbol{C}}$ is normal, (D.218) is equivalent to $c_0\boldsymbol{I}_n + c_1(\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top) + c_2\bar{\boldsymbol{C}}^\top\bar{\boldsymbol{C}} = \boldsymbol{O}_n$, which implies that $c_0\boldsymbol{1}_n^\top\boldsymbol{1}_n + c_1\boldsymbol{1}_n^\top(\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top)\boldsymbol{1}_n + c_2\boldsymbol{1}_n^\top\bar{\boldsymbol{C}}^\top\bar{\boldsymbol{C}}\boldsymbol{1}_n = 0$, which is equivalent to

$$c_0 + 2c_1 + c_2 = 0. \tag{D.220}$$

It follows from (D.218) and $\bar{\boldsymbol{C}} = (1/d)\dot{\boldsymbol{A}}(H)$ that

$$c_0\boldsymbol{I}_n + \frac{c_1}{d}\left(\dot{\boldsymbol{A}}(H) + \dot{\boldsymbol{A}}(H)^\top\right) + \frac{c_2}{d^2}\dot{\boldsymbol{A}}(H)\dot{\boldsymbol{A}}(H)^\top = \boldsymbol{O}_n,$$

which is equivalent to (D.217) because $c_2 \neq 0$, $c_0 d + c_2 = 0$, and (D.220) imply that

$$-\frac{c_0 d^2}{c_2} = d \quad \text{and} \quad -\frac{c_1 d}{c_2} = \frac{d-1}{2}. \qquad \blacksquare$$

Proof of Proposition 1.104

In what follows, I write $\bar{\boldsymbol{A}}$ for $\bar{\boldsymbol{A}}(G)$ and $\bar{\boldsymbol{C}}$ for $\bar{\boldsymbol{C}}(H)$. Suppose $\bar{\boldsymbol{A}} \neq \bar{\boldsymbol{C}}$ and the nine matrices of Proposition 1.104 are linearly independent. Let $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ be a pair of parameter points in Θ^2. We must show that $(\gamma_1, \zeta_1, \varsigma_1) = (\gamma_2, \zeta_2, \varsigma_2)$ is necessary for

$$\operatorname{var}\big(\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_1)\big) \mid \mathfrak{F}\big) = \operatorname{var}\big(\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_2)\big) \mid \mathfrak{F}\big). \tag{D.221}$$

Suppose (D.221) is true. Using Lemma B.4, we find that (D.221) is equivalent to

$$\begin{aligned}
&\varsigma_1^2\big(\boldsymbol{I}_n - \gamma_2 q(\bar{\boldsymbol{A}})\big)\big(\boldsymbol{I}_n + \zeta_1\bar{\boldsymbol{C}}\big)\big(\boldsymbol{I}_n + \zeta_1\bar{\boldsymbol{C}}^\top\big)\big(\boldsymbol{I}_n - \gamma_2 q(\bar{\boldsymbol{A}})^\top\big)\\
&\quad = \varsigma_2^2\big(\boldsymbol{I}_n - \gamma_1 q(\bar{\boldsymbol{A}})\big)\big(\boldsymbol{I}_n + \zeta_2\bar{\boldsymbol{C}}\big)\big(\boldsymbol{I}_n + \zeta_2\bar{\boldsymbol{C}}^\top\big)\big(\boldsymbol{I}_n - \gamma_1 q(\bar{\boldsymbol{A}})^\top\big),
\end{aligned}$$

which in turn is equivalent to

$$\begin{aligned}
\boldsymbol{O}_n = {} & \big((1+\gamma_2)^2\varsigma_1^2 - (1+\gamma_1)^2\varsigma_2^2\big)\boldsymbol{I}_n\\
& + \big(\gamma_1(1+\gamma_1)\varsigma_2^2 - \gamma_2(1+\gamma_2)\varsigma_1^2\big)\big(\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top\big)\\
& + \big(\gamma_2^2\varsigma_1^2 - \gamma_1^2\varsigma_2^2\big)\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\top\\
& + \big((1+\gamma_2)^2\zeta_1\varsigma_1^2 - (1+\gamma_1)^2\zeta_2\varsigma_2^2\big)\big(\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top\big)\\
& + \big((1+\gamma_2)^2\zeta_1^2\varsigma_1^2 - (1+\gamma_1)^2\zeta_2^2\varsigma_2^2\big)\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top\\
& + \big(\gamma_1(1+\gamma_1)\zeta_2\varsigma_2^2 - \gamma_2(1+\gamma_2)\zeta_1\varsigma_1^2\big)\big(\bar{\boldsymbol{A}}\big(\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top\big) + \big(\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top\big)\bar{\boldsymbol{A}}^\top\big)\\
& + \big(\gamma_1(1+\gamma_1)\zeta_2^2\varsigma_2^2 - \gamma_2(1+\gamma_2)\zeta_1^2\varsigma_1^2\big)\big(\bar{\boldsymbol{A}}\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top + \bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top\bar{\boldsymbol{A}}^\top\big)\\
& + \big(\gamma_2^2\zeta_1\varsigma_1^2 - \gamma_1^2\zeta_2\varsigma_2^2\big)\bar{\boldsymbol{A}}\big(\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top\big)\bar{\boldsymbol{A}}^\top\\
& + \big(\gamma_2^2\zeta_1^2\varsigma_1^2 - \gamma_1^2\zeta_2^2\varsigma_2^2\big)\bar{\boldsymbol{A}}\bar{\boldsymbol{C}}\big(\bar{\boldsymbol{A}}\bar{\boldsymbol{C}}\big)^\top.
\end{aligned}$$

We have

$$(1+\gamma_2)^2\varsigma_1^2 - (1+\gamma_1)^2\varsigma_2^2 = 0 \tag{D.222}$$

$$\gamma_1(1+\gamma_1)\varsigma_2^2 - \gamma_2(1+\gamma_2)\varsigma_1^2 = 0 \tag{D.223}$$

$$\gamma_2^2\varsigma_1^2 - \gamma_1^2\varsigma_2^2 = 0 \tag{D.224}$$

$$(1+\gamma_2)^2\zeta_1\varsigma_1^2 - (1+\gamma_1)^2\zeta_2\varsigma_2^2 = 0 \tag{D.225}$$

$$(1+\gamma_2)^2\zeta_1^2\varsigma_1^2 - (1+\gamma_1)^2\zeta_2^2\varsigma_2^2 = 0 \tag{D.226}$$

$$\gamma_1(1+\gamma_1)\zeta_2\varsigma_2^2 - \gamma_2(1+\gamma_2)\zeta_1\varsigma_1^2 = 0 \tag{D.227}$$

$$\gamma_1(1+\gamma_1)\zeta_2^2\varsigma_2^2 - \gamma_2(1+\gamma_2)\zeta_1^2\varsigma_1^2 = 0 \tag{D.228}$$

$$\gamma_2^2\zeta_1\varsigma_1^2 - \gamma_1^2\zeta_2\varsigma_2^2 = 0 \tag{D.229}$$

$$\gamma_2^2\zeta_1^2\varsigma_1^2 - \gamma_1^2\zeta_2^2\varsigma_2^2 = 0 \tag{D.230}$$

because the nine matrices of Proposition 1.104 are linearly independent. The system of equations (D.222) to (D.230) has a unique solution. Adding equations (D.222) and (D.223) gives

$$(1+\gamma_2)\varsigma_1^2 - (1+\gamma_1)\varsigma_2^2 = 0, \tag{D.231}$$

and adding equations (D.223) and (D.224) gives

$$\gamma_1 \varsigma_2^2 - \gamma_2 \varsigma_1^2 = 0. \tag{D.232}$$

Adding equations (D.231) and (D.232) gives $\varsigma_1^2 = \varsigma_2^2$, which is equivalent to $\varsigma_1 = \varsigma_2$ because $\varsigma_1 > 0$ and $\varsigma_2 > 0$. This equality together with equation (D.232) gives $(\gamma_1 - \gamma_2)\varsigma_1^2 = 0$, from which $\gamma_1 = \gamma_2$ follows because $\varsigma_1 > 0$. If $\gamma_1 = 0$, then $\gamma_1 = \gamma_2$, $\varsigma_1 = \varsigma_2$, and equation (D.225) give $(\zeta_1 - \zeta_2)\varsigma_1^2 = 0$, from which $\zeta_1 = \zeta_2$ follows because $\varsigma_1 > 0$. If $\gamma_1 \neq 0$, then $\gamma_1 = \gamma_2$, $\varsigma_1 = \varsigma_2$, and equation (D.229) give $(\zeta_1 - \zeta_2)\gamma_1^2\varsigma_1^2 = 0$, from which $\zeta_1 = \zeta_2$ follows because $\varsigma_1 > 0$. To sum up, I have established that $(\gamma_1, \zeta_1, \varsigma_1) = (\gamma_2, \zeta_2, \varsigma_2)$. ∎

Proof of Proposition 1.105

In what follows, I write $\bar{A}$ for $\bar{A}(G)$ and $\bar{C}$ for $\bar{C}(H)$. Suppose $\bar{A} = \bar{C}$, $\gamma\zeta \geq 0$, and the six matrices of Proposition 1.105 are linearly independent. Let $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ be a pair of parameter points in Θ^2. We must show that $(\gamma_1, \zeta_1, \varsigma_1) = (\gamma_2, \zeta_2, \varsigma_2)$ is necessary for

$$\operatorname{var}\big(\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_1)\big) \,\big|\, \mathfrak{F}\big) = \operatorname{var}\big(\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_2)\big) \,\big|\, \mathfrak{F}\big). \tag{D.233}$$

Suppose (D.233) is true. Using Lemma B.4, we find that (D.233) is equivalent to

$$\begin{aligned}
&\varsigma_1^2\big(\boldsymbol{I}_n - \gamma_2 q(\bar{\boldsymbol{C}})\big)\big(\boldsymbol{I}_n + \zeta_1\bar{\boldsymbol{C}}\big)\big(\boldsymbol{I}_n + \zeta_1\bar{\boldsymbol{C}}^{\mathsf{T}}\big)\big(\boldsymbol{I}_n - \gamma_2 q(\bar{\boldsymbol{C}})^{\mathsf{T}}\big)\\
&\quad = \varsigma_2^2\big(\boldsymbol{I}_n - \gamma_1 q(\bar{\boldsymbol{C}})\big)\big(\boldsymbol{I}_n + \zeta_2\bar{\boldsymbol{C}}\big)\big(\boldsymbol{I}_n + \zeta_2\bar{\boldsymbol{C}}^{\mathsf{T}}\big)\big(\boldsymbol{I}_n - \gamma_1 q(\bar{\boldsymbol{C}})^{\mathsf{T}}\big),
\end{aligned}$$

which in turn is equivalent to

$$\begin{aligned}
\boldsymbol{O}_n = {} & \big((1+\gamma_2)^2\varsigma_1^2 - (1+\gamma_1)^2\varsigma_2^2\big)\boldsymbol{I}_n\\
& + \big((1+\gamma_1)\big(\gamma_1 - (1+\gamma_1)\zeta_2\big)\varsigma_2^2 - (1+\gamma_2)\big(\gamma_2 - (1+\gamma_2)\zeta_1\big)\varsigma_1^2\big)\big(\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^{\mathsf{T}}\big)\\
& + \big(\big(\gamma_2 - (1+\gamma_2)\zeta_1\big)^2\varsigma_1^2 - \big(\gamma_1 - (1+\gamma_1)\zeta_2\big)^2\varsigma_2^2\big)\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^{\mathsf{T}}\\
& + \big(\gamma_1(1+\gamma_1)\zeta_2\varsigma_2^2 - \gamma_2(1+\gamma_2)\zeta_1\varsigma_1^2\big)\big(\bar{\boldsymbol{C}}^2 + \bar{\boldsymbol{C}}^{\mathsf{T}2}\big)\\
& + \big(\gamma_2\big(\gamma_2 - (1+\gamma_2)\zeta_1\big)\zeta_1\varsigma_1^2 - \gamma_1\big(\gamma_1 - (1+\gamma_1)\zeta_2\big)\zeta_2\varsigma_2^2\big)\bar{\boldsymbol{C}}\big(\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^{\mathsf{T}}\big)\bar{\boldsymbol{C}}^{\mathsf{T}}\\
& + \big(\gamma_2^2\zeta_1^2\varsigma_1^2 - \gamma_1^2\zeta_2^2\varsigma_2^2\big)\bar{\boldsymbol{C}}^2\bar{\boldsymbol{C}}^{\mathsf{T}2}.
\end{aligned}$$

We have

$$(1+\gamma_2)^2\varsigma_1^2 - (1+\gamma_1)^2\varsigma_2^2 = 0 \tag{D.234}$$

$$(1+\gamma_1)\big(\gamma_1 - (1+\gamma_1)\zeta_2\big)\varsigma_2^2 - (1+\gamma_2)\big(\gamma_2 - (1+\gamma_2)\zeta_1\big)\varsigma_1^2 = 0 \tag{D.235}$$

$$\big(\gamma_2 - (1+\gamma_2)\zeta_1\big)^2\varsigma_1^2 - \big(\gamma_1 - (1+\gamma_1)\zeta_2\big)^2\varsigma_2^2 = 0 \tag{D.236}$$

$$\gamma_1(1+\gamma_1)\zeta_2\varsigma_2^2 - \gamma_2(1+\gamma_2)\zeta_1\varsigma_1^2 = 0 \tag{D.237}$$

$$\gamma_2\big(\gamma_2 - (1+\gamma_2)\zeta_1\big)\zeta_1\varsigma_1^2 - \gamma_1\big(\gamma_1 - (1+\gamma_1)\zeta_2\big)\zeta_2\varsigma_2^2 = 0 \tag{D.238}$$

$$\gamma_2^2\zeta_1^2\varsigma_1^2 - \gamma_1^2\zeta_2^2\varsigma_2^2 = 0 \tag{D.239}$$

because the six matrices of Proposition 1.105 are linearly independent. The system of equations (D.234) to (D.239) has a unique solution because $\gamma_1\zeta_1 \geq 0$ and $\gamma_2\zeta_2 \geq 0$. To show this, I consider two cases:

(1) Suppose $\gamma\zeta = 0$, so that $\gamma_1\zeta_1 = \gamma_2\zeta_2 = 0$. I consider two cases:

(1.1) Suppose $\gamma_1 = 0$. Equation (D.239) gives $\gamma_2\zeta_1 = 0$ because $\varsigma_1 > 0$. We must have $\gamma_2 = 0$. Suppose, for the sake of contradiction, $\gamma_2 \neq 0$, from which $\zeta_1 = \zeta_2 = 0$ follows because $\gamma_2\zeta_1 = \gamma_2\zeta_2 = 0$. Adding equations (D.235) and (D.236) gives $\gamma_2\varsigma_1^2 = 0$. It follows that $\varsigma_1 = 0$, which contradicts $\varsigma_1 > 0$. This concludes the proof that $\gamma_2 = 0$. Equations (D.234) and $\gamma_1 = \gamma_2 = 0$ give $\varsigma_1^2 = \varsigma_2^2$, which is equivalent to $\varsigma_1 = \varsigma_2$ because $\varsigma_1 > 0$ and $\varsigma_2 > 0$. This equality and equation (D.235) give $(\zeta_1 - \zeta_2)\varsigma_1^2 = 0$, from which $\zeta_1 = \zeta_2$ follows because $\varsigma_1 > 0$.

(1.2) Suppose $\gamma_1 \neq 0$, from which $\zeta_1 = 0$ follows because $\gamma_1\zeta_1 = 0$. Equation (D.239) gives $\gamma_1\zeta_2 = 0$ because $\varsigma_2 > 0$. It follows that $\zeta_2 = 0$. Adding equations (D.235) and (D.236) gives

$$\gamma_1\varsigma_2^2 - \gamma_2\varsigma_1^2 = 0. \tag{D.240}$$

Adding equations (D.234), (D.235), and (D.240) gives $\varsigma_1^2 = \varsigma_2^2$, which is equivalent to $\varsigma_1 = \varsigma_2$ because $\varsigma_1 > 0$ and $\varsigma_2 > 0$. This equality and equation (D.240) give $(\gamma_1 - \gamma_2)\varsigma_1^2 = 0$, from which $\gamma_1 = \gamma_2$ follows because $\varsigma_1 > 0$.

(2) Suppose $\gamma\zeta > 0$, so that $\gamma_1\zeta_1 > 0$ and $\gamma_2\zeta_2 > 0$. Using (D.235), we find

$$\begin{aligned}
& & (1+\gamma_1)\big(\gamma_1 - (1+\gamma_1)\zeta_2\big)\varsigma_2^2 - \gamma_2(1+\gamma_2)\varsigma_1^2 + (1+\gamma_2)^2\zeta_1\varsigma_1^2 &= 0 \\
&\Rightarrow & (1+\gamma_1)\varsigma_2^2\Big(\gamma_1 - (1+\gamma_1)\zeta_2 - \gamma_1\frac{\zeta_2}{\zeta_1} + (1+\gamma_1)\zeta_1\Big) &= 0 \\
&\Leftrightarrow & \gamma_1 - (1+\gamma_1)\zeta_2 - \gamma_1\frac{\zeta_2}{\zeta_1} + (1+\gamma_1)\zeta_1 &= 0 \\
&\Leftrightarrow & (\zeta_1 - \zeta_2)\big(\gamma_1 + (1+\gamma_1)\zeta_1\big) &= 0 \\
&\Leftrightarrow & \zeta_1 - \zeta_2 &= 0.
\end{aligned}$$

The implication follows from equations (D.234) and (D.237), where $\zeta_1 \neq 0$ because $\gamma_1\zeta_1 > 0$. The first equivalence follows from $1 + \gamma_1 > 0$ (Assumption $\mathcal{P}$-γ) and $\varsigma_2 > 0$. The second equivalence is obvious. The last equivalence follows from $\gamma_1 + (1+\gamma_1)\zeta_1 \neq 0$, which is a consequence of $\gamma_1\zeta_1 > 0$. Indeed, if $\gamma_1 + (1+\gamma_1)\zeta_1 = 0$, then $\gamma_1\zeta_1 = -(1+\gamma_1)\zeta_1^2 \leq 0$ because $1 + \gamma_1 > 0$, which contradicts $\gamma_1\zeta_1 > 0$. Using $\zeta_1 = \zeta_2$ and $\zeta_1 \neq 0$, equation (D.237) gives

$$\gamma_1(1+\gamma_1)\varsigma_2^2 - \gamma_2(1+\gamma_2)\varsigma_1^2 = 0. \tag{D.241}$$

Adding equations (D.234) and (D.241) gives $(1+\gamma_1)\varsigma_2^2 = (1+\gamma_2)\varsigma_1^2$. This equality and equation (D.241) give $(\gamma_1 - \gamma_2)(1+\gamma_1)\varsigma_2^2 = 0$, from which $\gamma_1 =$

γ_2 follows because $1+\gamma_1 > 0$ and $\varsigma_2 > 0$. Equations (D.234) and $\gamma_1 = \gamma_2$ give $(\varsigma_1^2 - \varsigma_2^2)(1+\gamma_1)^2 = 0$, from which $\varsigma_1 = \varsigma_2$ follows because $1+\gamma_1 > 0$, $\varsigma_1 > 0$, and $\varsigma_2 > 0$.

To sum up, I have established that $(\gamma_1, \zeta_1, \varsigma_1) = (\gamma_2, \zeta_2, \varsigma_2)$. Finally, note that the system of equations (D.234) to (D.239) has infinitely many solutions without the restrictions $\gamma_1\zeta_1 \geq 0$ and $\gamma_2\zeta_2 \geq 0$, for example, for all $(c, s) \in \gamma(\Theta) \times \mathbb{R}_{++}$,

$$\left(\gamma_1, \gamma_2, \zeta_1, \zeta_2, \varsigma_1, \varsigma_2\right) = \left(0, c, 0, -\frac{c}{1+c}, s, (1+c)s\right)$$

is a solution. ■

Proof of Proposition 1.106

The proof is omitted because it is similar to the proof of Proposition 1.84. ■

Proof of Proposition 1.107

The proof is omitted because it is similar to the proof of Proposition 1.86. ■

Proof of Lemma 1.108

In what follows, I write $\bar{A}$ for $\bar{A}(G)$. Let $p > 1$ be an integer.

First, I show by contraposition that $\bar{A}^0, \ldots, \bar{A}^p$ are linearly independent if $q(\bar{A})\bar{A}^0, \ldots, q(\bar{A})\bar{A}^{p-1}$ are linearly independent. Suppose $\bar{A}^0, \ldots, \bar{A}^p$ are linearly dependent, that is, there exists a $(p+1)$-tuple $(a_1, \ldots, a_{p+1}) \neq \mathbf{0}_{p+1}$ such that $\sum_{k=1}^{p+1} a_k \bar{A}^{k-1} = \boldsymbol{O}_n$. Induction on p shows that

$$\sum_{k=1}^{p+1} a_k \bar{A}^{k-1} = \sum_{k=1}^{p} \left(\sum_{j=k+1}^{p+1} a_j \right) (\bar{A} - \boldsymbol{I}_n) \bar{A}^{k-1}$$

if $\sum_{k=1}^{p+1} a_k = 0$. Note that $\sum_{k=1}^{p+1} a_k \bar{A}^{k-1} = \boldsymbol{O}_n$ and $\bar{A}\mathbf{1}_n = \mathbf{1}_n$ imply that $\sum_{k=1}^{p+1} a_k = 0$. For all $k \in \{1, \ldots, p\}$, let $b_k := \sum_{j=k+1}^{p+1} a_j$. Note that $(a_1, \ldots, a_{p+1}) \neq \mathbf{0}_{p+1}$ implies that $(b_1, \ldots, b_p) \neq \mathbf{0}_p$ because $a_1 = -\sum_{j=2}^{p+1} a_j$, $a_{p+1} = b_p$, and for all $k \in \{2, \ldots, p\}$, $a_k = b_{k-1} - b_k$. We find

$$\boldsymbol{O}_n = \sum_{k=1}^{p+1} a_k \bar{A}^{k-1} = \sum_{k=1}^{p} \left(\sum_{j=k+1}^{p+1} a_j \right) (\bar{A} - \boldsymbol{I}_n) \bar{A}^{k-1} = \sum_{k=1}^{p} b_k q(\bar{A}) \bar{A}^{k-1},$$

that is, $q(\bar{A})\bar{A}^0, \ldots, q(\bar{A})\bar{A}^{p-1}$ are linearly dependent.

Second, I show that $q(\bar{A})\bar{A}^0, \ldots, q(\bar{A})\bar{A}^{p-1}$ are linearly independent if $\bar{A}^0, \ldots, \bar{A}^p$ are linearly independent. Suppose $q(\bar{A})\bar{A}^0, \ldots, q(\bar{A})\bar{A}^{p-1}$ are linearly dependent, that is, there exists a p-tuple $(c_1, \ldots, c_p) \neq \mathbf{0}_p$ such that $\sum_{k=1}^{p} c_k q(\bar{A}) \bar{A}^{k-1} = \boldsymbol{O}_n$.

Let $d_1 := -c_1$, $d_{p+1} := c_p$, and for all $k \in \{2, \ldots, p\}$, $d_k := c_{k-1} - c_k$. Note that $(c_1, \ldots, c_p) \neq \mathbf{0}_p$ implies that $(d_1, \ldots, d_{p+1}) \neq \mathbf{0}_{p+1}$ because $c_1 = -d_1$, $c_p = d_{p+1}$, and for all $k \in \{2, \ldots, p-1\}$, $c_k = -\sum_{j=1}^{k} d_j$. We find

$$\boldsymbol{O}_n = \sum_{k=1}^{p} c_k (\bar{\boldsymbol{A}} - \boldsymbol{I}_n) \bar{\boldsymbol{A}}^{k-1} = -c_1 \boldsymbol{I}_n + \sum_{k=2}^{p} (c_{k-1} - c_k) \bar{\boldsymbol{A}}^{k-1} + c_p \bar{\boldsymbol{A}}^p = \sum_{k=1}^{p+1} d_k \bar{\boldsymbol{A}}^{k-1},$$

that is, $\bar{\boldsymbol{A}}^0, \ldots, \bar{\boldsymbol{A}}^p$ are linearly dependent. ∎

Proof of Lemma 1.109

In what follows, I write $\bar{A}$ for $\bar{A}(G)$. Suppose G has a subdigraph of diameter (at least) three. It follows that there exist four pairwise distinct vertices w, x, y, and z in G such that $P := (w, x, y, z)$ is a shortest path in G (of length 3) from w to z. Suppose, for the sake of contradiction, $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}$, $\bar{\boldsymbol{A}}^2$, $\bar{\boldsymbol{A}}^3$ are linearly dependent. There exists a 4-tuple $(c_0, c_1, c_2, c_3) \neq (0, 0, 0, 0)$ such that

$$c_0 \boldsymbol{I}_n + c_1 \bar{\boldsymbol{A}} + c_2 \bar{\boldsymbol{A}}^2 + c_3 \bar{\boldsymbol{A}}^3 = \boldsymbol{O}_n. \tag{D.242}$$

Note that $c_3 \neq 0$. To prove this, suppose, for the sake of contradiction, $c_3 = 0$. Note that $(c_0, c_1, c_2) \neq (0, 0, 0)$ because $(c_0, c_1, c_2, c_3) \neq (0, 0, 0, 0)$ and $c_3 = 0$. It follows from (D.242) that $c_0 \boldsymbol{I}_n + c_1 \bar{\boldsymbol{A}} + c_2 \bar{\boldsymbol{A}}^2 = \boldsymbol{O}_n$, that is, $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}$, $\bar{\boldsymbol{A}}^2$ are linearly dependent. Note that both (w, x, y) and (x, y, z) are intransitive triples in G because P is a shortest path in G from w to z. Lemma 1.87 implies that $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}$, $\bar{\boldsymbol{A}}^2$ are linearly independent, and we have reached a contradiction. This concludes the proof that $c_3 \neq 0$. Note that $[\bar{\boldsymbol{A}}]_{w,z} = [\bar{\boldsymbol{A}}^2]_{w,z} = 0$ and $[\bar{\boldsymbol{A}}^3]_{w,z} > 0$ because P is a shortest path in G of length 3 from w to z. It follows from (D.242), $w \neq z$, $[\bar{\boldsymbol{A}}]_{w,z} = [\bar{\boldsymbol{A}}^2]_{w,z} = 0$, and $c_3 \neq 0$ that $[\bar{\boldsymbol{A}}^3]_{w,z} = 0$, a contradiction to $[\bar{\boldsymbol{A}}^3]_{w,z} > 0$. This concludes the proof that $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}$, $\bar{\boldsymbol{A}}^2$, $\bar{\boldsymbol{A}}^3$ are linearly independent. ∎

Proof of Proposition 1.111

The proof is omitted because it is similar to the proof of Proposition 1.89. ∎

Proof of Proposition 1.112

The proof is omitted because it is similar to the proof of Proposition 1.90. ∎

Proof of Proposition 1.113

The proof is omitted because it is similar to the proof of Proposition 1.91. ∎

Proof of Proposition 1.114

In what follows, I write $\bar{\boldsymbol{A}}$ for $\bar{\boldsymbol{A}}(G)$. I prove by contraposition that $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}+\bar{\boldsymbol{A}}^\top$, $\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\top$ are linearly independent if $q(\bar{\boldsymbol{A}})q(\bar{\boldsymbol{A}})^\top$, $q(\bar{\boldsymbol{A}})(q(\bar{\boldsymbol{A}})+q(\bar{\boldsymbol{A}})^\top)q(\bar{\boldsymbol{A}})^\top$, $q^2(\bar{\boldsymbol{A}})q^2(\bar{\boldsymbol{A}})^\top$ are linearly independent. Suppose $\boldsymbol{I}_n$, $\bar{\boldsymbol{A}}+\bar{\boldsymbol{A}}^\top$, $\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\top$ are linearly dependent, that is, there exists a triple $(c_1,c_2,c_3)\neq\boldsymbol{0}_3$ such that $c_1\boldsymbol{I}_n+c_2(\bar{\boldsymbol{A}}+\bar{\boldsymbol{A}}^\top)+c_3\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\top=\boldsymbol{O}_n$. Let

$$\boldsymbol{T}:=\begin{pmatrix}1&2&1\\0&1&1\\0&0&1\end{pmatrix},$$

and let $(d_1,d_2,d_3):=\boldsymbol{T}(c_1,c_2,c_3)$. Note that $(d_1,d_2,d_3)\neq\boldsymbol{0}_3$ because $\boldsymbol{T}$ is nonsingular and $(c_1,c_2,c_3)\neq\boldsymbol{0}_3$. We have $d_1\boldsymbol{I}_n+d_2(q(\bar{\boldsymbol{A}})+q(\bar{\boldsymbol{A}})^\top)+d_3q(\bar{\boldsymbol{A}})q(\bar{\boldsymbol{A}})^\top=\boldsymbol{O}_n$, which implies that $q(\bar{\boldsymbol{A}})q(\bar{\boldsymbol{A}})^\top$, $q(\bar{\boldsymbol{A}})(q(\bar{\boldsymbol{A}})+q(\bar{\boldsymbol{A}})^\top)q(\bar{\boldsymbol{A}})^\top$, $q^2(\bar{\boldsymbol{A}})q^2(\bar{\boldsymbol{A}})^\top$ are linearly dependent. ∎

Proof of Proposition 1.115

The proof is omitted because it is similar to the proof of Proposition 1.100. ∎

Proof of Proposition 1.116

In what follows, I write $\bar{\boldsymbol{A}}$ for $\bar{\boldsymbol{A}}(G)$ and $\bar{\boldsymbol{C}}$ for $\bar{\boldsymbol{C}}(H)$. Suppose $\bar{\boldsymbol{A}}\neq\bar{\boldsymbol{C}}$ and the nine matrices of Proposition 1.116 are linearly independent. Let $(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)$ be a pair of parameter points in Θ^2. We must show that $(\gamma_1,\zeta_1,\varsigma_1)=(\gamma_2,\zeta_2,\varsigma_2)$ is necessary for

$$\operatorname{var}\bigl(q(\bar{\boldsymbol{A}})\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_1))\bigm|\mathfrak{F}\bigr)=\operatorname{var}\bigl(q(\bar{\boldsymbol{A}})\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{\theta}_2))\bigm|\mathfrak{F}\bigr). \tag{D.243}$$

Suppose (D.243) is true. Using Lemma B.4, we find that (D.243) is equivalent to

$$\begin{aligned}&\varsigma_1^2\bigl(\boldsymbol{I}_n-\gamma_2q(\bar{\boldsymbol{A}})\bigr)q(\bar{\boldsymbol{A}})\bigl(\boldsymbol{I}_n+\zeta_1\bar{\boldsymbol{C}}\bigr)\bigl(\boldsymbol{I}_n+\zeta_1\bar{\boldsymbol{C}}^\top\bigr)q(\bar{\boldsymbol{A}})^\top\bigl(\boldsymbol{I}_n-\gamma_2q(\bar{\boldsymbol{A}})^\top\bigr)\\&\qquad=\varsigma_2^2\bigl(\boldsymbol{I}_n-\gamma_1q(\bar{\boldsymbol{A}})\bigr)q(\bar{\boldsymbol{A}})\bigl(\boldsymbol{I}_n+\zeta_2\bar{\boldsymbol{C}}\bigr)\bigl(\boldsymbol{I}_n+\zeta_2\bar{\boldsymbol{C}}^\top\bigr)q(\bar{\boldsymbol{A}})^\top\bigl(\boldsymbol{I}_n-\gamma_1q(\bar{\boldsymbol{A}})^\top\bigr),\end{aligned}$$

which in turn is equivalent to

$$\begin{aligned}\boldsymbol{O}_n&=(\varsigma_1^2-\varsigma_2^2)q(\bar{\boldsymbol{A}})q(\bar{\boldsymbol{A}})^\top\\&\quad+(\zeta_1\varsigma_1^2-\zeta_2\varsigma_2^2)q(\bar{\boldsymbol{A}})(\bar{\boldsymbol{C}}+\bar{\boldsymbol{C}}^\top)q(\bar{\boldsymbol{A}})^\top\\&\quad+(\zeta_1^2\varsigma_1^2-\zeta_2^2\varsigma_2^2)q(\bar{\boldsymbol{A}})\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top q(\bar{\boldsymbol{A}})^\top\\&\quad+(\gamma_1\varsigma_2^2-\gamma_2\varsigma_1^2)q(\bar{\boldsymbol{A}})\bigl(q(\bar{\boldsymbol{A}})+q(\bar{\boldsymbol{A}})^\top\bigr)q(\bar{\boldsymbol{A}})^\top\\&\quad+(\gamma_1\zeta_2\varsigma_2^2-\gamma_2\zeta_1\varsigma_1^2)q(\bar{\boldsymbol{A}})\bigl(q(\bar{\boldsymbol{A}})(\bar{\boldsymbol{C}}+\bar{\boldsymbol{C}}^\top)+(\bar{\boldsymbol{C}}+\bar{\boldsymbol{C}}^\top)q(\bar{\boldsymbol{A}})^\top\bigr)q(\bar{\boldsymbol{A}})^\top\\&\quad+(\gamma_1\zeta_2^2\varsigma_2^2-\gamma_2\zeta_1^2\varsigma_1^2)q(\bar{\boldsymbol{A}})\bigl(q(\bar{\boldsymbol{A}})\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top+\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top q(\bar{\boldsymbol{A}})^\top\bigr)q(\bar{\boldsymbol{A}})^\top\\&\quad+(\gamma_2^2\varsigma_1^2-\gamma_1^2\varsigma_2^2)q^2(\bar{\boldsymbol{A}})q^2(\bar{\boldsymbol{A}})^\top\\&\quad+(\gamma_2^2\zeta_1\varsigma_1^2-\gamma_1^2\zeta_2\varsigma_2^2)q^2(\bar{\boldsymbol{A}})(\bar{\boldsymbol{C}}+\bar{\boldsymbol{C}}^\top)q^2(\bar{\boldsymbol{A}})^\top\end{aligned}$$

$$+\left(\gamma_2^2\zeta_1^2\varsigma_1^2-\gamma_1^2\zeta_2^2\varsigma_2^2\right)q^2(\bar{\boldsymbol{A}})\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top q^2(\bar{\boldsymbol{A}})^\top.$$

We have

$$\varsigma_1^2-\varsigma_2^2=0 \tag{D.244}$$
$$\zeta_1\varsigma_1^2-\zeta_2\varsigma_2^2=0 \tag{D.245}$$
$$\zeta_1^2\varsigma_1^2-\zeta_2^2\varsigma_2^2=0 \tag{D.246}$$
$$\gamma_1\varsigma_2^2-\gamma_2\varsigma_1^2=0 \tag{D.247}$$
$$\gamma_1\zeta_2\varsigma_2^2-\gamma_2\zeta_1\varsigma_1^2=0 \tag{D.248}$$
$$\gamma_1\zeta_2^2\varsigma_2^2-\gamma_2\zeta_1^2\varsigma_1^2=0 \tag{D.249}$$
$$\gamma_2^2\varsigma_1^2-\gamma_1^2\varsigma_2^2=0 \tag{D.250}$$
$$\gamma_2^2\zeta_1\varsigma_1^2-\gamma_1^2\zeta_2\varsigma_2^2=0 \tag{D.251}$$
$$\gamma_2^2\zeta_1^2\varsigma_1^2-\gamma_1^2\zeta_2^2\varsigma_2^2=0 \tag{D.252}$$

because the nine matrices of Proposition 1.116 are linearly independent. The system of equations (D.244) to (D.252) has a unique solution. Equation (D.244) gives $\varsigma_1^2=\varsigma_2^2$, which is equivalent to $\varsigma_1=\varsigma_2$ because $\varsigma_1>0$ and $\varsigma_2>0$. This result and equation (D.245) give $(\zeta_1-\zeta_2)\varsigma_1^2=0$, from which $\zeta_1=\zeta_2$ follows because $\varsigma_1>0$. Similarly, equation (D.247) gives $\gamma_1=\gamma_2$. To sum up, I have established that $(\gamma_1,\zeta_1,\varsigma_1)=(\gamma_2,\zeta_2,\varsigma_2)$. ■

Proof of Proposition 1.117

In what follows, I write $\bar{\boldsymbol{A}}$ for $\bar{\boldsymbol{A}}(G)$ and $\bar{\boldsymbol{C}}$ for $\bar{\boldsymbol{C}}(H)$. I prove by contraposition that the nine matrices of Proposition 1.104 are linearly independent if the nine matrices of Proposition 1.116 are linearly independent. Suppose the nine matrices of Proposition 1.104 are linearly dependent, that is, there exists a 9-tuple $(c_1,\ldots,c_9)\neq\boldsymbol{0}_9$ such that

$$\begin{aligned}\boldsymbol{O}_n=&\,c_1\boldsymbol{I}_n+c_2\left(\bar{\boldsymbol{A}}+\bar{\boldsymbol{A}}^\top\right)+c_3\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\top+c_4\left(\bar{\boldsymbol{C}}+\bar{\boldsymbol{C}}^\top\right)+c_5\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top\\&+c_6\left(\bar{\boldsymbol{A}}\left(\bar{\boldsymbol{C}}+\bar{\boldsymbol{C}}^\top\right)+\left(\bar{\boldsymbol{C}}+\bar{\boldsymbol{C}}^\top\right)\bar{\boldsymbol{A}}^\top\right)+c_7\left(\bar{\boldsymbol{A}}\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top+\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top\bar{\boldsymbol{A}}^\top\right)\\&+c_8\bar{\boldsymbol{A}}\left(\bar{\boldsymbol{C}}+\bar{\boldsymbol{C}}^\top\right)\bar{\boldsymbol{A}}^\top+c_9\bar{\boldsymbol{A}}\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top\bar{\boldsymbol{A}}^\top.\end{aligned}$$

Let

$$\boldsymbol{T}:=\begin{pmatrix}1&2&1&0&0&0&0&0&0\\0&1&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0\\0&0&0&1&0&2&0&1&0\\0&0&0&0&1&0&2&0&1\\0&0&0&0&0&1&0&1&0\\0&0&0&0&0&0&1&0&1\\0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&1\end{pmatrix},$$

and let $(d_1,\ldots,d_9) := \boldsymbol{T}(c_1,\ldots,c_9)$. Note that $(d_1,\ldots,d_9) \neq \mathbf{0}_9$ because $\boldsymbol{T}$ is nonsingular and $(c_1,\ldots,c_9) \neq \mathbf{0}_9$. We have

$$\begin{aligned}\boldsymbol{O}_n &= d_1\boldsymbol{I}_n + d_4\big(\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top\big) + d_5\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top + d_2\big(q(\bar{\boldsymbol{A}}) + q(\bar{\boldsymbol{A}})^\top\big)\\ &\quad + d_6\big(q(\bar{\boldsymbol{A}})\big(\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top\big) + \big(\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top\big)q(\bar{\boldsymbol{A}})^\top\big) + d_7\big(q(\bar{\boldsymbol{A}})\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top + \bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top q(\bar{\boldsymbol{A}})^\top\big)\\ &\quad + d_3 q(\bar{\boldsymbol{A}})q(\bar{\boldsymbol{A}})^\top + d_8 q(\bar{\boldsymbol{A}})\big(\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top\big)q(\bar{\boldsymbol{A}})^\top + d_9 q(\bar{\boldsymbol{A}})\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top q(\bar{\boldsymbol{A}})^\top,\end{aligned}$$

which implies that the nine matrices of Proposition 1.116 are linearly dependent.■

Proof of Proposition 1.118

In what follows, I write $\bar{\boldsymbol{A}}$ for $\bar{\boldsymbol{A}}(G)$ and $\bar{\boldsymbol{C}}$ for $\bar{\boldsymbol{C}}(H)$. Suppose $\bar{\boldsymbol{A}} = \bar{\boldsymbol{C}}$, $\gamma\zeta \geq 0$, and the six matrices of Proposition 1.118 are linearly independent. Let $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ be a pair of parameter points in Θ^2. We must show that $(\gamma_1, \zeta_1, \varsigma_1) = (\gamma_2, \zeta_2, \varsigma_2)$ is necessary for

$$\operatorname{var}\big(q(\bar{\boldsymbol{A}})\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_1)\big) \mid \mathfrak{F}\big) = \operatorname{var}\big(q(\bar{\boldsymbol{A}})\boldsymbol{f}\big(\boldsymbol{y}(\boldsymbol{\theta}_2)\big) \mid \mathfrak{F}\big). \tag{D.253}$$

Suppose (D.253) is true. Using Lemma B.4, we find that (D.253) is equivalent to

$$\begin{aligned}&\varsigma_1^2\big(\boldsymbol{I}_n - \gamma_2 q(\bar{\boldsymbol{C}})\big)q(\bar{\boldsymbol{C}})\big(\boldsymbol{I}_n + \zeta_1\bar{\boldsymbol{C}}\big)\big(\boldsymbol{I}_n + \zeta_1\bar{\boldsymbol{C}}^\top\big)q(\bar{\boldsymbol{C}})^\top\big(\boldsymbol{I}_n - \gamma_2 q(\bar{\boldsymbol{C}})^\top\big)\\ &\quad = \varsigma_2^2\big(\boldsymbol{I}_n - \gamma_1 q(\bar{\boldsymbol{C}})\big)q(\bar{\boldsymbol{C}})\big(\boldsymbol{I}_n + \zeta_2\bar{\boldsymbol{C}}\big)\big(\boldsymbol{I}_n + \zeta_2\bar{\boldsymbol{C}}^\top\big)q(\bar{\boldsymbol{C}})^\top\big(\boldsymbol{I}_n - \gamma_1 q(\bar{\boldsymbol{C}})^\top\big),\end{aligned}$$

which in turn is equivalent to

$$\begin{aligned}\boldsymbol{O}_n &= (\varsigma_1^2 - \varsigma_2^2)q(\bar{\boldsymbol{C}})q(\bar{\boldsymbol{C}})^\top\\ &\quad + (\zeta_1\varsigma_1^2 - \zeta_2\varsigma_2^2)q(\bar{\boldsymbol{C}})\big(\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top\big)q(\bar{\boldsymbol{C}})^\top\\ &\quad + (\zeta_1^2\varsigma_1^2 - \zeta_2^2\varsigma_2^2)q(\bar{\boldsymbol{C}})\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top q(\bar{\boldsymbol{C}})^\top\\ &\quad + (\gamma_1\varsigma_2^2 - \gamma_2\varsigma_1^2)q(\bar{\boldsymbol{C}})\big(q(\bar{\boldsymbol{C}}) + q(\bar{\boldsymbol{C}})^\top\big)q(\bar{\boldsymbol{C}})^\top\\ &\quad + (\gamma_1\zeta_2\varsigma_2^2 - \gamma_2\zeta_1\varsigma_1^2)q(\bar{\boldsymbol{C}})\big(q(\bar{\boldsymbol{C}})\big(\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top\big) + \big(\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top\big)q(\bar{\boldsymbol{C}})^\top\big)q(\bar{\boldsymbol{C}})^\top\\ &\quad + (\gamma_1\zeta_2^2\varsigma_2^2 - \gamma_2\zeta_1^2\varsigma_1^2)q(\bar{\boldsymbol{C}})\big(q(\bar{\boldsymbol{C}})\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top + \bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top q(\bar{\boldsymbol{C}})^\top\big)q(\bar{\boldsymbol{C}})^\top\\ &\quad + (\gamma_2^2\varsigma_1^2 - \gamma_1^2\varsigma_2^2)q^2(\bar{\boldsymbol{C}})q^2(\bar{\boldsymbol{C}})^\top\\ &\quad + (\gamma_2^2\zeta_1\varsigma_1^2 - \gamma_1^2\zeta_2\varsigma_2^2)q^2(\bar{\boldsymbol{C}})\big(\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top\big)q^2(\bar{\boldsymbol{C}})^\top\\ &\quad + (\gamma_2^2\zeta_1^2\varsigma_1^2 - \gamma_1^2\zeta_2^2\varsigma_2^2)q^2(\bar{\boldsymbol{C}})\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top q^2(\bar{\boldsymbol{C}})^\top,\end{aligned}$$

which in turn is equivalent to

$$\begin{aligned}\boldsymbol{O}_n &= \big((1+\gamma_2)^2\varsigma_1^2 - (1+\gamma_1)^2\varsigma_2^2\big)q(\bar{\boldsymbol{C}})q(\bar{\boldsymbol{C}})^\top\\ &\quad + \Big(\big((1-\gamma_2^2)\zeta_1 - \gamma_2(1+\gamma_2)\big)\varsigma_1^2 - \big((1-\gamma_1^2)\zeta_2 - \gamma_1(1+\gamma_1)\big)\varsigma_2^2\Big)\\ &\qquad \times q(\bar{\boldsymbol{C}})\big(\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top\big)q(\bar{\boldsymbol{C}})^\top\\ &\quad + \Big(\big(\gamma_2^2(1+2\zeta_1) + \zeta_1^2\big)\varsigma_1^2 - \big(\gamma_1^2(1+2\zeta_2) + \zeta_2^2\big)\varsigma_2^2\Big)q(\bar{\boldsymbol{C}})\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top q(\bar{\boldsymbol{C}})^\top\\ &\quad + \big(\gamma_1(1+\gamma_1)\zeta_2\varsigma_2^2 - \gamma_2(1+\gamma_2)\zeta_1\varsigma_1^2\big)\end{aligned}$$

$$\begin{aligned}
&\quad\times q(\bar{\boldsymbol{C}})\big(q(\bar{\boldsymbol{C}})\big(\bar{\boldsymbol{C}}+\bar{\boldsymbol{C}}^\top\big)+\big(\bar{\boldsymbol{C}}+\bar{\boldsymbol{C}}^\top\big)q(\bar{\boldsymbol{C}})^\top\big)q(\bar{\boldsymbol{C}})^\top\\
&+\big(\gamma_1(\zeta_2-\gamma_1)\zeta_2\varsigma_2^2-\gamma_2(\zeta_1-\gamma_2)\zeta_1\varsigma_1^2\big)\\
&\quad\times q(\bar{\boldsymbol{C}})\big(q(\bar{\boldsymbol{C}})\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top+\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top q(\bar{\boldsymbol{C}})^\top\big)q(\bar{\boldsymbol{C}})^\top\\
&+\big(\gamma_2^2\zeta_1^2\varsigma_1^2-\gamma_1^2\zeta_2^2\varsigma_2^2\big)q^2(\bar{\boldsymbol{C}})\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top q^2(\bar{\boldsymbol{C}})^\top
\end{aligned}$$

because

$$q^2(\bar{\boldsymbol{C}})q^2(\bar{\boldsymbol{C}})^\top=q(\bar{\boldsymbol{C}})q(\bar{\boldsymbol{C}})^\top-q(\bar{\boldsymbol{C}})\big(\bar{\boldsymbol{C}}+\bar{\boldsymbol{C}}^\top\big)q(\bar{\boldsymbol{C}})^\top+q(\bar{\boldsymbol{C}})\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top q(\bar{\boldsymbol{C}})^\top$$

and

$$q(\bar{\boldsymbol{C}})\big(q(\bar{\boldsymbol{C}})+q(\bar{\boldsymbol{C}})^\top\big)q(\bar{\boldsymbol{C}})^\top=q(\bar{\boldsymbol{C}})\big(\bar{\boldsymbol{C}}+\bar{\boldsymbol{C}}^\top\big)q(\bar{\boldsymbol{C}})^\top-2q(\bar{\boldsymbol{C}})q(\bar{\boldsymbol{C}})^\top$$

and

$$\begin{aligned}
q^2(\bar{\boldsymbol{C}})\big(\bar{\boldsymbol{C}}+\bar{\boldsymbol{C}}^\top\big)q^2(\bar{\boldsymbol{C}})^\top&=q(\bar{\boldsymbol{C}})\big(q(\bar{\boldsymbol{C}})\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top+\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top q(\bar{\boldsymbol{C}})^\top\big)q(\bar{\boldsymbol{C}})^\top\\
&\quad-q(\bar{\boldsymbol{C}})\big(q(\bar{\boldsymbol{C}})\big(\bar{\boldsymbol{C}}+\bar{\boldsymbol{C}}^\top\big)+\big(\bar{\boldsymbol{C}}+\bar{\boldsymbol{C}}^\top\big)q(\bar{\boldsymbol{C}})^\top\big)q(\bar{\boldsymbol{C}})^\top\\
&\quad-q(\bar{\boldsymbol{C}})\big(\bar{\boldsymbol{C}}+\bar{\boldsymbol{C}}^\top\big)q(\bar{\boldsymbol{C}})^\top+2q(\bar{\boldsymbol{C}})\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top q(\bar{\boldsymbol{C}})^\top.
\end{aligned}$$

We have

$$(1+\gamma_2)^2\varsigma_1^2-(1+\gamma_1)^2\varsigma_2^2=0 \tag{D.254}$$

$$\big((1-\gamma_2^2)\zeta_1-\gamma_2(1+\gamma_2)\big)\varsigma_1^2-\big((1-\gamma_1^2)\zeta_2-\gamma_1(1+\gamma_1)\big)\varsigma_2^2=0 \tag{D.255}$$

$$\big(\gamma_2^2(1+2\zeta_1)+\zeta_1^2\big)\varsigma_1^2-\big(\gamma_1^2(1+2\zeta_2)+\zeta_2^2\big)\varsigma_2^2=0 \tag{D.256}$$

$$\gamma_1(1+\gamma_1)\zeta_2\varsigma_2^2-\gamma_2(1+\gamma_2)\zeta_1\varsigma_1^2=0 \tag{D.257}$$

$$\gamma_1(\zeta_2-\gamma_1)\zeta_2\varsigma_2^2-\gamma_2(\zeta_1-\gamma_2)\zeta_1\varsigma_1^2=0 \tag{D.258}$$

$$\gamma_2^2\zeta_1^2\varsigma_1^2-\gamma_1^2\zeta_2^2\varsigma_2^2=0 \tag{D.259}$$

because the six matrices of Proposition 1.118 are linearly independent. The system of equations (D.254) to (D.259) has a unique solution because $\gamma_1\zeta_1\geq 0$ and $\gamma_2\zeta_2\geq 0$. To show this, I consider two cases:

(1) Suppose $\gamma\zeta=0$, so that $\gamma_1\zeta_1=\gamma_2\zeta_2=0$. I consider two cases:

   (1.1) Suppose $\gamma_1=0$. Equation (D.259) gives $\gamma_2\zeta_1=0$ because $\varsigma_1>0$. We must have $\gamma_2=0$. Suppose, for the sake of contradiction, $\gamma_2\neq 0$, from which $\zeta_1=\zeta_2=0$ follows because $\gamma_2\zeta_1=\gamma_2\zeta_2=0$. Equation (D.256) gives $\gamma_2^2\varsigma_1^2=0$. It follows that $\varsigma_1=0$, which contradicts $\varsigma_1>0$. This concludes the proof that $\gamma_2=0$. Equations (D.254) and $\gamma_1=\gamma_2=0$ give $\varsigma_1^2=\varsigma_2^2$, which is equivalent to $\varsigma_1=\varsigma_2$ because $\varsigma_1>0$ and $\varsigma_2>0$. This equality, $\gamma_1=\gamma_2=0$, and equation (D.255) give $(\zeta_1-\zeta_2)\varsigma_1^2=0$, from which $\zeta_1=\zeta_2$ follows because $\varsigma_1>0$.

   (1.2) Suppose $\gamma_1\neq 0$, from which $\zeta_1=0$ follows because $\gamma_1\zeta_1=0$. Equation (D.259) gives $\gamma_1\zeta_2=0$ because $\varsigma_2>0$. It follows that $\zeta_2=0$. Adding equations (D.254) and (D.255) gives

$$(1+\gamma_2)\varsigma_1^2-(1+\gamma_1)\varsigma_2^2=0. \tag{D.260}$$

Adding equations (D.255), (D.256), and (D.260) gives $\varsigma_1^2 = \varsigma_2^2$, which is equivalent to $\varsigma_1 = \varsigma_2$ because $\varsigma_1 > 0$ and $\varsigma_2 > 0$. This equality and equation (D.260) give $(\gamma_1 - \gamma_2)\varsigma_1^2 = 0$, from which $\gamma_1 = \gamma_2$ follows because $\varsigma_1 > 0$.

(2) Suppose $\gamma\zeta > 0$, so that $\gamma_1\zeta_1 > 0$ and $\gamma_2\zeta_2 > 0$. I consider two cases:

(2.1) Suppose $\zeta_1 = -1$. Adding equations (D.257) and (D.258) gives

$$\gamma_1(1+\zeta_2)\zeta_2\varsigma_2^2 = \gamma_2(1+\zeta_1)\zeta_1\varsigma_1^2, \tag{D.261}$$

from which $\zeta_2 = -1$ follows because $\gamma_1 \neq 0$, $\zeta_2 \neq 0$, and $\varsigma_2 > 0$. Adding equations (D.256) and (D.259) gives $\varsigma_1^2 = \varsigma_1^2$, which is equivalent to $\varsigma_1 = \varsigma_2$ because $\varsigma_1 > 0$ and $\varsigma_2 > 0$. Adding equations (D.258) and (D.259) gives $(\gamma_1 - \gamma_2)\varsigma_1^2 = 0$, from which $\gamma_1 = \gamma_2$ follows because $\varsigma_1 > 0$.

(2.2) Suppose $\zeta_1 \neq -1$. It follows from equation (D.261) that $\zeta_2 \neq -1$ because $\gamma_2 \neq 0$, $\zeta_1 \neq 0$, and $\varsigma_1 > 0$. Adding equation (D.254), twice equation (D.255), and equation (D.256) gives

$$(1+\zeta_1)^2\varsigma_1^2 = (1+\zeta_2)^2\varsigma_2^2. \tag{D.262}$$

Multiplying both sides of equation (D.261) by $1+\zeta_2$, then substituting $(1+\zeta_1)^2\varsigma_1^2$ for $(1+\zeta_2)^2\varsigma_2^2$ (see equation (D.262)), and then dividing both sides by $\varsigma_1^2(1+\zeta_1)$ gives

$$\gamma_1(1+\zeta_1)\zeta_2 = \gamma_2\zeta_1(1+\zeta_2). \tag{D.263}$$

Using equations (D.257) and (D.263), equation (D.255) implies that

$$(1+\gamma_1)\gamma_2\zeta_1 = \gamma_1(1+\gamma_2)\zeta_2. \tag{D.264}$$

Note that $\zeta_1 + \gamma_1(1+\zeta_1) \neq 0$, which is a consequence of $\gamma_1\zeta_1 > 0$. Indeed, if $\zeta_1 + \gamma_1(1+\zeta_1) = 0$, then $\gamma_1\zeta_1 = -\gamma_1^2/(1+\gamma_1) \leq 0$ because $1+\gamma_1 > 0$ (Assumption $\mathcal{P}$-γ), which contradicts $\gamma_1\zeta_1 > 0$. Using equations (D.257), (D.261), and (D.264), equation (D.255) implies that

$$(\gamma_1 - \gamma_2)(1+\gamma_1)\frac{\gamma_2}{\gamma_1}(\zeta_1 + \gamma_1(1+\zeta_1)) = 0,$$

which in turn implies that $\gamma_1 = \gamma_2$ because $\gamma_1 \neq 0$, $\gamma_2 \neq 0$, $1+\gamma_1 > 0$, and $\zeta_1 + \gamma_1(1+\zeta_1) \neq 0$. Equation (D.254) and $\gamma_1 = \gamma_2$ give $\varsigma_1^2 = \varsigma_1^2$, which is equivalent to $\varsigma_1 = \varsigma_2$ because $\varsigma_1 > 0$ and $\varsigma_2 > 0$. Equation (D.257), $\gamma_1 = \gamma_2$, and $\varsigma_1 = \varsigma_2$ give $(\zeta_1 - \zeta_2)\gamma_1(1+\gamma_1)\varsigma_1^2 = 0$, which implies that $\zeta_1 = \zeta_2$ because $\gamma_1 \neq 0$, $1+\gamma_1 > 0$, and $\varsigma_1 > 0$.

To sum up, I have established that $(\gamma_1, \zeta_1, \varsigma_1) = (\gamma_2, \zeta_2, \varsigma_2)$. Finally, note that the system of equations (D.254) to (D.259) has infinitely many solutions without the restrictions $\gamma_1\zeta_1 \geq 0$ and $\gamma_2\zeta_2 \geq 0$, for example, for all $(c, s) \in \gamma(\Theta) \times \mathbb{R}_{++}$,

$$\left(\gamma_1, \gamma_2, \zeta_1, \zeta_2, \varsigma_1, \varsigma_2\right) = \left(0, c, 0, -\frac{c}{1+c}, s, (1+c)s\right)$$

is a solution. ■

Proof of Proposition 1.119

In what follows, I write $\bar{\boldsymbol{C}}$ for $\bar{\boldsymbol{C}}(H)$. I prove by contraposition that the six matrices of Proposition 1.105 are linearly independent if the six matrices of Proposition 1.118 are linearly independent. Suppose the six matrices of Proposition 1.105 are linearly dependent, that is, there exists a 6-tuple $(c_1, \ldots, c_6) \neq \mathbf{0}_6$ such that

$$c_1 \boldsymbol{I}_n + c_2\big(\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top\big) + c_3 \bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top + c_4\big(\bar{\boldsymbol{C}}^2 + \bar{\boldsymbol{C}}^{\top 2}\big) + c_5 \bar{\boldsymbol{C}}\big(\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top\big)\bar{\boldsymbol{C}}^\top + c_6 \bar{\boldsymbol{C}}^2 \bar{\boldsymbol{C}}^{\top 2} = \boldsymbol{O}_n.$$

Let

$$\boldsymbol{T} := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and let $(d_1, \ldots, d_6) := \boldsymbol{T}(c_1, \ldots, c_6)$. Note that $(d_1, \ldots, d_6) \neq \mathbf{0}_6$ because $\boldsymbol{T}$ is nonsingular and $(c_1, \ldots, c_6) \neq \mathbf{0}_6$. We have

$$\begin{aligned} \boldsymbol{O}_n = {} & d_1 \boldsymbol{I}_n + d_2\big(\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top\big) + d_3 \bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top + d_4\big(q(\bar{\boldsymbol{C}})\big(\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top\big) + \big(\bar{\boldsymbol{C}} + \bar{\boldsymbol{C}}^\top\big)q(\bar{\boldsymbol{C}})^\top\big) \\ & + d_5\big(q(\bar{\boldsymbol{C}})\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top + \bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top q(\bar{\boldsymbol{C}})^\top\big) + d_6 q(\bar{\boldsymbol{C}})\bar{\boldsymbol{C}}\bar{\boldsymbol{C}}^\top q(\bar{\boldsymbol{C}})^\top, \end{aligned}$$

which implies that the six matrices of Proposition 1.118 are linearly dependent. ■

Proof of Proposition 1.120

The proof is omitted. ■

Appendix E

Proofs of Chapter 2 Results

Proof of Proposition 2.1

The proof is omitted because it is similar to the proof of Proposition 1.16 (with $f = \mathrm{id}_{\mathbb{R}_+}$). ∎

Proof of Corollary 2.2

Note that

$$x^\star_{\theta_1} = \big(\mathrm{diag}(\bar{\beta} + \bar{\gamma}) - \mathrm{diag}(\bar{\gamma})\bar{A}(G)\big)^{-1}\bar{\alpha}$$

according to Proposition 2.1, which is equivalent to

$$x^\star_{\theta_1} = \mathrm{diag}(\bar{\beta} + \bar{\gamma})^{-1}\bar{\alpha} + \mathrm{diag}(\bar{\beta} + \bar{\gamma})^{-1}\,\mathrm{diag}(\bar{\gamma})\bar{A}(G)x^\star_{\theta_1}.$$

Using the preceding results, we find

$$x^\star_i(\theta_1) = e_i^\top x^\star_{\theta_1} = \frac{\bar{\alpha}_i}{\bar{\beta}_i + \bar{\gamma}_i} + \frac{\bar{\gamma}_i}{\bar{\beta}_i + \bar{\gamma}_i} e_i^\top \bar{A}(G)\big(\mathrm{diag}(\bar{\beta} + \bar{\gamma}) - \mathrm{diag}(\bar{\gamma})\bar{A}(G)\big)^{-1}\bar{\alpha},$$

where $\{e_k\}_{k=1}^n$ denotes the canonical basis of $\mathbb{R}^n$ with (column) basis vector $e_k := (\delta_{k,1}, \ldots, \delta_{k,n})$, where $\delta_{k,l}$ is Kronecker's delta of k and l. ∎

Proof of Proposition 2.3

Note that the unique and interior (Bayesian) NE of Γ is given by (Proposition 2.1)

$$x^\star_{\theta_1} = \big(\mathrm{diag}(\bar{\beta} + \bar{\gamma}) - \mathrm{diag}(\bar{\gamma})\bar{A}(G)\big)^{-1}\bar{\alpha},$$

from which it follows that

$$\forall i \in \mathcal{I} \quad x^\star_i(\theta_1) = e_i^\top\big(\mathrm{diag}(\bar{\beta} + \bar{\gamma}) - \mathrm{diag}(\bar{\gamma})\bar{A}(G)\big)^{-1}\bar{\alpha}.$$

Recall that, for all $i \in \mathcal{I}$, $\bar{\alpha}_i = \bar{\alpha}^C + \bar{\alpha}_i^I$, $\bar{\beta}_i = \bar{\beta}^C + \bar{\beta}_i^I$, and $\bar{\gamma}_i = \bar{\gamma}^C + \bar{\gamma}_i^I$.

Results 2.3.1, 2.3.2, and 2.3.3 are proved with the use of matrix calculus and the following fact.

Fact E.1 (*E.1.1*) *For all* $i \in \mathcal{I}$, $[(\operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\boldsymbol{\gamma}}) - \operatorname{diag}(\bar{\boldsymbol{\gamma}})\bar{\boldsymbol{A}}(G))^{-1}]_{i,i} > 0$.

(*E.1.2*) *For all* $(i,j) \in \mathcal{I}^2$ *with* $i \neq j$, $[(\operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\boldsymbol{\gamma}}) - \operatorname{diag}(\bar{\boldsymbol{\gamma}})\bar{\boldsymbol{A}}(G))^{-1}]_{i,j} \geq 0$.

Proof of Fact E.1 Note that

$$\begin{aligned}
&\big(\operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\boldsymbol{\gamma}}) - \operatorname{diag}(\bar{\boldsymbol{\gamma}})\bar{\boldsymbol{A}}(G)\big)^{-1} \\
&\quad = \big(\boldsymbol{I}_n - \operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\boldsymbol{\gamma}})^{-1}\operatorname{diag}(\bar{\boldsymbol{\gamma}})\bar{\boldsymbol{A}}(G)\big)^{-1}\operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\boldsymbol{\gamma}})^{-1} \\
&\quad \geq_c \operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\boldsymbol{\gamma}})^{-1}
\end{aligned} \tag{E.1}$$

because $\operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\boldsymbol{\gamma}})^{-1}\operatorname{diag}(\bar{\boldsymbol{\gamma}})\bar{\boldsymbol{A}}(G) \geq_c \boldsymbol{O}_n$ and $\rho(\operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\boldsymbol{\gamma}})^{-1}\operatorname{diag}(\bar{\boldsymbol{\gamma}})\bar{\boldsymbol{A}}(G)) < 1$ imply that $(\boldsymbol{I}_n - \operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\boldsymbol{\gamma}})^{-1}\operatorname{diag}(\bar{\boldsymbol{\gamma}})\bar{\boldsymbol{A}}(G))^{-1} \geq_c \boldsymbol{I}_n$ (Lemma B.6). Results E.1.1 and E.1.2 follow from (E.1). ■

Let us now prove Results 2.3.1, 2.3.2, and 2.3.3. Let $(i,j) \in \mathcal{I}^2$.

Proof of Result 2.3.1 We find

$$\begin{aligned}
\frac{\partial x_i^\star(\boldsymbol{\theta}_1)}{\partial \bar{\alpha}_j^I} &= \boldsymbol{e}_i^\top\big(\operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\boldsymbol{\gamma}}) - \operatorname{diag}(\bar{\boldsymbol{\gamma}})\bar{\boldsymbol{A}}(G)\big)^{-1}\frac{\partial \bar{\boldsymbol{\alpha}}}{\partial \bar{\alpha}_j^I} \\
&= \boldsymbol{e}_i^\top\big(\operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\boldsymbol{\gamma}}) - \operatorname{diag}(\bar{\boldsymbol{\gamma}})\bar{\boldsymbol{A}}(G)\big)^{-1}\boldsymbol{e}_j \\
&= \big[\big(\operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\boldsymbol{\gamma}}) - \operatorname{diag}(\bar{\boldsymbol{\gamma}})\bar{\boldsymbol{A}}(G)\big)^{-1}\big]_{i,j}.
\end{aligned}$$

It follows that $\partial x_i^\star(\boldsymbol{\theta}_1)/\partial \bar{\alpha}_j^I \geq 0$ if $i \neq j$ (Result E.1.2) and $\partial x_i^\star(\boldsymbol{\theta}_1)/\partial \bar{\alpha}_i^I > 0$ (Result E.1.1). ■

Proof of Result 2.3.2 We find

$$\begin{aligned}
\frac{\partial \boldsymbol{x}^\star_{\boldsymbol{\theta}_1}}{\partial \bar{\beta}_j^I} &= \frac{\partial}{\partial \bar{\beta}_j^I}\big(\operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\boldsymbol{\gamma}}) - \operatorname{diag}(\bar{\boldsymbol{\gamma}})\bar{\boldsymbol{A}}(G)\big)^{-1}\bar{\boldsymbol{\alpha}} \\
&= -\big(\operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\boldsymbol{\gamma}}) - \operatorname{diag}(\bar{\boldsymbol{\gamma}})\bar{\boldsymbol{A}}(G)\big)^{-1} \\
&\quad\times \frac{\partial}{\partial \bar{\beta}_j^I}\big(\operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\boldsymbol{\gamma}}) - \operatorname{diag}(\bar{\boldsymbol{\gamma}})\bar{\boldsymbol{A}}(G)\big)\underbrace{\big(\operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\boldsymbol{\gamma}}) - \operatorname{diag}(\bar{\boldsymbol{\gamma}})\bar{\boldsymbol{A}}(G)\big)^{-1}\bar{\boldsymbol{\alpha}}}_{=\boldsymbol{x}^\star_{\boldsymbol{\theta}_1}} \\
&= -\big(\operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\boldsymbol{\gamma}}) - \operatorname{diag}(\bar{\boldsymbol{\gamma}})\bar{\boldsymbol{A}}(G)\big)^{-1}\boldsymbol{e}_j\boldsymbol{e}_j^\top\boldsymbol{x}^\star_{\boldsymbol{\theta}_1} \\
&= -x_j^\star(\boldsymbol{\theta}_1)\big(\operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\boldsymbol{\gamma}}) - \operatorname{diag}(\bar{\boldsymbol{\gamma}})\bar{\boldsymbol{A}}(G)\big)^{-1}\boldsymbol{e}_j.
\end{aligned}$$

It follows that $\partial x_i^\star(\boldsymbol{\theta}_1)/\partial \bar{\beta}_j^I \leq 0$ if $i \neq j$ (Result E.1.2) and $\partial x_i^\star(\boldsymbol{\theta}_1)/\partial \bar{\beta}_i^I < 0$ (Result E.1.1) because $x_j^\star(\boldsymbol{\theta}_1) > 0$. ■

Proof of Result 2.3.3 We find

$$\frac{\partial \boldsymbol{x}^\star_{\boldsymbol{\theta}_1}}{\partial \bar{\gamma}_j^I} = \frac{\partial}{\partial \bar{\gamma}_j^I}\big(\operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\boldsymbol{\gamma}}) - \operatorname{diag}(\bar{\boldsymbol{\gamma}})\bar{\boldsymbol{A}}(G)\big)^{-1}\bar{\boldsymbol{\alpha}}$$

$$
\begin{aligned}
&= -\big(\mathrm{diag}(\bar{\boldsymbol{\beta}}+\bar{\gamma}) - \mathrm{diag}(\bar{\gamma})\bar{\boldsymbol{A}}(G)\big)^{-1} \\
&\quad \times \frac{\partial}{\partial \bar{\gamma}_j^I}\big(\mathrm{diag}(\bar{\boldsymbol{\beta}}+\bar{\gamma}) - \mathrm{diag}(\bar{\gamma})\bar{\boldsymbol{A}}(G)\big) \underbrace{\big(\mathrm{diag}(\bar{\boldsymbol{\beta}}+\bar{\gamma}) - \mathrm{diag}(\bar{\gamma})\bar{\boldsymbol{A}}(G)\big)^{-1}\bar{\boldsymbol{\alpha}}}_{=\boldsymbol{x}^\star_{\boldsymbol{\theta}_1}} \\
&= -\big(\mathrm{diag}(\bar{\boldsymbol{\beta}}+\bar{\gamma}) - \mathrm{diag}(\bar{\gamma})\bar{\boldsymbol{A}}(G)\big)^{-1}\big(\boldsymbol{e}_j\boldsymbol{e}_j^\mathsf{T} - \boldsymbol{e}_j\boldsymbol{e}_j^\mathsf{T}\bar{\boldsymbol{A}}(G)\big)\boldsymbol{x}^\star_{\boldsymbol{\theta}_1} \\
&= -\big(\mathrm{diag}(\bar{\boldsymbol{\beta}}+\bar{\gamma}) - \mathrm{diag}(\bar{\gamma})\bar{\boldsymbol{A}}(G)\big)^{-1}\boldsymbol{e}_j \underbrace{\boldsymbol{e}_j^\mathsf{T}\big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)\boldsymbol{x}^\star_{\boldsymbol{\theta}_1}}_{= x_j^\star(\boldsymbol{\theta}_1) - \sum_{k=1}^n \bar{a}_{j,k} x_k^\star(\boldsymbol{\theta}_1)} \\
&= -\left(x_j^\star(\boldsymbol{\theta}_1) - \sum_{k=1}^n \bar{a}_{j,k} x_k^\star(\boldsymbol{\theta}_1)\right)\big(\mathrm{diag}(\bar{\boldsymbol{\beta}}+\bar{\gamma}) - \mathrm{diag}(\bar{\gamma})\bar{\boldsymbol{A}}(G)\big)^{-1}\boldsymbol{e}_j.
\end{aligned}
$$

It follows that $\partial x_i^\star(\boldsymbol{\theta}_1)/\partial\bar{\gamma}_j^I \le 0$ (respectively, $\partial x_i^\star(\boldsymbol{\theta}_1)/\partial\bar{\gamma}_j^I \ge 0$) if $i \ne j$ and $x_j^\star(\boldsymbol{\theta}_1) \ge \sum_{k=1}^n \bar{a}_{j,k}x_k^\star(\boldsymbol{\theta}_1)$ (respectively, $x_j^\star(\boldsymbol{\theta}_1) \le \sum_{k=1}^n \bar{a}_{j,k}x_k^\star(\boldsymbol{\theta}_1)$) because $[(\mathrm{diag}(\bar{\boldsymbol{\beta}}+\bar{\gamma}) - \mathrm{diag}(\bar{\gamma})\bar{\boldsymbol{A}}(G))^{-1}]_{i,j} \ge 0$ (Result E.1.2) and $\partial x_i^\star(\boldsymbol{\theta}_1)/\partial\bar{\gamma}_i^I < 0$ (respectively, $\partial x_i^\star(\boldsymbol{\theta}_1)/\partial\bar{\gamma}_i^I > 0$) if $x_i^\star(\boldsymbol{\theta}_1) > \sum_{k=1}^n \bar{a}_{i,k}x_k^\star(\boldsymbol{\theta}_1)$ (respectively, $x_i^\star(\boldsymbol{\theta}_1) < \sum_{k=1}^n \bar{a}_{i,k}x_k^\star(\boldsymbol{\theta}_1)$) because $[(\mathrm{diag}(\bar{\boldsymbol{\beta}}+\bar{\gamma}) - \mathrm{diag}(\bar{\gamma})\bar{\boldsymbol{A}}(G))^{-1}]_{i,i} > 0$ (Result E.1.1). ∎

Proof of Proposition 2.4

Similar to the proof of Proposition 2.3, Results 2.4.1, 2.4.2, and 2.4.3 are proved with the use of matrix calculus. Let $(i,j) \in \mathcal{I}^2$.

Proof of Result 2.4.1 We find

$$
\begin{aligned}
\frac{\partial x_i^\star(\boldsymbol{\theta}_1)}{\partial \bar{\alpha}^C} &= \boldsymbol{e}_i^\mathsf{T}\big(\mathrm{diag}(\bar{\boldsymbol{\beta}}+\bar{\gamma}) - \mathrm{diag}(\bar{\gamma})\bar{\boldsymbol{A}}(G)\big)^{-1}\frac{\partial\bar{\boldsymbol{\alpha}}}{\partial\bar{\alpha}^C} \\
&= \boldsymbol{e}_i^\mathsf{T}\big(\mathrm{diag}(\bar{\boldsymbol{\beta}}+\bar{\gamma}) - \mathrm{diag}(\bar{\gamma})\bar{\boldsymbol{A}}(G)\big)^{-1}\boldsymbol{1}_n \\
&\ge \boldsymbol{e}_i^\mathsf{T}\,\mathrm{diag}(\bar{\boldsymbol{\beta}}+\bar{\gamma})^{-1}\boldsymbol{1}_n \\
&= \frac{1}{\bar{\beta}_i + \bar{\gamma}_i} \\
&> 0,
\end{aligned}
$$

where the first inequality follows from (E.1) and the second from $\bar{\beta}_i > 0$ and $\bar{\gamma}_i \ge 0$. ∎

Proof of Result 2.4.2 We find

$$
\begin{aligned}
\frac{\partial \boldsymbol{x}^\star_{\boldsymbol{\theta}_1}}{\partial\bar{\beta}^C} &= \frac{\partial}{\partial\bar{\beta}^C}\big(\mathrm{diag}(\bar{\boldsymbol{\beta}}+\bar{\gamma}) - \mathrm{diag}(\bar{\gamma})\bar{\boldsymbol{A}}(G)\big)^{-1}\bar{\boldsymbol{\alpha}} \\
&= -\big(\mathrm{diag}(\bar{\boldsymbol{\beta}}+\bar{\gamma}) - \mathrm{diag}(\bar{\gamma})\bar{\boldsymbol{A}}(G)\big)^{-1}
\end{aligned}
$$

$$\times \underbrace{\frac{\partial}{\partial \bar{\beta}^C}\big(\mathrm{diag}(\bar{\boldsymbol{\beta}}+\bar{\boldsymbol{\gamma}}) - \mathrm{diag}(\bar{\boldsymbol{\gamma}})\bar{\boldsymbol{A}}(G)\big)}_{=\boldsymbol{I}_n} \underbrace{\big(\mathrm{diag}(\bar{\boldsymbol{\beta}}+\bar{\boldsymbol{\gamma}}) - \mathrm{diag}(\bar{\boldsymbol{\gamma}})\bar{\boldsymbol{A}}(G)\big)^{-1}\bar{\boldsymbol{\alpha}}}_{=\boldsymbol{x}^\star_{\boldsymbol{\theta}_1}}$$

$$\begin{aligned}
&= -\big(\mathrm{diag}(\bar{\boldsymbol{\beta}}+\bar{\boldsymbol{\gamma}}) - \mathrm{diag}(\bar{\boldsymbol{\gamma}})\bar{\boldsymbol{A}}(G)\big)^{-1}\boldsymbol{x}^\star_{\boldsymbol{\theta}_1}\\
&\leq_c -\mathrm{diag}(\bar{\boldsymbol{\beta}}+\bar{\boldsymbol{\gamma}})^{-1}\boldsymbol{x}^\star_{\boldsymbol{\theta}_1}\\
&<_c \mathbf{0}_n,
\end{aligned}$$

where the first inequality follows from (E.1) and the second from $\bar{\boldsymbol{\beta}} >_c \mathbf{0}_n$, $\bar{\boldsymbol{\gamma}} \geq_c \mathbf{0}_n$, and $\boldsymbol{x}^\star_{\boldsymbol{\theta}_1} >_c \mathbf{0}_n$. It follows that $\partial x_i^\star(\boldsymbol{\theta}_1)/\partial\bar{\beta}^C < 0$. ■

Proof of Result 2.4.3 We find

$$\begin{aligned}
\frac{\partial \boldsymbol{x}^\star_{\boldsymbol{\theta}_1}}{\partial \bar{\gamma}^C} &= \frac{\partial}{\partial \bar{\gamma}^C}\big(\mathrm{diag}(\bar{\boldsymbol{\beta}}+\bar{\boldsymbol{\gamma}}) - \mathrm{diag}(\bar{\boldsymbol{\gamma}})\bar{\boldsymbol{A}}(G)\big)^{-1}\bar{\boldsymbol{\alpha}}\\
&= -\big(\mathrm{diag}(\bar{\boldsymbol{\beta}}+\bar{\boldsymbol{\gamma}}) - \mathrm{diag}(\bar{\boldsymbol{\gamma}})\bar{\boldsymbol{A}}(G)\big)^{-1}\\
&\quad\times \frac{\partial}{\partial \bar{\gamma}^C}\big(\mathrm{diag}(\bar{\boldsymbol{\beta}}+\bar{\boldsymbol{\gamma}}) - \mathrm{diag}(\bar{\boldsymbol{\gamma}})\bar{\boldsymbol{A}}(G)\big) \underbrace{\big(\mathrm{diag}(\bar{\boldsymbol{\beta}}+\bar{\boldsymbol{\gamma}}) - \mathrm{diag}(\bar{\boldsymbol{\gamma}})\bar{\boldsymbol{A}}(G)\big)^{-1}\bar{\boldsymbol{\alpha}}}_{=\boldsymbol{x}^\star_{\boldsymbol{\theta}_1}}\\
&= -\big(\mathrm{diag}(\bar{\boldsymbol{\beta}}+\bar{\boldsymbol{\gamma}}) - \mathrm{diag}(\bar{\boldsymbol{\gamma}})\bar{\boldsymbol{A}}(G)\big)^{-1}\big(\boldsymbol{I}_n - \bar{\boldsymbol{A}}(G)\big)\boldsymbol{x}^\star_{\boldsymbol{\theta}_1}.
\end{aligned}$$

It follows that the sign of $\partial x_i^\star(\boldsymbol{\theta}_1)/\partial\bar{\gamma}^C$ depends on $\bar{\boldsymbol{\beta}}$, $\bar{\boldsymbol{\gamma}}$, and G. ■

Proof of Proposition 2.5

Player i's (pure) strategy, $x_i\colon \Theta \to \mathbb{R}_+$, is given by the values it assumes on the finite type space $\Theta = \{\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_T\}$. For a particular type $\boldsymbol{\theta}_r \in \Theta$, that is, for a particular value (or realization) $\boldsymbol{\theta}_r$ of his signal $\boldsymbol{s}_i$, player i chooses action $x_i(\boldsymbol{\theta}_r)$ in order to maximize his conditional expected utility given the event $\{\boldsymbol{s}_i = \theta_r\}$,

$$\mathbb{E}\big(u_i\big(\mathrm{id}_\Omega, (x_1 \circ \boldsymbol{s}_1, \dots, x_n \circ \boldsymbol{s}_n)\big) \;\big|\; \boldsymbol{s}_i = \boldsymbol{\theta}_r\big).$$

A strategy profile $(x_1^\star, \dots, x_n^\star)$ is an interior BNE of $\Gamma(\alpha, \beta, \gamma)$ if and only if it satisfies three conditions: the interiority conditions, that is,

$$\forall (i,r) \in \mathcal{I} \times \{1, \dots, T\} \quad x_i^\star(\boldsymbol{\theta}_r) > 0;$$

the first-order conditions, that is,

$$\forall (i,r) \in \mathcal{I} \times \{1, \dots, T\} \quad \frac{\partial\, \mathbb{E}\big(u_i\big(\mathrm{id}_\Omega, (x_1^\star \circ \boldsymbol{s}_1, \dots, x_n^\star \circ \boldsymbol{s}_n)\big) \;\big|\; \boldsymbol{s}_i = \boldsymbol{\theta}_r\big)}{\partial x_i^\star(\boldsymbol{\theta}_r)} = 0; \quad \text{(E.2)}$$

and the second-order conditions, that is,

$$\forall (i,r) \in \mathcal{I} \times \{1, \dots, T\} \quad \frac{\partial^2\, \mathbb{E}\big(u_i\big(\mathrm{id}_\Omega, (x_1^\star \circ \boldsymbol{s}_1, \dots, x_n^\star \circ \boldsymbol{s}_n)\big) \;\big|\; \boldsymbol{s}_i = \boldsymbol{\theta}_r\big)}{\partial x_i^\star(\boldsymbol{\theta}_r)^2} < 0.$$

Let $(i, r) \in \mathcal{I} \times \{1, \ldots, T\}$. We find

$$
\begin{aligned}
&\mathbb{E}\big(u_i\big(\mathrm{id}_\Omega, (x_1^\star \circ \boldsymbol{s}_1, \ldots, x_n^\star \circ \boldsymbol{s}_n)\big) \bigm| \boldsymbol{s}_i = \boldsymbol{\theta}_r\big)\\
&\quad = \mathbb{E}(\alpha_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r) x_i^\star(\boldsymbol{\theta}_r) - \frac{1}{2}\,\mathbb{E}(\beta_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r) x_i^\star(\boldsymbol{\theta}_r)^2\\
&\qquad - \frac{1}{2}\,\mathbb{E}\left(\gamma_i x_i^\star(\boldsymbol{\theta}_r)^2 - 2\gamma_i x_i^\star(\boldsymbol{\theta}_r)\sum_{j=1}^n \bar{a}_{i,j}(x_j^\star \circ \boldsymbol{s}_j) + \gamma_i \left(\sum_{j=1}^n \bar{a}_{i,j}(x_j^\star \circ \boldsymbol{s}_j)\right)^2 \,\middle|\, \boldsymbol{s}_i = \boldsymbol{\theta}_r\right)\\
&\quad = \mathbb{E}(\alpha_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r) x_i^\star(\boldsymbol{\theta}_r) - \frac{1}{2}\,\mathbb{E}(\beta_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r) x_i^\star(\boldsymbol{\theta}_r)^2\\
&\qquad - \frac{1}{2}\,\mathbb{E}(\gamma_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r) x_i^\star(\boldsymbol{\theta}_r)^2 + \sum_{j=1}^n \bar{a}_{i,j}\,\mathbb{E}\big(\gamma_i (x_j^\star \circ \boldsymbol{s}_j) \bigm| \boldsymbol{s}_i = \boldsymbol{\theta}_r\big) x_i^\star(\boldsymbol{\theta}_r)\\
&\qquad - \frac{1}{2}\,\mathbb{E}\left(\gamma_i \left(\sum_{j=1}^n \bar{a}_{i,j}(x_j^\star \circ \boldsymbol{s}_j)\right)^2 \,\middle|\, \boldsymbol{s}_i = \boldsymbol{\theta}_r\right),
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
&\frac{\partial\,\mathbb{E}\big(u_i\big(\mathrm{id}_\Omega, (x_1^\star \circ \boldsymbol{s}_1, \ldots, x_n^\star \circ \boldsymbol{s}_n)\big) \bigm| \boldsymbol{s}_i = \boldsymbol{\theta}_r\big)}{\partial x_i^\star(\boldsymbol{\theta}_r)}\\
&\quad = \mathbb{E}(\alpha_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r) - \big(\mathbb{E}(\beta_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r) + \mathbb{E}(\gamma_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r)\big) x_i^\star(\boldsymbol{\theta}_r)\\
&\qquad + \sum_{j=1}^n \bar{a}_{i,j}\,\mathbb{E}\big(\gamma_i (x_j^\star \circ \boldsymbol{s}_j) \bigm| \boldsymbol{s}_i = \boldsymbol{\theta}_r\big)\\
&\quad = \mathbb{E}(\alpha_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r) - \mathbb{E}(\beta_i + \gamma_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r) x_i^\star(\boldsymbol{\theta}_r) \qquad \text{(E.3)}\\
&\qquad + \sum_{j=1}^n \bar{a}_{i,j}\,\mathbb{E}\big(\gamma_i (x_j^\star \circ \boldsymbol{s}_j) \bigm| \boldsymbol{s}_i = \boldsymbol{\theta}_r\big)
\end{aligned}
$$

and

$$
\frac{\partial^2\,\mathbb{E}\big(u_i\big(\mathrm{id}_\Omega, (x_1^\star \circ \boldsymbol{s}_1, \ldots, x_n^\star \circ \boldsymbol{s}_n)\big) \bigm| \boldsymbol{s}_i = \boldsymbol{\theta}_r\big)}{\partial x_i^\star(\boldsymbol{\theta}_r)^2} = -\,\mathbb{E}(\beta_i + \gamma_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r) < 0,
$$

where the inequality follows from $\beta_i > 0$ and $\gamma_i \geq 0$. Note that, for all $(i, r) \in \mathcal{I} \times \{1, \ldots, T\}$,

$$
\begin{aligned}
\mathbb{E}\big(\gamma_i (x_j^\star \circ \boldsymbol{s}_j) \bigm| \boldsymbol{s}_i = \boldsymbol{\theta}_r\big) &= \mathbb{E}\big(\mathbb{1}_\Omega\, \gamma_i (x_j^\star \circ \boldsymbol{s}_j) \bigm| \boldsymbol{s}_i = \boldsymbol{\theta}_r\big)\\
&= \mathbb{E}\left(\sum_{q=1}^T \mathbb{1}_{\{\boldsymbol{s}_j = \boldsymbol{\theta}_q\}}\, \gamma_i (x_j^\star \circ \boldsymbol{s}_j) \,\middle|\, \boldsymbol{s}_i = \boldsymbol{\theta}_r\right)\\
&= \sum_{q=1}^T \mathbb{E}\big(\mathbb{1}_{\{\boldsymbol{s}_j = \boldsymbol{\theta}_q\}}\, \gamma_i (x_j^\star \circ \boldsymbol{s}_j) \bigm| \boldsymbol{s}_i = \boldsymbol{\theta}_r\big)\\
&= \sum_{q=1}^T \frac{\mathbb{E}\big(\mathbb{1}_{\{\boldsymbol{s}_j = \boldsymbol{\theta}_q\}}\, \mathbb{1}_{\{\boldsymbol{s}_i = \boldsymbol{\theta}_r\}}\, \gamma_i (x_j^\star \circ \boldsymbol{s}_j)\big)}{\mathbb{P}(\boldsymbol{s}_i = \boldsymbol{\theta}_r)}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{q=1}^{T} \frac{\mathbb{E}\big(\mathbb{1}_{\{s_j=\theta_q\}\cap\{s_i=\theta_r\}}\, \gamma_i(x_j^\star \circ s_j)\big)}{\mathbb{P}(s_i = \theta_r)} \\
&= \sum_{q=1}^{T} \frac{\mathbb{P}(s_j = \theta_q, s_i = \theta_r)}{\mathbb{P}(s_i = \theta_r)} \frac{\mathbb{E}\big(\mathbb{1}_{\{s_j=\theta_q\}\cap\{s_i=\theta_r\}}\, \gamma_i(x_j^\star \circ s_j)\big)}{\mathbb{P}(s_j = \theta_q, s_i = \theta_r)} \\
&= \sum_{q=1}^{T} \mathbb{P}(s_j = \theta_q \mid s_i = \theta_r)\, \mathbb{E}\big(\gamma_i(x_j^\star \circ s_j) \bigm| s_j = \theta_q, s_i = \theta_r\big) \\
&= \sum_{q=1}^{T} \mathbb{P}(s_j = \theta_q \mid s_i = \theta_r)\, \mathbb{E}(\gamma_i \mid s_j = \theta_q, s_i = \theta_r) x_j^\star(\theta_q),
\end{aligned}
$$

where the first equality follows from

$$
\mathbb{1}_{\Omega} = \mathbb{1}_{\bigcup_{q=1}^{T} s_j^{-1}(\{\theta_q\})} = \sum_{q=1}^{T} \mathbb{1}_{s_j^{-1}(\{\theta_q\})} = \sum_{q=1}^{T} \mathbb{1}_{\{s_j=\theta_q\}} .
$$

Using (E.3) and the preceding result, the first-order conditions (E.2) are equivalent to

$$
\begin{aligned}
\forall (i,r) \in \mathcal{I} \times \{1,\dots,T\} \quad & \mathbb{E}(\alpha_i \mid s_i = \theta_r) - \mathbb{E}(\beta_i + \gamma_i \mid s_i = \theta_r) x_i^\star(\theta_r) \\
& + \sum_{j=1}^{n} \bar{a}_{i,j} \sum_{q=1}^{T} \mathbb{P}(s_j = \theta_q \mid s_i = \theta_r)\, \mathbb{E}(\gamma_i \mid s_j = \theta_q, s_i = \theta_r) x_j^\star(\theta_q) = 0,
\end{aligned}
$$

which are equivalent to

$$
\begin{aligned}
\forall (i,r) \in \mathcal{I} \times \{1,\dots,T\} \quad x_i^\star(\theta_r) = & \frac{\mathbb{E}(\alpha_i \mid s_i = \theta_r)}{\mathbb{E}(\beta_i + \gamma_i \mid s_i = \theta_r)} + \frac{1}{\mathbb{E}(\beta_i + \gamma_i \mid s_i = \theta_r)} \\
& \times \sum_{j=1}^{n} \bar{a}_{i,j} \sum_{q=1}^{T} \mathbb{P}(s_j = \theta_q \mid s_i = \theta_r)\, \mathbb{E}(\gamma_i \mid s_j = \theta_q, s_i = \theta_r) x_j^\star(\theta_q),
\end{aligned}
$$

which are equivalent to

$$
\forall i \in \mathcal{I} \quad \boldsymbol{x}_{\Theta,i}^\star = \operatorname{diag}\big(\boldsymbol{\beta}_i^\mu + \boldsymbol{\gamma}_i^\mu\big)^{-1} \boldsymbol{\alpha}_i^\mu + \operatorname{diag}\big(\boldsymbol{\beta}_i^\mu + \boldsymbol{\gamma}_i^\mu\big)^{-1} \sum_{j=1}^{n} \bar{a}_{i,j} (\boldsymbol{\Pi}_{i,j} \circ \boldsymbol{\Gamma}_{i,j}) \boldsymbol{x}_{\Theta,j}^\star,
$$

which are equivalent to

$$
\boldsymbol{x}_{\Theta}^\star = \operatorname{diag}\big(\boldsymbol{\beta}^\mu + \boldsymbol{\gamma}^\mu\big)^{-1} \boldsymbol{\alpha}^\mu + \operatorname{diag}\big(\boldsymbol{\beta}^\mu + \boldsymbol{\gamma}^\mu\big)^{-1} \Big(\big(\bar{\boldsymbol{A}}(G) \otimes \mathbf{1}_T \mathbf{1}_T^\top\big) \circ \boldsymbol{\Pi} \circ \boldsymbol{\Gamma} \Big) \boldsymbol{x}_{\Theta}^\star,
$$

which is equivalent to

$$
\Big(\boldsymbol{I}_{nT} - \operatorname{diag}\big(\boldsymbol{\beta}^\mu + \boldsymbol{\gamma}^\mu\big)^{-1} \Big(\big(\bar{\boldsymbol{A}}(G) \otimes \mathbf{1}_T \mathbf{1}_T^\top\big) \circ \boldsymbol{\Pi} \circ \boldsymbol{\Gamma} \Big) \Big) \boldsymbol{x}_{\Theta}^\star = \operatorname{diag}\big(\boldsymbol{\beta}^\mu + \boldsymbol{\gamma}^\mu\big)^{-1} \boldsymbol{\alpha}^\mu. \tag{E.4}
$$

In the remainder of the proof, we show that the system of equations (E.4) has a unique solution. We proceed in two steps.

First, we establish that the matrix

$$\boldsymbol{I}_{nT} - \operatorname{diag}\big(\boldsymbol{\beta}^{\mu} + \boldsymbol{\gamma}^{\mu}\big)^{-1}\Big(\big(\bar{\boldsymbol{A}}(G) \otimes \mathbf{1}_T\mathbf{1}_T^{\mathsf{T}}\big) \circ \boldsymbol{\Pi} \circ \boldsymbol{\Gamma}\Big) \tag{E.5}$$

is nonsingular. Let the square matrix $\boldsymbol{N}$ of order nT be defined by

$$\boldsymbol{N} := \operatorname{diag}\big(\boldsymbol{\beta}^{\mu} + \boldsymbol{\gamma}^{\mu}\big)^{-1}\Big(\big(\bar{\boldsymbol{A}}(G) \otimes \mathbf{1}_T\mathbf{1}_T^{\mathsf{T}}\big) \circ \boldsymbol{\Pi} \circ \boldsymbol{\Gamma}\Big).$$

We show that $\rho(\boldsymbol{N}) < 1$. To this end, note that $\rho(\boldsymbol{N}) \leq \|\boldsymbol{N}\|_\infty$ (Lemma B.7). The matrix $\boldsymbol{N}$ can be regarded as a block matrix with n^2 square blocks of orders T, where the block in block row $i \in \mathcal{I}$ and block column $j \in \mathcal{I}$ is equal to

$$\bar{a}_{i,j} \operatorname{diag}\big(\boldsymbol{\beta}_i^{\mu} + \boldsymbol{\gamma}_i^{\mu}\big)^{-1}(\boldsymbol{\Pi}_{i,j} \circ \boldsymbol{\Gamma}_{i,j}),$$

from which it follows that $\|\boldsymbol{N}\|_\infty$ is equal to

$$\max\left\{\sum_{j=1}^{n} \bar{a}_{i,j} \sum_{q=1}^{T} \big[\operatorname{diag}\big(\boldsymbol{\beta}_i^{\mu} + \boldsymbol{\gamma}_i^{\mu}\big)^{-1}(\boldsymbol{\Pi}_{i,j} \circ \boldsymbol{\Gamma}_{i,j})\big]_{r,q} \;\middle|\; (i,r) \in \mathcal{I} \times \{1,\ldots,T\}\right\}.$$

We find, for all $(i,r) \in \mathcal{I} \times \{1,\ldots,T\}$,

$$\begin{aligned}
&\sum_{j=1}^{n} \bar{a}_{i,j} \sum_{q=1}^{T} \big[\operatorname{diag}\big(\boldsymbol{\beta}_i^{\mu} + \boldsymbol{\gamma}_i^{\mu}\big)^{-1}(\boldsymbol{\Pi}_{i,j} \circ \boldsymbol{\Gamma}_{i,j})\big]_{r,q} \\
&\quad= \sum_{j=1}^{n} \bar{a}_{i,j} \sum_{q=1}^{T} \big[\operatorname{diag}\big(\boldsymbol{\beta}_i^{\mu} + \boldsymbol{\gamma}_i^{\mu}\big)^{-1}\big]_{r,r} \big[\boldsymbol{\Pi}_{i,j} \circ \boldsymbol{\Gamma}_{i,j}\big]_{r,q} \\
&\quad= \sum_{j=1}^{n} \bar{a}_{i,j} \sum_{q=1}^{T} \frac{1}{\mathbb{E}(\beta_i + \gamma_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r)} \mathbb{P}(\boldsymbol{s}_j = \boldsymbol{\theta}_q \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r)\, \mathbb{E}(\gamma_i \mid \boldsymbol{s}_j = \boldsymbol{\theta}_q, \boldsymbol{s}_i = \boldsymbol{\theta}_r) \\
&\quad= \frac{1}{\mathbb{E}(\beta_i + \gamma_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r)} \sum_{j=1}^{n} \bar{a}_{i,j} \sum_{q=1}^{T} \mathbb{P}(\boldsymbol{s}_j = \boldsymbol{\theta}_q \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r)\, \mathbb{E}(\gamma_i \mid \boldsymbol{s}_j = \boldsymbol{\theta}_q, \boldsymbol{s}_i = \boldsymbol{\theta}_r) \\
&\quad= \frac{1}{\mathbb{E}(\beta_i + \gamma_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r)} \sum_{j=1}^{n} \bar{a}_{i,j}\, \mathbb{E}(\gamma_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r) \\
&\quad= \frac{\mathbb{E}(\gamma_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r)}{\mathbb{E}(\beta_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r) + \mathbb{E}(\gamma_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r)} \\
&\quad< 1,
\end{aligned}$$

where the second to last equality follows from

$$\sum_{q=1}^{T} \mathbb{P}(\boldsymbol{s}_j = \boldsymbol{\theta}_q \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r)\, \mathbb{E}(\gamma_i \mid \boldsymbol{s}_j = \boldsymbol{\theta}_q, \boldsymbol{s}_i = \boldsymbol{\theta}_r)$$

$$
\begin{aligned}
&= \sum_{q=1}^{T} \frac{\mathbb{P}(\boldsymbol{s}_j = \boldsymbol{\theta}_q, \boldsymbol{s}_i = \boldsymbol{\theta}_r)}{\mathbb{P}(\boldsymbol{s}_i = \boldsymbol{\theta}_r)} \frac{\mathbb{E}\big(\mathbb{1}_{\{\boldsymbol{s}_j=\boldsymbol{\theta}_q\}\cap\{\boldsymbol{s}_i=\boldsymbol{\theta}_r\}}\,\gamma_i\big)}{\mathbb{P}(\boldsymbol{s}_j = \boldsymbol{\theta}_q, \boldsymbol{s}_i = \boldsymbol{\theta}_r)} \\
&= \sum_{q=1}^{T} \frac{\mathbb{E}\big(\mathbb{1}_{\{\boldsymbol{s}_j=\boldsymbol{\theta}_q\}}\,\mathbb{1}_{\{\boldsymbol{s}_i=\boldsymbol{\theta}_r\}}\,\gamma_i\big)}{\mathbb{P}(\boldsymbol{s}_i = \boldsymbol{\theta}_r)} \\
&= \sum_{q=1}^{T} \mathbb{E}\big(\mathbb{1}_{\{\boldsymbol{s}_j=\boldsymbol{\theta}_q\}}\,\gamma_i \,\big|\, \boldsymbol{s}_i = \boldsymbol{\theta}_r\big) \\
&= \mathbb{E}\left(\left(\sum_{q=1}^{T} \mathbb{1}_{\{\boldsymbol{s}_j=\boldsymbol{\theta}_q\}}\right)\gamma_i \,\middle|\, \boldsymbol{s}_i = \boldsymbol{\theta}_r\right) \\
&= \mathbb{E}\left(\mathbb{1}_{\bigcup_{q=1}^{T}\{\boldsymbol{s}_j=\boldsymbol{\theta}_q\}}\,\gamma_i \,\middle|\, \boldsymbol{s}_i = \boldsymbol{\theta}_r\right) \\
&= \mathbb{E}(\mathbb{1}_{\Omega}\,\gamma_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r) \\
&= \mathbb{E}(\gamma_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r)
\end{aligned}
$$

and the inequality from $\beta_i > 0$ and $\gamma_i \geq 0$. This concludes the proof of $\rho(\boldsymbol{N}) < 1$. It follows that matrix (E.5) is nonsingular (Lemma B.3) and the system of equations (E.4) has a unique solution,

$$
\boldsymbol{x}_{\Theta}^{\star} = \Big(\boldsymbol{I}_{nT} - \operatorname{diag}\big(\boldsymbol{\beta}^{\mu} + \boldsymbol{\gamma}^{\mu}\big)^{-1}\Big(\big(\bar{\boldsymbol{A}}(G) \otimes \mathbf{1}_T\mathbf{1}_T^{\mathsf{T}}\big) \circ \boldsymbol{\Pi} \circ \boldsymbol{\Gamma}\Big)\Big)^{-1} \operatorname{diag}\big(\boldsymbol{\beta}^{\mu} + \boldsymbol{\gamma}^{\mu}\big)^{-1}\boldsymbol{\alpha}^{\mu} \tag{E.6}
$$

$$
= \Big(\operatorname{diag}\big(\boldsymbol{\beta}^{\mu} + \boldsymbol{\gamma}^{\mu}\big) - \big(\bar{\boldsymbol{A}}(G) \otimes \mathbf{1}_T\mathbf{1}_T^{\mathsf{T}}\big) \circ \boldsymbol{\Pi} \circ \boldsymbol{\Gamma}\Big)^{-1} \boldsymbol{\alpha}^{\mu}. \tag{E.7}
$$

Second, we establish that $\boldsymbol{x}_{\Theta}^{\star} >_c \mathbf{0}_{nT}$. Note that matrix (E.5) is a nonsingular M-matrix whose inverse is bounded below by $\boldsymbol{I}_{nT}$ because the matrix $\boldsymbol{N}$ is nonnegative with $\rho(\boldsymbol{N}) < 1$ (Lemma B.6). Note also that

$$
\operatorname{diag}\big(\boldsymbol{\beta}^{\mu} + \boldsymbol{\gamma}^{\mu}\big)^{-1}\boldsymbol{\alpha}^{\mu} >_c \mathbf{0}_{nT}.
$$

The foregoing two results together with (E.6) imply that $\boldsymbol{x}_{\Theta}^{\star} >_c \mathbf{0}_{nT}$ (Lemma B.1).

This concludes the proof that the strategy profile $(x_1^{\star}, \ldots, x_n^{\star})$ given by (E.7) is the unique interior BNE of $\Gamma(\alpha, \beta, \gamma)$.

Finally, note that $\Gamma(\alpha, \beta, \gamma)$ has no boundary BNEs. In order to prove this, suppose, for the sake of contradiction, the strategy profile $(\tilde{x}_1^{\star}, \ldots, \tilde{x}_n^{\star})$ is a boundary BNE of $\Gamma(\alpha, \beta, \gamma)$. There exists a pair $(k, s) \in \mathcal{I} \times \{1, \ldots, T\}$ such that $\tilde{x}_k^{\star}(\boldsymbol{\theta}_s) = 0$ and

$$
\frac{\partial\, \mathbb{E}\big(u_k\big(\mathrm{id}_{\Omega}, (\tilde{x}_1^{\star} \circ \boldsymbol{s}_1, \ldots, \tilde{x}_n^{\star} \circ \boldsymbol{s}_n)\big) \,\big|\, \boldsymbol{s}_k = \boldsymbol{\theta}_s\big)}{\partial \tilde{x}_k^{\star}(\boldsymbol{\theta}_s)} \leq 0. \tag{E.8}
$$

Using (E.3), we find

$$
\frac{\partial\, \mathbb{E}\big(u_k\big(\mathrm{id}_{\Omega}, (\tilde{x}_1^{\star} \circ \boldsymbol{s}_1, \ldots, \tilde{x}_n^{\star} \circ \boldsymbol{s}_n)\big) \,\big|\, \boldsymbol{s}_k = \boldsymbol{\theta}_s\big)}{\partial \tilde{x}_k^{\star}(\boldsymbol{\theta}_s)} = \mathbb{E}(\alpha_k \mid \boldsymbol{s}_k = \boldsymbol{\theta}_s)
$$

$$+\sum_{j=1}^{n} \bar{a}_{k,j}\,\mathbb{E}\big(\gamma_k(\tilde{x}_j^\star \circ \boldsymbol{s}_j) \,\big|\, \boldsymbol{s}_k = \boldsymbol{\theta}_s\big) > 0$$

because $\alpha_k > 0$ and $\gamma_k \geq 0$, which contradicts (E.8). Consequently, the strategy profile $(\tilde{x}_1^\star, \ldots, \tilde{x}_n^\star)$ cannot be a boundary BNE of $\Gamma(\alpha, \beta, \gamma)$. ■

Proof of Proposition 2.6

The statement follows from the first-order conditions for the unique and interior BNE strategy profile $(x_1^\star, \ldots, x_n^\star)$ of $\Gamma(\alpha, \beta, \gamma)$. The first-order conditions (E.2) are equivalent to (see (E.3))

$$\begin{aligned}\forall (i,r) \in \mathcal{I} \times \{1, \ldots, T\} \quad & \mathbb{E}\big((\beta_i + \gamma_i)(x_i^\star \circ \boldsymbol{s}_i) \,\big|\, \boldsymbol{s}_i = \boldsymbol{\theta}_r\big) \\ &= \mathbb{E}(\alpha_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r) + \sum_{j=1}^{n} \bar{a}_{i,j}\,\mathbb{E}\big(\gamma_i(x_j^\star \circ \boldsymbol{s}_j) \,\big|\, \boldsymbol{s}_i = \boldsymbol{\theta}_r\big). \end{aligned} \tag{E.9}$$

It follows that

$$\begin{aligned}\forall i \in \mathcal{I} \quad & \sum_{r=1}^{T} \mathbb{P}(\boldsymbol{s}_i = \boldsymbol{\theta}_r)\,\mathbb{E}\big((\beta_i + \gamma_i)(x_i^\star \circ \boldsymbol{s}_i) \,\big|\, \boldsymbol{s}_i = \boldsymbol{\theta}_r\big) \\ = & \sum_{r=1}^{T} \mathbb{P}(\boldsymbol{s}_i = \boldsymbol{\theta}_r)\,\mathbb{E}(\alpha_i \mid \boldsymbol{s}_i = \boldsymbol{\theta}_r) + \sum_{j=1}^{n} \bar{a}_{i,j} \sum_{r=1}^{T} \mathbb{P}(\boldsymbol{s}_i = \boldsymbol{\theta}_r)\,\mathbb{E}\big(\gamma_i(x_j^\star \circ \boldsymbol{s}_j) \,\big|\, \boldsymbol{s}_i = \boldsymbol{\theta}_r\big), \end{aligned}$$

which are equivalent to

$$\forall i \in \mathcal{I} \quad \mathbb{E}\big((\beta_i + \gamma_i)(x_i^\star \circ \boldsymbol{s}_i)\big) = \mathbb{E}(\alpha_i) + \mathbb{E}\left(\gamma_i \sum_{j=1}^{n} \bar{a}_{i,j}(x_j^\star \circ \boldsymbol{s}_j)\right), \tag{E.10}$$

which in turn are equivalent to

$$\mathbb{E}\left(\operatorname{diag}(\boldsymbol{\beta} + \boldsymbol{\gamma}) \begin{pmatrix} x_1^\star \circ \boldsymbol{s}_1 \\ \vdots \\ x_n^\star \circ \boldsymbol{s}_n \end{pmatrix}\right) = \mathbb{E}(\boldsymbol{\alpha}) + \mathbb{E}\left(\operatorname{diag}(\boldsymbol{\gamma}) \bar{A}(G) \begin{pmatrix} x_1^\star \circ \boldsymbol{s}_1 \\ \vdots \\ x_n^\star \circ \boldsymbol{s}_n \end{pmatrix}\right). \quad ■$$

Proof of Proposition 2.7

We find

$$\begin{aligned} & w^\star\big(\Gamma(\alpha, \beta, \gamma)\big) \\ &= \sum_{i=1}^{n} \mathbb{E}\big(u_i\big(\mathrm{id}_\Omega, (x_1^\star \circ \boldsymbol{s}_1, \ldots, x_n^\star \circ \boldsymbol{s}_n)\big)\big) \\ &= \sum_{i=1}^{n} \mathbb{E}\left(\alpha_i(x_i^\star \circ \boldsymbol{s}_i) - \frac{\beta_i}{2}(x_i^\star \circ \boldsymbol{s}_i)^2 - \frac{\gamma_i}{2}\left(x_i^\star \circ \boldsymbol{s}_i - \sum_{j=1}^{n} \bar{a}_{i,j}(x_j^\star \circ \boldsymbol{s}_j)\right)^2\right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \mathbb{E}\big(\alpha_i (x_i^\star \circ \boldsymbol{s}_i)\big) - \frac{1}{2} \sum_{i=1}^{n} \mathbb{E}\big((\beta_i + \gamma_i)(x_i^\star \circ \boldsymbol{s}_i)^2\big) \\
&\quad + \sum_{i=1}^{n} \sum_{j=1}^{n} \bar{a}_{i,j}\, \mathbb{E}\big(\gamma_i (x_i^\star \circ \boldsymbol{s}_i)(x_j^\star \circ \boldsymbol{s}_j)\big) \\
&\quad - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \bar{a}_{i,j} \bar{a}_{i,k}\, \mathbb{E}\big(\gamma_i (x_j^\star \circ \boldsymbol{s}_j)(x_k^\star \circ \boldsymbol{s}_k)\big) \\
&= \frac{1}{2} \sum_{i=1}^{n} \mathbb{E}\big((\beta_i + \gamma_i)(x_i^\star \circ \boldsymbol{s}_i)^2\big) - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \bar{a}_{i,j} \bar{a}_{i,k}\, \mathbb{E}\big(\gamma_i (x_j^\star \circ \boldsymbol{s}_j)(x_k^\star \circ \boldsymbol{s}_k)\big),
\end{aligned}
$$

where the last equality follows from

$$
\forall i \in \mathcal{I} \quad \mathbb{E}\big((\beta_i + \gamma_i)(x_i^\star \circ \boldsymbol{s}_i)^2\big) = \mathbb{E}\big(\alpha_i (x_i^\star \circ \boldsymbol{s}_i)\big) + \sum_{j=1}^{n} \bar{a}_{i,j}\, \mathbb{E}\big(\gamma_i (x_i^\star \circ \boldsymbol{s}_i)(x_j^\star \circ \boldsymbol{s}_j)\big). \tag{E.11}
$$

In order to prove (E.11), note that the first-order conditions (E.2) for the unique and interior BNE strategy profile $(x_1^\star, \ldots, x_n^\star)$ of $\Gamma(\alpha, \beta, \gamma)$ are equivalent to (E.9), which imply that

$$
\begin{aligned}
\forall i \in \mathcal{I} \quad & \sum_{r=1}^{T} \mathbb{P}(\boldsymbol{s}_i = \boldsymbol{\theta}_r)\, \mathbb{E}\big((\beta_i + \gamma_i)(x_i^\star \circ \boldsymbol{s}_i)^2 \,\big|\, \boldsymbol{s}_i = \boldsymbol{\theta}_r\big) \\
&= \sum_{r=1}^{T} \mathbb{P}(\boldsymbol{s}_i = \boldsymbol{\theta}_r)\, \mathbb{E}\big(\alpha_i (x_i^\star \circ \boldsymbol{s}_i) \,\big|\, \boldsymbol{s}_i = \boldsymbol{\theta}_r\big) \\
&\quad + \sum_{j=1}^{n} \bar{a}_{i,j} \sum_{r=1}^{T} \mathbb{P}(\boldsymbol{s}_i = \boldsymbol{\theta}_r)\, \mathbb{E}\big(\gamma_i (x_i^\star \circ \boldsymbol{s}_i)(x_j^\star \circ \boldsymbol{s}_j) \,\big|\, \boldsymbol{s}_i = \boldsymbol{\theta}_r\big).
\end{aligned} \tag{E.12}
$$

Finally, note that (E.11) and (E.12) are equivalent. ∎

Proof of Corollary 2.8

Let the (column) vectors $\bar{\boldsymbol{\beta}}$ and $\bar{\boldsymbol{\gamma}}$ be defined as in Section 2.3. We find

$$
\boldsymbol{\beta}^{\mu} = \bar{\boldsymbol{\beta}} \otimes \mathbf{1}_T \quad \text{and} \quad \boldsymbol{\gamma}^{\mu} = \bar{\boldsymbol{\gamma}} \otimes \mathbf{1}_T. \tag{E.13}
$$

Note that, for all $(i,j) \in \mathcal{I}^2$, $\boldsymbol{\Gamma}_{i,j} = \bar{\gamma}_i \mathbf{1}_T \mathbf{1}_T^\top$. We find

$$
\begin{aligned}
\boldsymbol{\Gamma} &= \begin{pmatrix}
\bar{\gamma}_1 \mathbf{1}_T \mathbf{1}_T^\top & \ldots & \bar{\gamma}_1 \mathbf{1}_T \mathbf{1}_T^\top & \ldots & \bar{\gamma}_1 \mathbf{1}_T \mathbf{1}_T^\top \\
\vdots & & \vdots & & \vdots \\
\bar{\gamma}_i \mathbf{1}_T \mathbf{1}_T^\top & \ldots & \bar{\gamma}_i \mathbf{1}_T \mathbf{1}_T^\top & \ldots & \bar{\gamma}_i \mathbf{1}_T \mathbf{1}_T^\top \\
\vdots & & \vdots & & \vdots \\
\bar{\gamma}_n \mathbf{1}_T \mathbf{1}_T^\top & \ldots & \bar{\gamma}_n \mathbf{1}_T \mathbf{1}_T^\top & \ldots & \bar{\gamma}_n \mathbf{1}_T \mathbf{1}_T^\top
\end{pmatrix} \\
&= \big(\operatorname{diag}(\bar{\boldsymbol{\gamma}}) \otimes \boldsymbol{I}_T\big)\big(\mathbf{1}_n \mathbf{1}_n^\top \otimes \mathbf{1}_T \mathbf{1}_T^\top\big)
\end{aligned}
$$

$$= \left(\operatorname{diag}(\bar{\gamma})\mathbf{1}_n\mathbf{1}_n^\top\right) \otimes \mathbf{1}_T\mathbf{1}_T^\top,$$

which implies that

$$\begin{aligned}
\left(\bar{\boldsymbol{A}}(G) \otimes \mathbf{1}_T\mathbf{1}_T^\top\right) \circ \boldsymbol{\Pi} \circ \boldsymbol{\Gamma} = \boldsymbol{\Gamma} \circ \left(\bar{\boldsymbol{A}}(G) \otimes \mathbf{1}_T\mathbf{1}_T^\top\right) \circ \boldsymbol{\Pi} \\
&= \Big(\left(\operatorname{diag}(\bar{\gamma})\mathbf{1}_n\mathbf{1}_n^\top\right) \otimes \mathbf{1}_T\mathbf{1}_T^\top\Big) \circ \left(\bar{\boldsymbol{A}}(G) \otimes \mathbf{1}_T\mathbf{1}_T^\top\right) \circ \boldsymbol{\Pi} \\
&= \Big(\left(\operatorname{diag}(\bar{\gamma})\bar{\boldsymbol{A}}(G)\right) \otimes \mathbf{1}_T\mathbf{1}_T^\top\Big) \circ \boldsymbol{\Pi}. \qquad \text{(E.14)}
\end{aligned}$$

Using results (E.13) and (E.14), the statement of the corollary follows from Proposition 2.5. ■

Proof of Corollary 2.9

The statement follows from Corollary 2.8. ■

Proof of Corollary 2.10

Suppose, for all $i \in \mathcal{I}$, there exist $\bar{\beta}_i \in \mathbb{R}_{++}$ and $\bar{\gamma}_i \in \mathbb{R}_+$ such that $\beta_i(\Omega) = \{\bar{\beta}_i\}$ and $\gamma_i(\Omega) = \{\bar{\gamma}_i\}$. Let the (column) vectors $\bar{\boldsymbol{\beta}}$ and $\bar{\gamma}$ be defined as in Section 2.3. Using Proposition 2.6, we find

$$\begin{aligned}
& \mathbb{E}\left(\operatorname{diag}(\boldsymbol{\beta}+\gamma)\begin{pmatrix} x_1^\star \circ \boldsymbol{s}_1 \\ \vdots \\ x_n^\star \circ \boldsymbol{s}_n \end{pmatrix}\right) = \mathbb{E}(\boldsymbol{\alpha}) + \mathbb{E}\left(\operatorname{diag}(\gamma)\bar{\boldsymbol{A}}(G)\begin{pmatrix} x_1^\star \circ \boldsymbol{s}_1 \\ \vdots \\ x_n^\star \circ \boldsymbol{s}_n \end{pmatrix}\right) \\
\Leftrightarrow\quad & \mathbb{E}\left(\operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\gamma})\begin{pmatrix} x_1^\star \circ \boldsymbol{s}_1 \\ \vdots \\ x_n^\star \circ \boldsymbol{s}_n \end{pmatrix}\right) = \mathbb{E}(\boldsymbol{\alpha}) + \mathbb{E}\left(\operatorname{diag}(\bar{\gamma})\bar{\boldsymbol{A}}(G)\begin{pmatrix} x_1^\star \circ \boldsymbol{s}_1 \\ \vdots \\ x_n^\star \circ \boldsymbol{s}_n \end{pmatrix}\right) \\
\Leftrightarrow\quad & \operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\gamma})\,\mathbb{E}\left(\begin{pmatrix} x_1^\star \circ \boldsymbol{s}_1 \\ \vdots \\ x_n^\star \circ \boldsymbol{s}_n \end{pmatrix}\right) = \mathbb{E}(\boldsymbol{\alpha}) + \operatorname{diag}(\bar{\gamma})\bar{\boldsymbol{A}}(G)\,\mathbb{E}\left(\begin{pmatrix} x_1^\star \circ \boldsymbol{s}_1 \\ \vdots \\ x_n^\star \circ \boldsymbol{s}_n \end{pmatrix}\right) \\
\Leftrightarrow\quad & \mathbb{E}(\boldsymbol{\alpha}) = \left(\operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\gamma}) - \operatorname{diag}(\bar{\gamma})\bar{\boldsymbol{A}}(G)\right)\mathbb{E}\left(\begin{pmatrix} x_1^\star \circ \boldsymbol{s}_1 \\ \vdots \\ x_n^\star \circ \boldsymbol{s}_n \end{pmatrix}\right). \qquad \text{(E.15)}
\end{aligned}$$

Note that $\operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\gamma}) - \operatorname{diag}(\bar{\gamma})\bar{\boldsymbol{A}}(G)$ is nonsingular. Indeed,

$$\operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\gamma}) - \operatorname{diag}(\bar{\gamma})\bar{\boldsymbol{A}}(G) = \operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\gamma})\left(\boldsymbol{I}_n - \operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\gamma})^{-1}\operatorname{diag}(\bar{\gamma})\bar{\boldsymbol{A}}(G)\right)$$

and

$$\rho\left(\operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\gamma})^{-1}\operatorname{diag}(\bar{\gamma})\bar{\boldsymbol{A}}(G)\right) \le \underbrace{\left\|\operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\gamma})^{-1}\operatorname{diag}(\bar{\gamma})\right\|_\infty}_{<1}\underbrace{\|\bar{\boldsymbol{A}}(G)\|_\infty}_{=1} < 1,$$

from which the claim follows (Lemma B.3). We conclude that (E.15) is equivalent to $\mathbb{E}((x_1^\star \circ \boldsymbol{s}_1, \ldots, x_n^\star \circ \boldsymbol{s}_n)) = (\operatorname{diag}(\bar{\boldsymbol{\beta}}+\bar{\gamma}) - \operatorname{diag}(\bar{\gamma})\bar{\boldsymbol{A}}(G))^{-1}\,\mathbb{E}(\boldsymbol{\alpha})$. ■

Proof of Lemma 2.11

Suppose Assumption S-I is satisfied. Let $i \in \mathcal{I}$. Note that, for all $j \in \mathcal{N}_G^+(i)$, $\mathbb{E}(x_j^\star \circ \boldsymbol{s}_j \mid \boldsymbol{s}_i) = \mathbb{E}(x_j^\star \circ \boldsymbol{s}_j)$. Using this result and the first-order conditions for the unique and interior BNE in pure strategies $(x_1^\star, \ldots, x_n^\star)$ of $\Gamma(\alpha)$, we find

$$x_i^\star \circ \boldsymbol{s}_i = \frac{1}{\bar{\beta}_i + \bar{\gamma}_i}\left(\mathbb{E}(\alpha_i \mid \boldsymbol{s}_i) + \bar{\gamma}_i \sum_{j \in \mathcal{I}} \bar{a}_{i,j}\, \mathbb{E}(x_j^\star \circ \boldsymbol{s}_j)\right),$$

from which

$$x_i^\star \circ \boldsymbol{s}_i - \mathbb{E}(x_i^\star \circ \boldsymbol{s}_i) = \frac{\mathbb{E}(\alpha_i \mid \boldsymbol{s}_i) - \mathbb{E}(\alpha_i)}{\bar{\beta}_i + \bar{\gamma}_i}$$

follows. We conclude that

$$\operatorname{var}(x_i^\star \circ \boldsymbol{s}_i) = \frac{\operatorname{var}\big(\mathbb{E}(\alpha_i \mid \boldsymbol{s}_i)\big)}{(\bar{\beta}_i + \bar{\gamma}_i)^2}. \qquad \blacksquare$$

Proof of Corollary 2.12

The statement follows from Proposition 2.7 and Lemma 2.11. ■

Proof of Corollary 2.13

Suppose, for all $i \in \mathcal{I}$, there exist $\bar{\alpha}_i \in \mathbb{R}_{++}$ and $\bar{\gamma}_i \in \mathbb{R}_+$ such that $\alpha_i(\Omega) = \{\bar{\alpha}_i\}$ and $\gamma_i(\Omega) = \{\bar{\gamma}_i\}$. Let the (column) vectors $\bar{\boldsymbol{\alpha}}$ and $\bar{\boldsymbol{\gamma}}$ be defined as in Section 2.3. We find

$$\boldsymbol{\alpha}^\mu = \bar{\boldsymbol{\alpha}} \otimes \mathbf{1}_T \quad \text{and} \quad \boldsymbol{\gamma}^\mu = \bar{\boldsymbol{\gamma}} \otimes \mathbf{1}_T. \tag{E.16}$$

Using results (E.14) and (E.16), the statement of the corollary follows from Proposition 2.5. ■

Proof of Corollary 2.14

The statement follows from Corollary 2.13. ■

Proof of Corollary 2.15

The statement follows from Proposition 2.7. ■

Proof of Corollary 2.16

The statement follows from Proposition 2.6. ■

Proof of Corollary 2.17

Suppose, for all $i \in \mathcal{I}$, there exist $\bar{\alpha}_i \in \mathbb{R}_{++}$ and $\bar{\beta}_i \in \mathbb{R}_{++}$ such that $\alpha_i(\Omega) = \{\bar{\alpha}_i\}$ and $\beta_i(\Omega) = \{\bar{\beta}_i\}$. Let the (column) vectors $\bar{\boldsymbol{\alpha}}$ and $\bar{\boldsymbol{\beta}}$ be defined as in Section 2.3. We find

$$\boldsymbol{\alpha}^{\mu} = \bar{\boldsymbol{\alpha}} \otimes \mathbf{1}_T \quad \text{and} \quad \boldsymbol{\beta}^{\mu} = \bar{\boldsymbol{\beta}} \otimes \mathbf{1}_T. \tag{E.17}$$

Using results (E.17), the statement of the corollary follows from Proposition 2.5. ■

Proof of Corollary 2.18

The statement follows from Proposition 2.6. ■

Proof of Corollary 2.19

The statement follows from Proposition 2.7. ■

Appendix F

Proofs of Chapter 3 Results

Proof of Proposition 3.8

The statement follows from Lemma B.3. ■

Proof of Lemma 3.9

Note that bounded linear operators of finite rank on a Banach space are compact and compact linear operators on a Banach space are completely continuous. It follows that square matrices represent completely continuous operators (with respect to any norm). The statement then follows from Suzuki (1976). ■

Proof of Proposition 3.10

Suppose Condition 3.8.1 is satisfied. Let $k\colon \mathcal{V}(D) \to \mathcal{I}(|\mathcal{V}(D)|)$ be a bijection. There exists a unique permutation π of $\mathcal{I}(|\mathcal{V}(D)|)$ such that $h = \pi \circ k$. Let $\boldsymbol{P}_\pi := (\boldsymbol{e}_{\pi(1)}, \ldots, \boldsymbol{e}_{\pi(|\mathcal{V}(D)|)})^\top$, that is, $\boldsymbol{P}_\pi$ is the permutation matrix of π. First, note that $\boldsymbol{\alpha}_k(D) = \boldsymbol{P}_\pi \boldsymbol{\alpha}_h(D)$ and $\boldsymbol{\lambda}_k(D) = \boldsymbol{P}_\pi \boldsymbol{\lambda}_h(D)$. Indeed, we find, for all $i \in \mathcal{I}(|\mathcal{V}(D)|)$, $[\boldsymbol{\alpha}_k(D)]_i = (\alpha(D) \circ k^{-1})(i) = (\alpha(D) \circ h^{-1})(\pi(i)) = [\boldsymbol{\alpha}_h(D)]_{\pi(i)} = [\boldsymbol{P}_\pi \boldsymbol{\alpha}_h(D)]_i$ because $k^{-1} = h^{-1} \circ \pi$. Second, note that $\dot{\boldsymbol{A}}_k(D) = \boldsymbol{P}_\pi \dot{\boldsymbol{A}}_h(D) \boldsymbol{P}_\pi^{-1}$. Indeed, we find, for all $(i,j) \in \mathcal{I}(|\mathcal{V}(D)|)^2$, $[\dot{\boldsymbol{A}}_k(D)]_{i,j} = 1$ if and only if $(k^{-1}(i), k^{-1}(j)) \in \mathcal{A}(D)$ if and only if $[\dot{\boldsymbol{A}}_h(D)]_{\pi(i),\pi(j)} = 1$ if and only if $[\boldsymbol{P}_\pi \dot{\boldsymbol{A}}_h(D) \boldsymbol{P}_\pi^{-1}]_{i,j} = 1$. Third, note that $\sigma(\operatorname{diag}(\boldsymbol{\lambda}_h(D))\dot{\boldsymbol{A}}_h(D)) = \sigma(\operatorname{diag}(\boldsymbol{\lambda}_k(D))\dot{\boldsymbol{A}}_k(D))$ because $\operatorname{diag}(\boldsymbol{\lambda}_h(D))\dot{\boldsymbol{A}}_h(D)$ and $\operatorname{diag}(\boldsymbol{\lambda}_k(D))\dot{\boldsymbol{A}}_k(D)$ are similar. Using the preceding results, we find

$$\begin{aligned}
\boldsymbol{b}_k\big(\boldsymbol{\alpha}_k(D), \boldsymbol{\lambda}_k(D), D\big) &= \big(\boldsymbol{I}_{|\mathcal{V}(D)|} - \operatorname{diag}\big(\boldsymbol{\lambda}_k(D)\big)\dot{\boldsymbol{A}}_k(D)\big)^{-1}\boldsymbol{\alpha}_k(D) \\
&= \big(\boldsymbol{I}_{|\mathcal{V}(D)|} - \boldsymbol{P}_\pi \operatorname{diag}\big(\boldsymbol{\lambda}_h(D)\big)\boldsymbol{P}_\pi^{-1}\boldsymbol{P}_\pi \dot{\boldsymbol{A}}_h(D)\boldsymbol{P}_\pi^{-1}\big)^{-1}\boldsymbol{P}_\pi\boldsymbol{\alpha}_h(D) \\
&= \boldsymbol{P}_\pi\big(\boldsymbol{I}_{|\mathcal{V}(D)|} - \operatorname{diag}\big(\boldsymbol{\lambda}_h(D)\big)\dot{\boldsymbol{A}}_h(D)\big)^{-1}\boldsymbol{P}_\pi^{-1}\boldsymbol{P}_\pi\boldsymbol{\alpha}_h(D) \\
&= \boldsymbol{P}_\pi \boldsymbol{b}_h\big(\boldsymbol{\alpha}_h(D), \boldsymbol{\lambda}_h(D), D\big).
\end{aligned}$$

■

Proof of Proposition 3.11

Suppose Condition 3.8.1 is satisfied. Let E be a nonempty digraph of order $|\mathcal{V}(D)|$ that is isomorphic to D by means of the digraph isomorphism $f\colon \mathcal{V}(D) \to \mathcal{V}(E)$. Let $k\colon \mathcal{V}(E) \to \mathcal{I}(|\mathcal{V}(D)|)$ be a bijection. Note that $\alpha(D) = \alpha(E) \circ f$ and $\lambda(D) = \lambda(E) \circ f$. There exists a unique permutation π of $\mathcal{I}(|\mathcal{V}(D)|)$ such that $h = \pi \circ k \circ f$. See Figure 3.1 for an illustration. Let $\boldsymbol{P}_\pi$ denote the permutation matrix of π. First, note that $\boldsymbol{\alpha}_k(E) = \boldsymbol{P}_\pi \boldsymbol{\alpha}_h(D)$ and $\boldsymbol{\lambda}_k(E) = \boldsymbol{P}_\pi \boldsymbol{\lambda}_h(D)$. Indeed, we find, for all $i \in \mathcal{I}(|\mathcal{V}(D)|)$, $[\boldsymbol{\alpha}_k(E)]_i = (\alpha(E) \circ k^{-1})(i) = (\alpha(D) \circ f^{-1} \circ f \circ h^{-1} \circ \pi)(i) = (\alpha(D) \circ h^{-1})(\pi(i)) = [\boldsymbol{\alpha}_h(D)]_{\pi(i)} = [\boldsymbol{P}_\pi \boldsymbol{\alpha}_h(D)]_i$ because $k^{-1} = f \circ h^{-1} \circ \pi$. Second, note that $\dot{\boldsymbol{A}}_k(E) = \boldsymbol{P}_\pi \dot{\boldsymbol{A}}_h(D) \boldsymbol{P}_\pi^{-1}$. Indeed, we find, for all $(i,j) \in \mathcal{I}(|\mathcal{V}(D)|)^2$, $[\dot{\boldsymbol{A}}_k(E)]_{i,j} = 1$ if and only if $(k^{-1}(i), k^{-1}(j)) \in \mathcal{A}(E)$ if and only if $(f^{-1}(k^{-1}(i)), f^{-1}(k^{-1}(j))) \in \mathcal{A}(D)$ if and only if $[\dot{\boldsymbol{A}}_h(D)]_{\pi(i),\pi(j)} = 1$ if and only if $[\boldsymbol{P}_\pi \dot{\boldsymbol{A}}_h(D) \boldsymbol{P}_\pi^{-1}]_{i,j} = 1$. Third, note that $\sigma(\operatorname{diag}(\boldsymbol{\lambda}_h(D))\dot{\boldsymbol{A}}_h(D)) = \sigma(\operatorname{diag}(\boldsymbol{\lambda}_k(E))\dot{\boldsymbol{A}}_k(E))$ because $\operatorname{diag}(\boldsymbol{\lambda}_h(D))\dot{\boldsymbol{A}}_h(D)$ and $\operatorname{diag}(\boldsymbol{\lambda}_k(E))\dot{\boldsymbol{A}}_k(E)$ are similar. Using the preceding results, we find

$$\begin{aligned}
\boldsymbol{b}_k\big(\boldsymbol{\alpha}_k(E), \boldsymbol{\lambda}_k(E), E\big) &= \big(\boldsymbol{I}_{|\mathcal{V}(D)|} - \operatorname{diag}\big(\boldsymbol{\lambda}_k(E)\big)\dot{\boldsymbol{A}}_k(E)\big)^{-1} \boldsymbol{\alpha}_k(E) \\
&= \big(\boldsymbol{I}_{|\mathcal{V}(D)|} - \boldsymbol{P}_\pi \operatorname{diag}\big(\boldsymbol{\lambda}_h(D)\big)\boldsymbol{P}_\pi^{-1} \boldsymbol{P}_\pi \dot{\boldsymbol{A}}_h(D) \boldsymbol{P}_\pi^{-1}\big)^{-1} \boldsymbol{P}_\pi \boldsymbol{\alpha}_h(D) \\
&= \boldsymbol{P}_\pi \big(\boldsymbol{I}_{|\mathcal{V}(D)|} - \operatorname{diag}\big(\boldsymbol{\lambda}_h(D)\big)\dot{\boldsymbol{A}}_h(D)\big)^{-1} \boldsymbol{\alpha}_h(D) \\
&= \boldsymbol{P}_\pi \boldsymbol{b}_h\big(\boldsymbol{\alpha}_h(D), \boldsymbol{\lambda}_h(D), D\big). \qquad \blacksquare
\end{aligned}$$

Proof of Proposition 3.12

Suppose Condition C-ρ is satisfied. It follows that both $\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D)$ and $\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D))^{-1}$ are nonsingular (Lemma B.2). It follows also that $\boldsymbol{I}_n + \operatorname{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D))^{-1}$ is nonsingular (Lemmata B.2 and B.9). The identity $\boldsymbol{\lambda} = \boldsymbol{\lambda}^+ - \boldsymbol{\lambda}^-$ implies that $\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D) = \boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D) + \operatorname{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)$, which in turn implies that

$$\begin{aligned}
\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D) = \Big(&\boldsymbol{I}_n + \operatorname{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)\big(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D)\big)^{-1}\Big) \\
&\times \big(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D)\big).
\end{aligned} \tag{F.1}$$

Note that $\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)$ is nonsingular because both factors on the right-hand side of (F.1) are nonsingular. We have

$$\begin{aligned}
&\boldsymbol{I}_n - \Big(\operatorname{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)\big(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D)\big)^{-1}\Big)^2 \\
&\quad = \Big(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)\big(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D)\big)^{-1}\Big) \\
&\qquad \times \Big(\boldsymbol{I}_n + \operatorname{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)\big(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D)\big)^{-1}\Big).
\end{aligned} \tag{F.2}$$

Note that $\boldsymbol{I}_n - (\mathrm{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)(\boldsymbol{I}_n - \mathrm{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D))^{-1})^2$ is nonsingular because both factors on the right-hand side of (F.2) are nonsingular. Combining results (F.1) and (F.2) shows that the inverse of $\boldsymbol{I}_n - \mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)$ is given by

$$\Big(\boldsymbol{I}_n - \mathrm{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D)\Big)^{-1}\bigg(\boldsymbol{I}_n - \Big(\mathrm{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)\big(\boldsymbol{I}_n - \mathrm{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D)\big)^{-1}\Big)^2\bigg)^{-1} \times \Big(\boldsymbol{I}_n - \mathrm{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)\big(\boldsymbol{I}_n - \mathrm{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D)\big)^{-1}\Big).$$

Note that, for all $\boldsymbol{A} \in \mathcal{M}(n,\mathbb{R})$, $\rho(\boldsymbol{A}^2) = \rho(\boldsymbol{A})^2$, which is a consequence of the *spectral mapping theorem* (see, for example, Kubrusly 2011, Theorem 6.19 and Corollary 6.20). It follows that $\rho((\mathrm{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)(\boldsymbol{I}_n - \mathrm{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D))^{-1})^2) < 1$ (Condition C-ρ), which implies that $\boldsymbol{I}_n - (\mathrm{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)(\boldsymbol{I}_n - \mathrm{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D))^{-1})^2$ is a nonsingular M-matrix because $\boldsymbol{I}_n - \mathrm{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D)$ is a nonsingular M-matrix with nonnegative inverse. Finally, note that $\boldsymbol{I}_n - \mathrm{diag}(\boldsymbol{\lambda}^-)\dot{\boldsymbol{A}}(D)(\boldsymbol{I}_n - \mathrm{diag}(\boldsymbol{\lambda}^+)\dot{\boldsymbol{A}}(D))^{-1}$ is a nonsingular M-matrix. ∎

Proof of Corollary 3.13

Results 3.13.1 and 3.13.2 follow from Proposition 3.12. ∎

Proof of Proposition 3.14

Results 3.14.1 and 3.14.2 follow from Propositions 3.8 and 3.12 and Lemmata B.1 and B.6. ∎

Proof of Corollary 3.15

Results 3.15.1 to 3.15.4 follow from Proposition 3.14. As regards Results 3.15.1 and 3.15.2, note that if $\boldsymbol{\lambda} \geq_c \boldsymbol{0}_n$, $\rho(\mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$, and $\boldsymbol{\alpha} \geq_c \boldsymbol{0}_n$, then $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) \geq_c \boldsymbol{\alpha}$ because $(\boldsymbol{I}_n - \mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D))^{-1} \geq_c \boldsymbol{I}_n$ (Lemma B.6). ∎

Proof of Lemma 3.16

Let $\boldsymbol{a} \in \mathbb{R}^n_+$, and let $\boldsymbol{A} \in \mathcal{M}(n,\mathbb{R})$ be nonnegative. Assume that $\boldsymbol{A}\boldsymbol{a} <_c \boldsymbol{a}$. It follows that $\boldsymbol{a} \neq \boldsymbol{0}_n$. Note that $\rho(\boldsymbol{A}) = \rho(\boldsymbol{A}^\mathsf{T})$ because $\boldsymbol{A}$ and $\boldsymbol{A}^\mathsf{T}$ are similar and, therefore, have the same spectrum. Note also that $\rho(\boldsymbol{A}^\mathsf{T})$ is an eigenvalue of $\boldsymbol{A}^\mathsf{T}$ to which there corresponds a nonnegative (and nonzero) eigenvector $\boldsymbol{v}$ (Lemma B.5). The foregoing two facts imply that $\boldsymbol{A}^\mathsf{T}\boldsymbol{v} = \rho(\boldsymbol{A})\boldsymbol{v}$. Let $\boldsymbol{z} := \boldsymbol{a} - \boldsymbol{A}\boldsymbol{a}$, where $\boldsymbol{z} >_c \boldsymbol{0}_n$ by assumption. We find

$$\boldsymbol{z}^\mathsf{T}\boldsymbol{v} = \boldsymbol{a}^\mathsf{T}\boldsymbol{v} - \boldsymbol{a}^\mathsf{T}\boldsymbol{A}^\mathsf{T}\boldsymbol{v} = \boldsymbol{a}^\mathsf{T}\boldsymbol{v} - \rho(\boldsymbol{A})\boldsymbol{a}^\mathsf{T}\boldsymbol{v} = \big(1 - \rho(\boldsymbol{A})\big)\boldsymbol{a}^\mathsf{T}\boldsymbol{v} \tag{F.3}$$

with $\boldsymbol{z}^\mathsf{T}\boldsymbol{v} > 0$ because $\boldsymbol{z} >_c \boldsymbol{0}_n$, $\boldsymbol{v} \geq_c \boldsymbol{0}_n$, and $\boldsymbol{v} \neq \boldsymbol{0}_n$. It follows from $\boldsymbol{a} \geq_c \boldsymbol{0}_n$, $\boldsymbol{v} \geq_c \boldsymbol{0}_n$, $\boldsymbol{z}^\mathsf{T}\boldsymbol{v} > 0$, and (F.3) that $\boldsymbol{a}^\mathsf{T}\boldsymbol{v} > 0$. We conclude that $\rho(\boldsymbol{A}) < 1$ because $(1 - \rho(\boldsymbol{A}))\boldsymbol{a}^\mathsf{T}\boldsymbol{v} > 0$ and $\boldsymbol{a}^\mathsf{T}\boldsymbol{v} > 0$. ∎

Proof of Corollary 3.17

The statement follows from Lemma 3.16. ■

Proof of Corollary 3.18

The statement follows from Lemma 3.16. ■

Proof of Lemma 3.19

Suppose $\boldsymbol{\lambda} \geq_c \mathbf{0}_n$ and $\rho(\mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$. Let $(i, j) \in \mathcal{I}(n)^2$ with $i \neq j$. Note that $[(\boldsymbol{I}_n - \mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D))^{-1}]_{i,j} = \boldsymbol{e}_i^\top(\boldsymbol{I}_n - \mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D))^{-1}\boldsymbol{e}_j = b_i(\boldsymbol{e}_j, \boldsymbol{\lambda}, D)$.

Proof of Result 3.19.1 We have $(\boldsymbol{I}_n - \mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D))^{-1} \geq_c \boldsymbol{I}_n$ (Lemma B.6), from which it follows that $b_i(\boldsymbol{e}_i, \boldsymbol{\lambda}, D) \geq 1$ and $b_i(\boldsymbol{e}_j, \boldsymbol{\lambda}, D) \geq 0$. ■

Proof of Result 3.19.2 Suppose there exists a walk $(i_0, \ldots, i_p)$ in D of length $p \in \mathbb{Z}_{++}$ from i to j such that $(\lambda_{i_0}, \ldots, \lambda_{i_{p-1}}) >_c \mathbf{0}_p$.

The proof of $b_i(\boldsymbol{e}_j, \boldsymbol{\lambda}, D) > 0$ is based on the following auxiliary result.

Fact F.1 *For all $q \in \mathbb{Z}_{++}$, $[(\mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D))^q]_{i,j} > 0$ if and only if there exists a walk $(i_0, \ldots, i_q)$ in D of length q from i to j such that $(\lambda_{i_0}, \ldots, \lambda_{i_{q-1}}) >_c \mathbf{0}_q$.*

Proof of Fact F.1 The proof is by induction on q. Note that, for all $q \in \mathbb{Z}_{++}$, $[\dot{\boldsymbol{A}}(D)^q]_{i,j} > 0$ if and only if there exists a walk in D of length q from i to j (see, for example, Festinger 1949, pp. 154–55).

First, the *base case*:

($\Rightarrow$) Assume that $[\mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)]_{i,j} > 0$. We find $\lambda_i > 0$ and $\dot{a}_{i,j}(D) > 0$ because $[\mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)]_{i,j} = \lambda_i \dot{a}_{i,j}(D)$, and (i, j) is a walk in D from i to j with the required property.

($\Leftarrow$) Assume that (i, j) is a walk in D and $\lambda_i > 0$. It follows that $\dot{a}_{i,j}(D) > 0$ and $[\mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)]_{i,j} = \lambda_i \dot{a}_{i,j}(D) > 0$.

Second, the *inductive step*: Let $q \in \mathbb{Z}_{++}$. Assume that the following statement is true:

> The inequality $[(\mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D))^q]_{i,j} > 0$ is true if and only if there exists a walk $(i_0, \ldots, i_q)$ in D of length q from i to j such that $(\lambda_{i_0}, \ldots, \lambda_{i_{q-1}}) >_c \mathbf{0}_q$. $(**)$

We need to show that statement $(**)$ is true for $q + 1$.

($\Rightarrow$) Assume that $[(\mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D))^{q+1}]_{i,j} > 0$. We have

$$\left[(\mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D))^{q+1}\right]_{i,j} = \sum_{k=1}^{n} \left[(\mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D))^{q}\right]_{i,k} \left[\mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)\right]_{k,j}$$

$$= \sum_{k=1}^{n} \lambda_k \dot{a}_{k,j}(D)\big[\big(\mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)\big)^q\big]_{i,k}.$$

It follows from $\boldsymbol{\lambda} \geq_c \mathbf{0}_n$ (which together with $\dot{\boldsymbol{A}}(D) \geq_c \boldsymbol{O}_n$ implies that $\mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D) \geq_c \boldsymbol{O}_n$) and $[(\mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D))^{q+1}]_{i,j} > 0$ that there exists a $\bar{k} \in \mathcal{I}(n)$ such that $\lambda_{\bar{k}} > 0$, $\dot{a}_{\bar{k},j}(D) > 0$, and $[(\mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D))^q]_{i,\bar{k}} > 0$. The third inequality and statement $(**)$ imply that there exists a walk $(i_0, \ldots, i_q)$ in D of length q from i to $\bar{k}$ such that $(\lambda_{i_0}, \ldots, \lambda_{i_{q-1}}) >_c \mathbf{0}_q$. It follows from $\lambda_{\bar{k}} > 0$ and $\dot{a}_{\bar{k},j}(D) > 0$ that $(i_0, \ldots, i_q, j)$ is a walk in D with the required property.

($\Leftarrow$) Assume that there exists a walk $(i_0, \ldots, i_q, i_{q+1})$ in D of length $q+1$ from i to j such that $(\lambda_{i_0}, \ldots, \lambda_{i_q}) >_c \mathbf{0}_{q+1}$. Note that $(i_0, \ldots, i_q)$ is a walk in D of length q from i to i_q with $(\lambda_{i_0}, \ldots, \lambda_{i_{q-1}}) >_c \mathbf{0}_q$. It follows from statement $(**)$ that $\big[(\mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D))^q\big]_{i,i_q} > 0$. Note also that $\dot{a}_{i_q,j}(D) > 0$ because (i_q, j) is an arc in the walk $(i_0, \ldots, i_q, i_{q+1})$. We find

$$\begin{aligned}
\big[\big(\mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)\big)^{q+1}\big]_{i,j} &= \sum_{k=1}^{n} \lambda_k \dot{a}_{k,j}(D)\big[\big(\mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)\big)^q\big]_{i,k} \\
&\geq \lambda_{i_q} \dot{a}_{i_q,j}(D)\big[\big(\mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)\big)^q\big]_{i,i_q} \\
&> 0,
\end{aligned}$$

where the first inequality follows from $\boldsymbol{\lambda} \geq_c \mathbf{0}_n$ and $\dot{\boldsymbol{A}}(D) \geq_c \boldsymbol{O}_n$ and the second from $\lambda_{i_q} > 0$, $\dot{a}_{i_q,j}(D) > 0$, and $[(\mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D))^q]_{i,i_q} > 0$. ■

We find

$$\begin{aligned}
b_i(\boldsymbol{e}_j, \boldsymbol{\lambda}, D) &= \big[\big(\boldsymbol{I}_n - \mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)\big)^{-1}\big]_{i,j} \\
&= \sum_{k=0}^{\infty} \big[\big(\mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)\big)^k\big]_{i,j} \\
&\geq \big[\big(\mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)\big)^p\big]_{i,j} \\
&> 0,
\end{aligned}$$

where the second equality follows from Lemma B.2, the first inequality from $\boldsymbol{\lambda} \geq_c \mathbf{0}_n$ and $\dot{\boldsymbol{A}}(D) \geq_c \boldsymbol{O}_n$, and the second inequality from Fact F.1 and the assumption that $(i_0, \ldots, i_p)$ is a walk in D of length p from i to j such that $(\lambda_{i_0}, \ldots, \lambda_{i_{p-1}}) >_c \mathbf{0}_p$. ■

Proof of Result 3.19.3 Suppose $\boldsymbol{\lambda} \geq_c \mathbf{0}_n$, $\rho(\mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$, and there does not exist a walk in D from i to j. If $\boldsymbol{\lambda} \neq \mathbf{0}_n$, then

$$b_i(\boldsymbol{e}_j, \boldsymbol{\lambda}, D) = \sum_{k=0}^{\infty} \big[\big(\mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)\big)^k\big]_{i,j} = 0,$$

where the first equality follows from Lemma B.2 and the second from Fact F.1. If $\boldsymbol{\lambda} = \mathbf{0}_n$, then $b_i(\boldsymbol{e}_j, \boldsymbol{\lambda}, D) = \delta_{i,j} = 0$. ■

Proof of Proposition 3.22

Suppose Condition I-α-λ is satisfied and $1 \notin \sigma(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D))$. Let $(i,j) \in \mathcal{I}(n)^2$. We find

$$\frac{\partial \boldsymbol{b}(\boldsymbol{\alpha},\boldsymbol{\lambda},D)}{\partial \boldsymbol{\alpha}} = \frac{\partial \big(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)\big)^{-1}\boldsymbol{\alpha}}{\partial \boldsymbol{\alpha}} = \big(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)\big)^{-1},$$

which implies that

$$\frac{\partial\, b_i(\boldsymbol{\alpha},\boldsymbol{\lambda},D)}{\partial \alpha_j} = \boldsymbol{e}_i^\top \big(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)\big)^{-1}\boldsymbol{e}_j = b_i(\boldsymbol{e}_j,\boldsymbol{\lambda},D).$$

Results 3.22.1 and 3.22.2 follow from Lemma 3.19. ■

Proof of Proposition 3.23

Suppose Condition I-α-λ is satisfied and $1 \notin \sigma(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D))$. Let $(i,j) \in \mathcal{I}(n)^2$. We find

$$\begin{aligned}
\frac{\partial\, b_i(\boldsymbol{\alpha},\boldsymbol{\lambda},D)}{\partial \lambda_j} &= \frac{\partial \boldsymbol{e}_i^\top \boldsymbol{b}(\boldsymbol{\alpha},\boldsymbol{\lambda},D)}{\partial \lambda_j} \\
&= -\boldsymbol{e}_i^\top \big(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)\big)^{-1} \frac{\partial \big(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)\big)}{\partial \lambda_j} \\
&\qquad \times \big(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)\big)^{-1}\boldsymbol{\alpha} \\
&= \boldsymbol{e}_i^\top \big(\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)\big)^{-1}\boldsymbol{e}_j\boldsymbol{e}_j^\top \dot{\boldsymbol{A}}(D)\boldsymbol{b}(\boldsymbol{\alpha},\boldsymbol{\lambda},D) \\
&= b_i(\boldsymbol{e}_j,\boldsymbol{\lambda},D) \sum_{k \in \mathcal{N}_D^+(j)} b_k(\boldsymbol{\alpha},\boldsymbol{\lambda},D).
\end{aligned}$$

Results 3.23.1 and 3.23.2 follow from Result 3.15.1, in particular, $\boldsymbol{b}(\boldsymbol{\alpha},\boldsymbol{\lambda},D) \geq_c \boldsymbol{\alpha}$ if $\boldsymbol{\alpha} \geq_c \boldsymbol{0}_n$ and $\boldsymbol{\lambda} \in \mathbb{R}_+^n$ is such that $\rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$, and Lemma 3.19. ■

Proof of Proposition 3.24

Assume that $1 \notin \sigma(\operatorname{diag}(\boldsymbol{\lambda}(D))\dot{\boldsymbol{A}}(D)) \cup \sigma(\operatorname{diag}(\boldsymbol{\lambda}(E))\dot{\boldsymbol{A}}(E))$. According to Definition C and Proposition 3.8,

$$\boldsymbol{b}\big(\boldsymbol{\alpha}(D),\boldsymbol{\lambda}(D),D\big) = \boldsymbol{\alpha}(D) + \operatorname{diag}\big(\boldsymbol{\lambda}(D)\big)\dot{\boldsymbol{A}}(D)\boldsymbol{b}\big(\boldsymbol{\alpha}(D),\boldsymbol{\lambda}(D),D\big)$$

and

$$\boldsymbol{b}\big(\boldsymbol{\alpha}(E),\boldsymbol{\lambda}(E),E\big) = \boldsymbol{\alpha}(E) + \operatorname{diag}\big(\boldsymbol{\lambda}(E)\big)\dot{\boldsymbol{A}}(E)\boldsymbol{b}\big(\boldsymbol{\alpha}(E),\boldsymbol{\lambda}(E),E\big).$$

We find

$$\boldsymbol{b}\big(\boldsymbol{\alpha}(E),\boldsymbol{\lambda}(E),E\big) - \boldsymbol{b}\big(\boldsymbol{\alpha}(D),\boldsymbol{\lambda}(D),D\big)$$

$$
\begin{aligned}
&= \boldsymbol{\alpha}(E) - \boldsymbol{\alpha}(D) \\
&\quad + \operatorname{diag}\big(\boldsymbol{\lambda}(E)\big)\dot{\boldsymbol{A}}(E)\boldsymbol{b}\big(\boldsymbol{\alpha}(E),\boldsymbol{\lambda}(E),E\big) - \operatorname{diag}\big(\boldsymbol{\lambda}(D)\big)\dot{\boldsymbol{A}}(D)\boldsymbol{b}\big(\boldsymbol{\alpha}(D),\boldsymbol{\lambda}(D),D\big) \\
&= \boldsymbol{\alpha}(E) - \boldsymbol{\alpha}(D) \\
&\quad + \operatorname{diag}\big(\boldsymbol{\lambda}(E)\big)\dot{\boldsymbol{A}}(E)\boldsymbol{b}\big(\boldsymbol{\alpha}(E),\boldsymbol{\lambda}(E),E\big) - \operatorname{diag}\big(\boldsymbol{\lambda}(D)\big)\dot{\boldsymbol{A}}(D)\boldsymbol{b}\big(\boldsymbol{\alpha}(E),\boldsymbol{\lambda}(E),E\big) \\
&\quad + \operatorname{diag}\big(\boldsymbol{\lambda}(D)\big)\dot{\boldsymbol{A}}(D)\boldsymbol{b}\big(\boldsymbol{\alpha}(E),\boldsymbol{\lambda}(E),E\big) - \operatorname{diag}\big(\boldsymbol{\lambda}(D)\big)\dot{\boldsymbol{A}}(D)\boldsymbol{b}\big(\boldsymbol{\alpha}(D),\boldsymbol{\lambda}(D),D\big) \\
&= \boldsymbol{\alpha}(E) - \boldsymbol{\alpha}(D) + \big(\operatorname{diag}\big(\boldsymbol{\lambda}(E)\big)\dot{\boldsymbol{A}}(E) - \operatorname{diag}\big(\boldsymbol{\lambda}(D)\big)\dot{\boldsymbol{A}}(D)\big)\boldsymbol{b}\big(\boldsymbol{\alpha}(E),\boldsymbol{\lambda}(E),E\big) \\
&\quad + \operatorname{diag}\big(\boldsymbol{\lambda}(D)\big)\dot{\boldsymbol{A}}(D)\big(\boldsymbol{b}\big(\boldsymbol{\alpha}(E),\boldsymbol{\lambda}(E),E\big) - \boldsymbol{b}\big(\boldsymbol{\alpha}(D),\boldsymbol{\lambda}(D),D\big)\big),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\big(\boldsymbol{I}_n - \operatorname{diag}\big(\boldsymbol{\lambda}(D)\big)\dot{\boldsymbol{A}}(D)\big)\big(\boldsymbol{b}\big(\boldsymbol{\alpha}(E),\boldsymbol{\lambda}(E),E\big) - \boldsymbol{b}\big(\boldsymbol{\alpha}(D),\boldsymbol{\lambda}(D),D\big)\big) &= \boldsymbol{\alpha}(E) - \boldsymbol{\alpha}(D) \\
+ \big(\operatorname{diag}\big(\boldsymbol{\lambda}(E)\big)\dot{\boldsymbol{A}}(E) - \operatorname{diag}\big(\boldsymbol{\lambda}(D)\big)\dot{\boldsymbol{A}}(D)\big)\boldsymbol{b}\big(\boldsymbol{\alpha}(E),\boldsymbol{\lambda}(E),E\big)&,
\end{aligned}
$$

which in turn implies that

$$
\begin{aligned}
&\boldsymbol{b}\big(\boldsymbol{\alpha}(E),\boldsymbol{\lambda}(E),E\big) - \boldsymbol{b}\big(\boldsymbol{\alpha}(D),\boldsymbol{\lambda}(D),D\big) \\
&\quad = \big(\boldsymbol{I}_n - \operatorname{diag}\big(\boldsymbol{\lambda}(D)\big)\dot{\boldsymbol{A}}(D)\big)^{-1}\big(\boldsymbol{\alpha}(E) - \boldsymbol{\alpha}(D)\big) \\
&\qquad + \big(\boldsymbol{I}_n - \operatorname{diag}\big(\boldsymbol{\lambda}(D)\big)\dot{\boldsymbol{A}}(D)\big)^{-1}\big(\operatorname{diag}\big(\boldsymbol{\lambda}(E)\big)\dot{\boldsymbol{A}}(E) - \operatorname{diag}\big(\boldsymbol{\lambda}(D)\big)\dot{\boldsymbol{A}}(D)\big) \\
&\qquad\quad \times \boldsymbol{b}\big(\boldsymbol{\alpha}(E),\boldsymbol{\lambda}(E),E\big) \\
&\quad = \boldsymbol{b}\big(\boldsymbol{\alpha}(E) - \boldsymbol{\alpha}(D),\boldsymbol{\lambda}(D),D\big) \\
&\qquad + \boldsymbol{b}\big(\big(\operatorname{diag}\big(\boldsymbol{\lambda}(E)\big)\dot{\boldsymbol{A}}(E) - \operatorname{diag}\big(\boldsymbol{\lambda}(D)\big)\dot{\boldsymbol{A}}(D)\big)\boldsymbol{b}\big(\boldsymbol{\lambda}(E),\boldsymbol{\alpha}(E),E\big),\boldsymbol{\lambda}(D),D\big).
\end{aligned}
$$

This concludes the proof of (3.7).

Fact F.2 *If E is a superdigraph of D and $\boldsymbol{\lambda}(E) \geq_c \boldsymbol{\lambda}(D) \geq_c \boldsymbol{0}_n$, then*

$$
\rho\big(\operatorname{diag}\big(\boldsymbol{\lambda}(E)\big)\dot{\boldsymbol{A}}(E)\big) \geq \rho\big(\operatorname{diag}\big(\boldsymbol{\lambda}(D)\big)\dot{\boldsymbol{A}}(D)\big).
$$

Proof of Fact F.2 Suppose E is a superdigraph of D and $\boldsymbol{\lambda}(E) \geq_c \boldsymbol{\lambda}(D) \geq_c \boldsymbol{0}_n$. We have $\operatorname{diag}(\boldsymbol{\lambda}(E))\dot{\boldsymbol{A}}(E) \geq_c \operatorname{diag}(\boldsymbol{\lambda}(D))\dot{\boldsymbol{A}}(D) \geq_c \boldsymbol{O}_n$, from which it follows that $\rho(\operatorname{diag}(\boldsymbol{\lambda}(E))\dot{\boldsymbol{A}}(E)) \geq \rho(\operatorname{diag}(\boldsymbol{\lambda}(D))\dot{\boldsymbol{A}}(D))$ (see, for example, Varga 2000, Theorem 2.21). ■

Results 3.24.1 and 3.24.2 follow from (3.7), Fact F.2, and Corollary 3.15. ■

Proof of Proposition 3.25

Suppose Conditions C-α and C-λ are satisfied and $1 \notin \sigma(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D \boxminus x)) \cup \sigma(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D))$. Let $\boldsymbol{A} := \operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)$ and $\boldsymbol{A}_x := \operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D \boxminus x)$. Let the function $\widehat{x}\colon \mathcal{I}(n) \setminus \{x\} \to \mathcal{I}(n)$ be defined by

$$
\widehat{x}(i) := \begin{cases} x - 1 & \text{if } i < x, \\ x & \text{if } i > x. \end{cases}
$$

The proof of (3.8) rests upon four auxiliary results, which are stated as Facts F.3 to F.6.

Fact F.3 *For all* $(i,j) \in \mathcal{I}(n)^2$, $[\boldsymbol{A}\,\mathrm{adj}(\boldsymbol{I}_n - \boldsymbol{A})]_{i,j} = [\mathrm{adj}(\boldsymbol{I}_n - \boldsymbol{A})]_{i,j} - \delta_{i,j}\det(\boldsymbol{I}_n - \boldsymbol{A})$ $= [\mathrm{adj}(\boldsymbol{I}_n - \boldsymbol{A})\boldsymbol{A}]_{i,j}$.

Fact F.4 *For all* $i \in \mathcal{I}(n)$, $[\mathrm{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)]_{i,x} = [\mathrm{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)]_{x,i} = \delta_{i,x}\det(\boldsymbol{I}_n - \boldsymbol{A}_x)$.

Fact F.5 *For all* $(i,j) \in (\mathcal{I}(n) \setminus \{x\})^2$, *if* $j < x$, *then*

$$\det\big([\boldsymbol{I}_n - \boldsymbol{A}_x]_{-j,-i}\big) = (-1)^{x+\widehat{x}(i)-1}\det\big([\boldsymbol{I}_n - \boldsymbol{A}_x]_{-\{j,x\},-\{i,x\}}\big),$$

and if $j > x$, *then*

$$\det\big([\boldsymbol{I}_n - \boldsymbol{A}_x]_{-j,-i}\big) = (-1)^{x+\widehat{x}(i)}\det\big([\boldsymbol{I}_n - \boldsymbol{A}_x]_{-\{j,x\},-\{i,x\}}\big).$$

Fact F.6 *For all* $i \in \mathcal{I}(n)$, $[\mathrm{adj}(\boldsymbol{I}_n - \boldsymbol{A})]_{i,x} = [\mathrm{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)]_{i,x} + [\mathrm{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)\boldsymbol{A}]_{i,x}$.

Proof of Fact F.3 The statement follows from

$$(\boldsymbol{I}_n - \boldsymbol{A})\,\mathrm{adj}(\boldsymbol{I}_n - \boldsymbol{A}) = \mathrm{adj}(\boldsymbol{I}_n - \boldsymbol{A})(\boldsymbol{I}_n - \boldsymbol{A}) = \det(\boldsymbol{I}_n - \boldsymbol{A})\boldsymbol{I}_n,$$

which implies that

$$\boldsymbol{A}\,\mathrm{adj}(\boldsymbol{I}_n - \boldsymbol{A}) = \mathrm{adj}(\boldsymbol{I}_n - \boldsymbol{A})\boldsymbol{A} = \mathrm{adj}(\boldsymbol{I}_n - \boldsymbol{A}) - \det(\boldsymbol{I}_n - \boldsymbol{A})\boldsymbol{I}_n. \qquad \blacksquare$$

Proof of Fact F.4 Let $i \in \mathcal{I}(n)$. I only prove $[\mathrm{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)]_{i,x} = \delta_{i,x}\det(\boldsymbol{I}_n - \boldsymbol{A}_x)$. The proof of $[\mathrm{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)]_{x,i} = \delta_{i,x}\det(\boldsymbol{I}_n - \boldsymbol{A}_x)$ is analogous. First, consider the case $i \neq x$. Note that $[\mathrm{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)]_{i,x} = (-1)^{i+x}\det([\boldsymbol{I}_n - \boldsymbol{A}_x]_{-x,-i})$, where the $\widehat{x}(i)$th column of $[\boldsymbol{I}_n - \boldsymbol{A}_x]_{-x,-i}$ is equal to $\boldsymbol{0}_{n-1}$. Consequently, $\det\big([\boldsymbol{I}_n - \boldsymbol{A}_x]_{-x,-i}\big)$ $= 0$, from which $[\mathrm{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)]_{i,x} = 0$ follows. Second, consider the case $i = x$. The Laplace expansion of the determinant of $\boldsymbol{I}_n - \boldsymbol{A}_x$ along the xth column yields

$$\begin{aligned}\det(\boldsymbol{I}_n - \boldsymbol{A}_x) &= \sum_{j=1}^{n}(-1)^{j+x}\det\big([\boldsymbol{I}_n - \boldsymbol{A}_x]_{-j,-x}\big)[\boldsymbol{I}_n - \boldsymbol{A}_x]_{j,x}\\ &= (-1)^{x+x}\det\big([\boldsymbol{I}_n - \boldsymbol{A}_x]_{-x,-x}\big)\\ &= [\mathrm{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)]_{x,x}\end{aligned}$$

because, for all $j \in \mathcal{I}(n)$, $[\boldsymbol{I}_n - \boldsymbol{A}_x]_{j,x} = \delta_{j,x}$. $\blacksquare$

Proof of Fact F.5 Let $(i,j) \in (\mathcal{I}(n) \setminus \{x\})^2$. The Laplace expansion of the determinant of $[\boldsymbol{I}_n - \boldsymbol{A}_x]_{-j,-i}$ along the $\widehat{x}(i)$th column yields

$$\begin{aligned}&\det\big([\boldsymbol{I}_n - \boldsymbol{A}_x]_{-j,-i}\big)\\ &\qquad = \sum_{k=1}^{n-1}(-1)^{k+\widehat{x}(i)}\det\big([[\boldsymbol{I}_n - \boldsymbol{A}_x]_{-j,-i}]_{-k,-\widehat{x}(i)}\big)\,[[\boldsymbol{I}_n - \boldsymbol{A}_x]_{-j,-i}]_{k,\widehat{x}(i)}.\end{aligned}$$

Note that

$$\big[[\boldsymbol{I}_n - \boldsymbol{A}_x]_{-j,-i}\big]_{-k,-\widehat{x}(i)} = \begin{cases} [\boldsymbol{I}_n - \boldsymbol{A}_x]_{-\{j,k\},-\{i,x\}} & \text{if } k < j, \\ [\boldsymbol{I}_n - \boldsymbol{A}_x]_{-\{j,k+1\},-\{i,x\}} & \text{if } k \geq j, \end{cases}$$

and

$$\big[[\boldsymbol{I}_n - \boldsymbol{A}_x]_{-j,-i}\big]_{k,\widehat{x}(i)} = \big[[\boldsymbol{I}_n - \boldsymbol{A}_x]_{-j,\bullet}\big]_{k,x} = \big[[\boldsymbol{e}_x]_{-j}\big]_k = \begin{cases} \delta_{k,x-1} & \text{if } j < x, \\ \delta_{k,x} & \text{if } j > x. \end{cases}$$

Using the preceding three results, if $j < x$, then

$$\det\big([\boldsymbol{I}_n - \boldsymbol{A}_x]_{-j,-i}\big) = (-1)^{x+\widehat{x}(i)-1} \det\big([\boldsymbol{I}_n - \boldsymbol{A}_x]_{-\{j,x\},-\{i,x\}}\big),$$

and if $j > x$, then

$$\det\big([\boldsymbol{I}_n - \boldsymbol{A}_x]_{-j,-i}\big) = (-1)^{x+\widehat{x}(i)} \det\big([\boldsymbol{I}_n - \boldsymbol{A}_x]_{-\{j,x\},-\{i,x\}}\big). \qquad \blacksquare$$

Proof of Fact F.6 Let $i \in \mathcal{I}(n)$. First, consider the case $i = x$. We find

$$\begin{aligned} [\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A})]_{x,x} &= (-1)^{x+x} \det\big([\boldsymbol{I}_n - \boldsymbol{A}]_{-x,-x}\big) \\ &= (-1)^{x+x} \det\big([\boldsymbol{I}_n - \boldsymbol{A}_x]_{-x,-x}\big) \\ &= [\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)]_{x,x} \end{aligned}$$

and

$$[\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)\boldsymbol{A}]_{x,x} = \sum_{j=1, j\neq x}^{n} \underbrace{[\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)]_{x,j}}_{=0 \text{ (Fact F.4)}} [\boldsymbol{A}]_{j,x} + [\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)]_{x,x} \underbrace{[\boldsymbol{A}]_{x,x}}_{=0} = 0, \tag{F.4}$$

which imply that $[\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A})]_{x,x} = [\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)]_{x,x} + [\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)\boldsymbol{A}]_{x,x}$.

Second, consider the case $i \neq x$. Note that $[\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)]_{i,x} = 0$ (Fact F.4). Let $\boldsymbol{B}(x,i) := [\boldsymbol{I}_n - \boldsymbol{A}]_{-x,-i}$. For all $k \in \mathcal{I}(n)$, let $\boldsymbol{a}_k$ denote the kth column of $\boldsymbol{A}$. Note that $[\boldsymbol{B}(x,i)]_{\bullet,\widehat{x}(i)} = [\boldsymbol{e}_x - \boldsymbol{a}_x]_{-x}$ because $i \neq x$. Let $\boldsymbol{C}(x,i)$ and $\boldsymbol{D}(x,i)$ be the unique square matrices of order $n-1$ that satisfy

$$\begin{aligned} \forall j \in \mathcal{I}(n-1) \quad [\boldsymbol{C}(x,i)]_{\bullet,j} &= (1 - \delta_{j,\widehat{x}(i)})[\boldsymbol{B}(x,i)]_{\bullet,j} + \delta_{j,\widehat{x}(i)}[\boldsymbol{e}_x]_{-x}, \\ \forall j \in \mathcal{I}(n-1) \quad [\boldsymbol{D}(x,i)]_{\bullet,j} &= (1 - \delta_{j,\widehat{x}(i)})[\boldsymbol{B}(x,i)]_{\bullet,j} - \delta_{j,\widehat{x}(i)}[\boldsymbol{a}_x]_{-x}. \end{aligned}$$

As an illustration, if $i \notin \{1, x-1, x+1, n\}$, then

$$\begin{aligned} \boldsymbol{B}(x,i) &= [(\boldsymbol{e}_1 - \boldsymbol{a}_1, \ldots, \boldsymbol{e}_{x-1} - \boldsymbol{a}_{x-1}, \boldsymbol{e}_x - \boldsymbol{a}_x, \boldsymbol{e}_{x+1} - \boldsymbol{a}_{x+1}, \ldots, \boldsymbol{e}_n - \boldsymbol{a}_n)]_{-x,\bullet}, \\ \boldsymbol{C}(x,i) &= [(\boldsymbol{e}_1 - \boldsymbol{a}_1, \ldots, \boldsymbol{e}_{x-1} - \boldsymbol{a}_{x-1}, \boldsymbol{e}_x \qquad, \boldsymbol{e}_{x+1} - \boldsymbol{a}_{x+1}, \ldots, \boldsymbol{e}_n - \boldsymbol{a}_n)]_{-x,\bullet}, \\ \boldsymbol{D}(x,i) &= [(\boldsymbol{e}_1 - \boldsymbol{a}_1, \ldots, \boldsymbol{e}_{x-1} - \boldsymbol{a}_{x-1}, \qquad - \boldsymbol{a}_x, \boldsymbol{e}_{x+1} - \boldsymbol{a}_{x+1}, \ldots, \boldsymbol{e}_n - \boldsymbol{a}_n)]_{-x,\bullet}. \end{aligned}$$

We find

$$
\begin{aligned}
[\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A})]_{i,x} &= (-1)^{i+x} \det\big([\boldsymbol{I}_n - \boldsymbol{A}]_{-x,-i}\big) \\
&= (-1)^{i+x} \det\big(\boldsymbol{B}(x,i)\big) \\
&= (-1)^{i+x} \big(\det\big(\boldsymbol{C}(x,i)\big) + \det\big(\boldsymbol{D}(x,i)\big)\big) \\
&= (-1)^{i+x} \det\big(\boldsymbol{D}(x,i)\big) \\
&= (-1)^{i+x} \sum_{j=1}^{n-1} (-1)^{j+\widehat{x}(i)} \det\big([\boldsymbol{D}(x,i)]_{-j,-\widehat{x}(i)}\big) [\boldsymbol{D}(x,i)]_{j,\widehat{x}(i)} \\
&= (-1)^{i+x} \sum_{j=1}^{x-1} (-1)^{j+\widehat{x}(i)} \det\big([\boldsymbol{I}_n - \boldsymbol{A}]_{-\{j,x\},-\{i,x\}}\big) [-\boldsymbol{A}]_{j,x} \\
&\quad + (-1)^{i+x} \sum_{j=x}^{n-1} (-1)^{j+\widehat{x}(i)} \det\big([\boldsymbol{I}_n - \boldsymbol{A}]_{-\{j+1,x\},-\{i,x\}}\big) [-\boldsymbol{A}]_{j+1,x} \\
&= \sum_{j=1}^{x-1} (-1)^{i+j} (-1)^{x+\widehat{x}(i)-1} \det\big([\boldsymbol{I}_n - \boldsymbol{A}]_{-\{j,x\},-\{i,x\}}\big) [\boldsymbol{A}]_{j,x} \\
&\quad + \sum_{j=x+1}^{n} (-1)^{i+j} (-1)^{x+\widehat{x}(i)} \det\big([\boldsymbol{I}_n - \boldsymbol{A}]_{-\{j,x\},-\{i,x\}}\big) [\boldsymbol{A}]_{j,x} \\
&= \sum_{j=1}^{x-1} (-1)^{i+j} \det\big([\boldsymbol{I}_n - \boldsymbol{A}_x]_{-j,-i}\big) [\boldsymbol{A}]_{j,x} \\
&\quad + (-1)^{i+x} \det\big([\boldsymbol{I}_n - \boldsymbol{A}_x]_{-x,-i}\big) [\boldsymbol{A}]_{x,x} \\
&\quad + \sum_{j=x+1}^{n} (-1)^{i+j} \det\big([\boldsymbol{I}_n - \boldsymbol{A}_x]_{-j,-i}\big) [\boldsymbol{A}]_{j,x} \\
&= \sum_{j=1}^{n} (-1)^{i+j} \det\big([\boldsymbol{I}_n - \boldsymbol{A}_x]_{-j,-i}\big) [\boldsymbol{A}]_{j,x} \\
&= \sum_{j=1}^{n} [\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)]_{i,j} [\boldsymbol{A}]_{j,x} \\
&= [\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)\boldsymbol{A}]_{i,x}.
\end{aligned} \tag{F.5}
$$

The first two equalities are obvious. The third equality follows from the fact that $\det(\boldsymbol{B}(x,i))$ is a multilinear function of the columns of $\boldsymbol{B}(x,i)$. The fourth equality follows from $\det(\boldsymbol{C}(x,i)) = 0$ (the $\widehat{x}(i)$th column of $\boldsymbol{C}(x,i)$ is equal to $\mathbf{0}_{n-1}$). The fifth equality is according to the Laplace expansion of the determinant of $\boldsymbol{D}(x,i)$ along the $\widehat{x}(i)$th column. The sixth equality is according to

$$
\begin{aligned}
&\forall j \in \mathcal{I}(x-1) && [\boldsymbol{D}(x,i)]_{j,\widehat{x}(i)} = [-\boldsymbol{A}]_{j,x}, \\
&\forall j \in \mathcal{I}(n-1) \setminus \mathcal{I}(x-1) && [\boldsymbol{D}(x,i)]_{j,\widehat{x}(i)} = [-\boldsymbol{A}]_{j+1,x},
\end{aligned}
$$

and

$$
\forall j \in \mathcal{I}(x-1) \qquad [\boldsymbol{D}(x,i)]_{-j,-\widehat{x}(i)} = [\boldsymbol{I}_n - \boldsymbol{A}]_{-\{j,x\},-\{i,x\}},
$$

$$\forall j \in \mathcal{I}(n-1) \setminus \mathcal{I}(x-1) \quad [\boldsymbol{D}(x,i)]_{-j,-\widehat{x}(i)} = [\boldsymbol{I}_n - \boldsymbol{A}]_{-\{j+1,x\},-\{i,x\}}.$$

The seventh equality is obvious. The eighth equality follows from Fact F.5 and the equality $[\boldsymbol{A}]_{x,x} = 0$. The last three equalities are obvious. ■

We have

$$\begin{aligned}
\boldsymbol{b}(\boldsymbol{\alpha},\boldsymbol{\lambda},D) - \boldsymbol{b}(\boldsymbol{\alpha},\boldsymbol{\lambda},D \boxminus x) &= \boldsymbol{b}\big(\operatorname{diag}(\boldsymbol{\lambda})\big(\dot{\boldsymbol{A}}(D) - \dot{\boldsymbol{A}}(D \boxminus x)\big)\boldsymbol{b}(\boldsymbol{\alpha},\boldsymbol{\lambda},D),\boldsymbol{\lambda},D \boxminus x\big) \\
&= (\boldsymbol{I}_n - \boldsymbol{A}_x)^{-1}(\boldsymbol{A} - \boldsymbol{A}_x)(\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{\alpha} \\
&= \frac{\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)(\boldsymbol{A} - \boldsymbol{A}_x)\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A})\boldsymbol{\alpha}}{\det(\boldsymbol{I}_n - \boldsymbol{A})\det(\boldsymbol{I}_n - \boldsymbol{A}_x)}, \qquad \text{(F.6)}
\end{aligned}$$

where the first equality is according to Proposition 3.24. We find, for all $(i,j) \in \mathcal{I}(n)^2$,

$$\begin{aligned}
&[\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)(\boldsymbol{A} - \boldsymbol{A}_x)\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A})]_{i,j} \\
&\quad = \sum_{l=1}^{n}[\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)(\boldsymbol{A} - \boldsymbol{A}_x)]_{i,l}[\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A})]_{l,j} \\
&\quad = \sum_{l=1}^{n}\sum_{k=1}^{n}[\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)]_{i,k}[\boldsymbol{A} - \boldsymbol{A}_x]_{k,l}[\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A})]_{l,j} \\
&\quad = \sum_{l=1}^{n}[\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)]_{i,x}[\boldsymbol{A}]_{x,l}[\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A})]_{l,j} \\
&\qquad + \sum_{k=1}^{n}[\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)]_{i,k}[\boldsymbol{A}]_{k,x}[\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A})]_{x,j} \\
&\quad = [\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)]_{i,x}[\boldsymbol{A}\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A})]_{x,j} + [\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)\boldsymbol{A}]_{i,x}[\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A})]_{x,j} \\
&\quad = [\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)]_{i,x}\big([\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A})]_{x,j} - \delta_{j,x}\det(\boldsymbol{I}_n - \boldsymbol{A})\big) \\
&\qquad + [\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)\boldsymbol{A}]_{i,x}[\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A})]_{x,j} \\
&\quad = \big([\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)]_{i,x} + [\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)\boldsymbol{A}]_{i,x}\big)[\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A})]_{x,j} \\
&\qquad - \delta_{j,x}\det(\boldsymbol{I}_n - \boldsymbol{A})[\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)]_{i,x} \\
&\quad = [\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A})]_{i,x}[\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A})]_{x,j} - \delta_{i,x}\delta_{j,x}\det(\boldsymbol{I}_n - \boldsymbol{A})\det(\boldsymbol{I}_n - \boldsymbol{A}_x). \qquad \text{(F.7)}
\end{aligned}$$

The first two equalities are obvious. The third equality follows from the fact that

$$\forall (k,l) \in \mathcal{I}(n)^2 \quad [\boldsymbol{A} - \boldsymbol{A}_x]_{k,l} = [\boldsymbol{A}]_{k,l}\,\mathbb{1}_{\{k,l\}}(x),$$

where $[\boldsymbol{A}]_{x,x} = 0$. The fourth equality is obvious. The fifth equality follows from Fact F.3. The sixth equality is obvious. The last equality follows from Facts F.4 and F.6. Using (F.7), we find, for all $i \in \mathcal{I}(n)$,

$$\begin{aligned}
&[\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)(\boldsymbol{A} - \boldsymbol{A}_x)\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A})\boldsymbol{\alpha}]_i \\
&\quad = \sum_{j=1}^{n}[\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)(\boldsymbol{A} - \boldsymbol{A}_x)\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A})]_{i,j}[\boldsymbol{\alpha}]_j
\end{aligned}$$

$$
\begin{aligned}
&= [\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A})]_{i,x} \sum_{j=1}^{n} [\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A})]_{x,j} [\boldsymbol{\alpha}]_j \\
&\quad - \delta_{i,x} \det(\boldsymbol{I}_n - \boldsymbol{A}) \det(\boldsymbol{I}_n - \boldsymbol{A}_x) \sum_{j=1}^{n} \delta_{j,x} [\boldsymbol{\alpha}]_j \\
&= [\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A})]_{i,x} [\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A})\boldsymbol{\alpha}]_x - \delta_{i,x} \det(\boldsymbol{I}_n - \boldsymbol{A}) \det(\boldsymbol{I}_n - \boldsymbol{A}_x) [\boldsymbol{\alpha}]_x. \quad \text{(F.8)}
\end{aligned}
$$

Note that

$$
\boldsymbol{e}_x^\top (\boldsymbol{I}_n - \boldsymbol{A})^{-1} \boldsymbol{e}_x = \frac{[\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A})]_{x,x}}{\det(\boldsymbol{I}_n - \boldsymbol{A})} = \frac{\det(\boldsymbol{I}_n - \boldsymbol{A}_x)}{\det(\boldsymbol{I}_n - \boldsymbol{A})}, \tag{F.9}
$$

where the last equality follows from the Laplace expansion of the determinant of $\boldsymbol{I}_n - \boldsymbol{A}_x$ along the xth column (see also the proof of Fact F.4):

$$
\begin{aligned}
\det(\boldsymbol{I}_n - \boldsymbol{A}_x) &= \sum_{j=1}^{n} (-1)^{j+x} \det\big([\boldsymbol{I}_n - \boldsymbol{A}_x]_{-j,-x}\big) [\boldsymbol{I}_n - \boldsymbol{A}_x]_{j,x} \\
&= (-1)^{x+x} \det\big([\boldsymbol{I}_n - \boldsymbol{A}_x]_{-x,-x}\big) \\
&= (-1)^{x+x} \det\big([\boldsymbol{I}_n - \boldsymbol{A}]_{-x,-x}\big) \\
&= [\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A})]_{x,x}. \qquad \text{(F.10)}
\end{aligned}
$$

Finally, using (F.6), (F.8), and (F.9), we find, for all $i \in \mathcal{I}(n)$,

$$
\begin{aligned}
&[\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) - \boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D \boxminus x)]_i \\
&\quad = \frac{[\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)(\boldsymbol{A} - \boldsymbol{A}_x) \operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A})\boldsymbol{\alpha}]_i}{\det(\boldsymbol{I}_n - \boldsymbol{A}) \det(\boldsymbol{I}_n - \boldsymbol{A}_x)} \\
&\quad = \frac{[\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A})]_{i,x} [\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A})\boldsymbol{\alpha}]_x - \delta_{i,x} \det(\boldsymbol{I}_n - \boldsymbol{A}) \det(\boldsymbol{I}_n - \boldsymbol{A}_x) [\boldsymbol{\alpha}]_x}{\det(\boldsymbol{I}_n - \boldsymbol{A}) \det(\boldsymbol{I}_n - \boldsymbol{A}_x)} \\
&\quad = \left[\frac{\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A})\boldsymbol{\alpha}}{\det(\boldsymbol{I}_n - \boldsymbol{A})}\right]_x \frac{\det(\boldsymbol{I}_n - \boldsymbol{A})}{\det(\boldsymbol{I}_n - \boldsymbol{A}_x)} \left[\frac{\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A})}{\det(\boldsymbol{I}_n - \boldsymbol{A})}\right]_{i,x} - \delta_{i,x} [\boldsymbol{\alpha}]_x \\
&\quad = \left[(\boldsymbol{I}_n - \boldsymbol{A})^{-1} \boldsymbol{\alpha}\right]_x \frac{1}{\boldsymbol{e}_x^\top (\boldsymbol{I}_n - \boldsymbol{A})^{-1} \boldsymbol{e}_x} \left[(\boldsymbol{I}_n - \boldsymbol{A})^{-1}\right]_{i,x} - \delta_{i,x} [\boldsymbol{\alpha}]_x \\
&\quad = \left[(\boldsymbol{I}_n - \boldsymbol{A})^{-1} \boldsymbol{\alpha}\right]_x \frac{1}{\left[(\boldsymbol{I}_n - \boldsymbol{A})^{-1} \boldsymbol{e}_x\right]_x} \left[(\boldsymbol{I}_n - \boldsymbol{A})^{-1} \boldsymbol{e}_x\right]_i - [\boldsymbol{\alpha}]_x [\boldsymbol{e}_x]_i \\
&\quad = [\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)]_x \frac{1}{[\boldsymbol{b}(\boldsymbol{e}_x, \boldsymbol{\lambda}, D)]_x} [\boldsymbol{b}(\boldsymbol{e}_x, \boldsymbol{\lambda}, D)]_i - [\boldsymbol{\alpha}]_x [\boldsymbol{e}_x]_i \\
&\quad = \frac{b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)}{b_x(\boldsymbol{e}_x, \boldsymbol{\lambda}, D)} [\boldsymbol{b}(\boldsymbol{e}_x, \boldsymbol{\lambda}, D)]_i - \alpha_x [\boldsymbol{e}_x]_i,
\end{aligned}
$$

which implies that

$$
\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) - \boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D \boxminus x) = \frac{b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)}{b_x(\boldsymbol{e}_x, \boldsymbol{\lambda}, D)} \boldsymbol{b}(\boldsymbol{e}_x, \boldsymbol{\lambda}, D) - \alpha_x \boldsymbol{e}_x.
$$

This concludes the proof of (3.8). Let $i \in \mathcal{I}(n) \setminus \{x\}$.

Proof of Result 3.25.1 Suppose $\boldsymbol{\lambda} \geq_c \mathbf{0}_n$, $\rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$, and $\boldsymbol{\alpha} \geq_c \mathbf{0}_n$. We find

$$\Delta_x(D \boxminus x, D) = \alpha_x - b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) \leq 0$$

because $b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) \geq \alpha_x$ (Result 3.15.1) and

$$\Delta_i(D \boxminus x, D) = -\frac{b_i(\boldsymbol{e}_x, \boldsymbol{\lambda}, D)}{b_x(\boldsymbol{e}_x, \boldsymbol{\lambda}, D)}\, b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) \leq 0$$

because $b_x(\boldsymbol{e}_x, \boldsymbol{\lambda}, D) \geq 1$ (Result 3.19.1), $b_i(\boldsymbol{e}_x, \boldsymbol{\lambda}, D) \geq 0$ (Result 3.19.1), and $b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) \geq 0$ (Result 3.15.1). ■

Proof of Result 3.25.2 Suppose $\lambda_x > 0$, $[\boldsymbol{\lambda}]_{-x} \geq_c \mathbf{0}_{n-1}$, $\rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$, there exists a $k \in \mathcal{N}_D^+(x)$ with $\alpha_k > 0$, and $[\boldsymbol{\alpha}]_{-k} \geq_c \mathbf{0}_{n-1}$. We find

$$\Delta_x(D \boxminus x, D) = \alpha_x - b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) = -\lambda_x \sum_{j \in \mathcal{N}_D^+(x)} b_j(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) \leq -\lambda_x\, b_k(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) < 0,$$

where the first inequality follows from $\lambda_x > 0$ and $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) \geq_c \mathbf{0}_n$ (Result 3.15.1) and the second from $b_k(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) \geq \alpha_k > 0$ (Result 3.15.1). ■

Proof of Result 3.25.3 Suppose $\boldsymbol{\lambda} \geq_c \mathbf{0}_n$, there exists a walk $(i_0, \ldots, i_p)$ in D of length p from i to x such that $(\lambda_{i_0}, \ldots, \lambda_{i_{p-1}}) >_c \mathbf{0}_p$, $\rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$, $\alpha_x > 0$, and $[\boldsymbol{\alpha}]_{-x} \geq_c \mathbf{0}_{n-1}$. We find

$$\Delta_i(D \boxminus x, D) = -\frac{b_i(\boldsymbol{e}_x, \boldsymbol{\lambda}, D)}{b_x(\boldsymbol{e}_x, \boldsymbol{\lambda}, D)}\, b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) < 0$$

because $b_x(\boldsymbol{e}_x, \boldsymbol{\lambda}, D) \geq 1$ (Result 3.19.1), $b_i(\boldsymbol{e}_x, \boldsymbol{\lambda}, D) > 0$ (Result 3.19.2), and $b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) \geq \alpha_x > 0$ (Result 3.15.1). ■

Proof of Lemma 3.26

Suppose Condition C-λ is satisfied and $\rho(\operatorname{diag}(|\boldsymbol{\lambda}|)\dot{\boldsymbol{A}}(D)) < 1$. First, note that $\boldsymbol{O}_n \leq_c |\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)| = \operatorname{diag}(|\boldsymbol{\lambda}|)\dot{\boldsymbol{A}}(D)$, which implies that $\rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) \leq \rho(\operatorname{diag}(|\boldsymbol{\lambda}|)\dot{\boldsymbol{A}}(D))$ (see, for example, Varga 2000, Theorem 2.21). Second, note that $\boldsymbol{O}_n \leq_c |\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D \boxminus x)| \leq_c \operatorname{diag}(|\boldsymbol{\lambda}|)\dot{\boldsymbol{A}}(D)$ because $\dot{\boldsymbol{A}}(D \boxminus x) \leq_c \dot{\boldsymbol{A}}(D)$, which implies that $\rho(\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D \boxminus x)) \leq \rho(\operatorname{diag}(|\boldsymbol{\lambda}|)\dot{\boldsymbol{A}}(D))$ (Theorem 2.21). Let $\boldsymbol{B}$ be equal to $\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)$ or $\operatorname{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D \boxminus x)$. Note that $\rho(\boldsymbol{B}) < 1$. I show that $1 \notin \sigma(\boldsymbol{B})$. Suppose, for the sake of contradiction, $1 \in \sigma(\boldsymbol{B})$. It follows that $\rho(\boldsymbol{B}) \geq 1$, which contradicts $\rho(\boldsymbol{B}) < 1$. This concludes the proof of $1 \notin \sigma(\boldsymbol{B})$. ■

Proof of Proposition 3.27

Let $i \in \mathcal{I}(n) \setminus \{x\}$, and let $\bar{\boldsymbol{\lambda}} \in \mathbb{R}_+^n$ be such that both inequalities $\rho(\operatorname{diag}(\bar{\boldsymbol{\lambda}})\dot{\boldsymbol{A}}(D)) < 1$ and $\operatorname{diag}(\bar{\boldsymbol{\lambda}})\dot{\boldsymbol{A}}(D)\mathbf{1}_n \leq_c \mathbf{1}_n$ are true. Note that $\boldsymbol{I}_n - \operatorname{diag}(\bar{\boldsymbol{\lambda}})\dot{\boldsymbol{A}}(D)$ is a nonsingular

M-matrix whose inverse is bounded below by $\boldsymbol{I}_n$ because $\operatorname{diag}(\bar{\boldsymbol{\lambda}})\dot{\boldsymbol{A}}(D)$ is nonnegative and $\rho(\operatorname{diag}(\bar{\boldsymbol{\lambda}})\dot{\boldsymbol{A}}(D)) < 1$ (Lemma B.6). Note also that all row sums of $\boldsymbol{I}_n - \operatorname{diag}(\bar{\boldsymbol{\lambda}})\dot{\boldsymbol{A}}(D)$ are nonnegative because $\operatorname{diag}(\bar{\boldsymbol{\lambda}})\dot{\boldsymbol{A}}(D)\mathbf{1}_n \leq_c \mathbf{1}_n$. It follows that (see, for example, Berman and Plemmons 1994, Lemma 3.14 on p. 254)

$$0 \leq \left[\left(\boldsymbol{I}_n - \operatorname{diag}(\bar{\boldsymbol{\lambda}})\dot{\boldsymbol{A}}(D)\right)^{-1}\right]_{i,x} \leq \left[\left(\boldsymbol{I}_n - \operatorname{diag}(\bar{\boldsymbol{\lambda}})\dot{\boldsymbol{A}}(D)\right)^{-1}\right]_{x,x},$$

where $[(\boldsymbol{I}_n - \operatorname{diag}(\bar{\boldsymbol{\lambda}})\dot{\boldsymbol{A}}(D))^{-1}]_{x,x} \geq 1$. We conclude that

$$S_{D,i,x}(\bar{\boldsymbol{\lambda}}) = \frac{b_i(\boldsymbol{e}_x, \bar{\boldsymbol{\lambda}}, D)}{b_x(\boldsymbol{e}_x, \bar{\boldsymbol{\lambda}}, D)} = \frac{\left[\left(\boldsymbol{I}_n - \operatorname{diag}(\bar{\boldsymbol{\lambda}})\dot{\boldsymbol{A}}(D)\right)^{-1}\right]_{i,x}}{\left[\left(\boldsymbol{I}_n - \operatorname{diag}(\bar{\boldsymbol{\lambda}})\dot{\boldsymbol{A}}(D)\right)^{-1}\right]_{x,x}} \in [0,1]. \qquad \blacksquare$$

Proof of Proposition 3.29

Let $i \in \mathcal{I}(n) \setminus \{x\}$. Suppose there exists a walk in D from i to x. Let $\bar{\lambda} \in \mathbb{R}_{++}$ be such that $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$, and let $\boldsymbol{C}(\bar{\lambda}) := (\boldsymbol{I}_n - \bar{\lambda}\dot{\boldsymbol{A}}(D))^{-1}$. Note that

$$S_{D,i,x}(\bar{\lambda}\mathbf{1}_n) = \frac{b_i(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)}{b_x(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)} = \frac{\boldsymbol{e}_i^\top\left(\boldsymbol{I}_n - \bar{\lambda}\dot{\boldsymbol{A}}(D)\right)^{-1}\boldsymbol{e}_x}{\boldsymbol{e}_x^\top\left(\boldsymbol{I}_n - \bar{\lambda}\dot{\boldsymbol{A}}(D)\right)^{-1}\boldsymbol{e}_x^\top} = \frac{\boldsymbol{e}_i^\top\boldsymbol{C}(\bar{\lambda})\boldsymbol{e}_x}{\boldsymbol{e}_x^\top\boldsymbol{C}(\bar{\lambda})\boldsymbol{e}_x}.$$

We find

$$\begin{aligned}
&\partial\big(t \mapsto S_{D,i,x}(t\mathbf{1}_n)\big)(\bar{\lambda}) \\
&\quad= \frac{\boldsymbol{e}_i^\top\boldsymbol{C}(\bar{\lambda})^2\boldsymbol{e}_x\boldsymbol{e}_x^\top\boldsymbol{C}(\bar{\lambda})\boldsymbol{e}_x - \boldsymbol{e}_i^\top\boldsymbol{C}(\bar{\lambda})\boldsymbol{e}_x\boldsymbol{e}_x^\top\boldsymbol{C}(\bar{\lambda})^2\boldsymbol{e}_x}{\bar{\lambda}\left(\boldsymbol{e}_x^\top\boldsymbol{C}(\bar{\lambda})\boldsymbol{e}_x\right)^2} \\
&\quad= \frac{1}{\bar{\lambda}[\boldsymbol{C}(\bar{\lambda})]_{x,x}^2}\sum_{k=1}^n[\boldsymbol{C}(\bar{\lambda})]_{k,x}\big([\boldsymbol{C}(\bar{\lambda})]_{x,x}[\boldsymbol{C}(\bar{\lambda})]_{i,k} - [\boldsymbol{C}(\bar{\lambda})]_{i,x}[\boldsymbol{C}(\bar{\lambda})]_{x,k}\big) \\
&\quad= \frac{1}{\bar{\lambda}[\boldsymbol{C}(\bar{\lambda})]_{x,x}^2}\sum_{k=1}^n[\boldsymbol{C}(\bar{\lambda})]_{k,x}\det\begin{pmatrix}[\boldsymbol{C}(\bar{\lambda})]_{x,x} & [\boldsymbol{C}(\bar{\lambda})]_{x,k} \\ [\boldsymbol{C}(\bar{\lambda})]_{i,x} & [\boldsymbol{C}(\bar{\lambda})]_{i,k}\end{pmatrix} \\
&\quad= \frac{1}{\bar{\lambda}[\boldsymbol{C}(\bar{\lambda})]_{x,x}^2}\sum_{k\in\mathcal{I}(n)\setminus\{x\}}[\boldsymbol{C}(\bar{\lambda})]_{k,x}\det\big([\boldsymbol{C}(\bar{\lambda})]_{\{x,i\},\{x,k\}}\big) \\
&\quad= \frac{1}{\bar{\lambda}[\boldsymbol{C}(\bar{\lambda})]_{x,x}^2}[\boldsymbol{C}(\bar{\lambda})]_{i,x}\det\big([\boldsymbol{C}(\bar{\lambda})]_{\{x,i\},\{x,i\}}\big) \\
&\qquad+ \frac{1}{\bar{\lambda}[\boldsymbol{C}(\bar{\lambda})]_{x,x}^2}\sum_{k\in\mathcal{I}(n)\setminus\{x,i\}}[\boldsymbol{C}(\bar{\lambda})]_{k,x}\det\big([\boldsymbol{C}(\bar{\lambda})]_{\{x,i\},\{x,k\}}\big),
\end{aligned} \tag{F.11}$$

where the first equality follows from

$$\frac{\partial\boldsymbol{C}(\bar{\lambda})}{\partial\bar{\lambda}} = \boldsymbol{C}(\bar{\lambda})\dot{\boldsymbol{A}}(D)\boldsymbol{C}(\bar{\lambda})$$

and

$$\dot{\boldsymbol{A}}(D)\boldsymbol{C}(\bar{\lambda}) = \boldsymbol{C}(\bar{\lambda})\dot{\boldsymbol{A}}(D) = \frac{1}{\bar{\lambda}}\big(\boldsymbol{C}(\bar{\lambda}) - \boldsymbol{I}_n\big).$$

First, note that $\boldsymbol{C}(\bar{\lambda})$ is an inverse M-matrix that is bounded below by $\boldsymbol{I}_n$ (Lemma B.6). It follows that $[\boldsymbol{C}(\bar{\lambda})]_{x,x} \geq 1$ and, for all $k \in \mathcal{I}(n) \setminus \{x\}$, $[\boldsymbol{C}(\bar{\lambda})]_{k,x} \geq 0$. Second, note that $[\boldsymbol{C}(\bar{\lambda})]_{i,x} > 0$ because $\bar{\lambda} > 0$,

$$[\boldsymbol{C}(\bar{\lambda})]_{i,x} = b_i(\boldsymbol{e}_x, \bar{\lambda}, D) = \sum_{k=0}^{\infty} \bar{\lambda}^k d^k_{D,i\to x},$$

and there exists a walk in D from i to x, that is, $d^k_{D,i\to x} > 0$ for some $k \in \mathbb{Z}_{++}$. Third, note that all principal minors of an inverse M-matrix are positive (see, for example, Johnson 1982, Corollary 1). It follows that $\det([\boldsymbol{C}(\bar{\lambda})]_{\{x,i\},\{x,i\}}) > 0$. Fourth, note that all almost principal minors of an inverse M-matrix are nonnegative (see, for example, Willoughby 1977, pp. 77–78). It follows that for all $k \in \mathcal{I}(n) \setminus \{x, i\}$, $\det([\boldsymbol{C}(\bar{\lambda})]_{\{x,i\},\{x,k\}}) \geq 0$. The foregoing four facts and (F.11) imply that $\partial(t \mapsto S_{D,i,x}(t\mathbf{1}_n))(\bar{\lambda}) > 0$. ∎

Proof of Proposition 3.30

Let $(i, j) \in (\mathcal{I}(n) \setminus \{x\}) \times \mathcal{I}(n)$, $\bar{\boldsymbol{\lambda}} := (\bar{\lambda}_1, \ldots, \bar{\lambda}_n) \in \mathbb{R}^n_+$ with $\rho(\operatorname{diag}(\bar{\boldsymbol{\lambda}})\dot{\boldsymbol{A}}(D)) < 1$, and $\boldsymbol{C}(\bar{\boldsymbol{\lambda}}) := (\boldsymbol{I}_n - \operatorname{diag}(\bar{\boldsymbol{\lambda}})\dot{\boldsymbol{A}}(D))^{-1}$. Note that

$$S_{D,i,x}(\bar{\boldsymbol{\lambda}}) = \frac{b_i(\boldsymbol{e}_x, \bar{\boldsymbol{\lambda}}, D)}{b_x(\boldsymbol{e}_x, \bar{\boldsymbol{\lambda}}, D)} = \frac{\boldsymbol{e}_i^\top \big(\boldsymbol{I}_n - \operatorname{diag}(\bar{\boldsymbol{\lambda}})\dot{\boldsymbol{A}}(D)\big)^{-1}\boldsymbol{e}_x}{\boldsymbol{e}_x^\top \big(\boldsymbol{I}_n - \operatorname{diag}(\bar{\boldsymbol{\lambda}})\dot{\boldsymbol{A}}(D)\big)^{-1}\boldsymbol{e}_x^\top} = \frac{\boldsymbol{e}_i^\top \boldsymbol{C}(\bar{\boldsymbol{\lambda}})\boldsymbol{e}_x}{\boldsymbol{e}_x^\top \boldsymbol{C}(\bar{\boldsymbol{\lambda}})\boldsymbol{e}_x}.$$

We find

$$\begin{aligned}
&\frac{\partial S_{D,i,x}(\bar{\boldsymbol{\lambda}})}{\partial \bar{\lambda}_j} \\
&\quad = \frac{\boldsymbol{e}_i^\top \boldsymbol{C}(\bar{\boldsymbol{\lambda}})\boldsymbol{e}_j\boldsymbol{e}_j^\top \dot{\boldsymbol{A}}(D)\boldsymbol{C}(\bar{\boldsymbol{\lambda}})\boldsymbol{e}_x\boldsymbol{e}_x^\top \boldsymbol{C}(\bar{\boldsymbol{\lambda}})\boldsymbol{e}_x - \boldsymbol{e}_i^\top \boldsymbol{C}(\bar{\boldsymbol{\lambda}})\boldsymbol{e}_x\boldsymbol{e}_x^\top \boldsymbol{C}(\bar{\boldsymbol{\lambda}})\boldsymbol{e}_j\boldsymbol{e}_j^\top \dot{\boldsymbol{A}}(D)\boldsymbol{C}(\bar{\boldsymbol{\lambda}})\boldsymbol{e}_x}{\big(\boldsymbol{e}_x^\top \boldsymbol{C}(\bar{\boldsymbol{\lambda}})\boldsymbol{e}_x\big)^2} \\
&\quad = \frac{[\dot{\boldsymbol{A}}(D)\boldsymbol{C}(\bar{\boldsymbol{\lambda}})]_{j,x}}{[\boldsymbol{C}(\bar{\boldsymbol{\lambda}})]^2_{x,x}}\big([\boldsymbol{C}(\bar{\boldsymbol{\lambda}})]_{x,x}[\boldsymbol{C}(\bar{\boldsymbol{\lambda}})]_{i,j} - [\boldsymbol{C}(\bar{\boldsymbol{\lambda}})]_{i,x}[\boldsymbol{C}(\bar{\boldsymbol{\lambda}})]_{x,j}\big) \\
&\quad = \frac{[\dot{\boldsymbol{A}}(D)\boldsymbol{C}(\bar{\boldsymbol{\lambda}})]_{j,x}}{[\boldsymbol{C}(\bar{\boldsymbol{\lambda}})]^2_{x,x}} \det\begin{pmatrix} [\boldsymbol{C}(\bar{\boldsymbol{\lambda}})]_{x,x} & [\boldsymbol{C}(\bar{\boldsymbol{\lambda}})]_{x,j} \\ [\boldsymbol{C}(\bar{\boldsymbol{\lambda}})]_{i,x} & [\boldsymbol{C}(\bar{\boldsymbol{\lambda}})]_{i,j} \end{pmatrix} \\
&\quad = \frac{[\dot{\boldsymbol{A}}(D)\boldsymbol{C}(\bar{\boldsymbol{\lambda}})]_{j,x}}{[\boldsymbol{C}(\bar{\boldsymbol{\lambda}})]^2_{x,x}} \det\big([\boldsymbol{C}(\bar{\boldsymbol{\lambda}})]_{\{x,i\},\{x,j\}}\big),
\end{aligned} \tag{F.12}$$

where the first equality follows from

$$\frac{\partial \boldsymbol{C}(\bar{\boldsymbol{\lambda}})}{\partial \bar{\lambda}_j} = \boldsymbol{C}(\bar{\boldsymbol{\lambda}})\boldsymbol{e}_j\boldsymbol{e}_j^\top \dot{\boldsymbol{A}}(D)\boldsymbol{C}(\bar{\boldsymbol{\lambda}}).$$

First, note that $\boldsymbol{C}(\bar{\boldsymbol{\lambda}})$ is an inverse M-matrix that is bounded below by $\boldsymbol{I}_n$ (Lemma B.6). It follows that $[\boldsymbol{C}(\bar{\boldsymbol{\lambda}})]_{x,x} \geq 1$ and $[\dot{\boldsymbol{A}}(D)\boldsymbol{C}(\bar{\boldsymbol{\lambda}})]_{j,x} \geq 0$. Second, note that if $\bar{\lambda}_j > 0$, then

$$[\dot{\boldsymbol{A}}(D)\boldsymbol{C}(\bar{\boldsymbol{\lambda}})]_{j,x} = \frac{[\boldsymbol{C}(\bar{\boldsymbol{\lambda}})]_{j,x} - \delta_{j,x}}{\bar{\lambda}_j}$$

as $\operatorname{diag}(\bar{\boldsymbol{\lambda}})\dot{\boldsymbol{A}}(D)\boldsymbol{C}(\bar{\boldsymbol{\lambda}}) = \boldsymbol{C}(\bar{\boldsymbol{\lambda}}) - \boldsymbol{I}_n$. Third, note that $[\boldsymbol{C}(\bar{\boldsymbol{\lambda}})]_{j,x} = b_j(\boldsymbol{e}_x, \bar{\boldsymbol{\lambda}}, D) > 0$ if there exists a walk $(i_0, \ldots, i_p)$ in D of length p from j to x such that $(\bar{\lambda}_{i_0}, \ldots, \bar{\lambda}_{i_{p-1}}) >_c \boldsymbol{0}_p$ (Lemma 3.19). Fourth, note that $\det([\boldsymbol{C}(\bar{\lambda})]_{\{x,i\},\{x,x\}}) = 0$. Fifth, note that all principal minors of an inverse M-matrix are positive (see, for example, Johnson 1982, Corollary 1). It follows that $\det([\boldsymbol{C}(\bar{\lambda})]_{\{x,i\},\{x,i\}}) > 0$. Sixth, note that all almost principal minors of an inverse M-matrix are nonnegative (see, for example, Willoughby 1977, pp. 77–78). It follows that $\det([\boldsymbol{C}(\bar{\lambda})]_{\{x,i\},\{x,j\}}) \geq 0$ (if $i \neq j$). The foregoing six facts and (F.12) imply Results 3.30.1, 3.30.2, and 3.30.3. ∎

Proof of Proposition 3.31

Suppose α is as in Example 3.2, λ is as in Example 3.5, and $1 \notin \sigma(\bar{\lambda}\dot{\boldsymbol{A}}(D)) \cup \sigma(\bar{\lambda}\dot{\boldsymbol{A}}(D \boxminus x))$. Let $\boldsymbol{A} := \bar{\lambda}\dot{\boldsymbol{A}}(D)$ and $\boldsymbol{A}_x := \bar{\lambda}\dot{\boldsymbol{A}}(D \boxminus x)$. First, consider the case $\bar{\lambda} = 0$. We find

$$\boldsymbol{b}\big(\dot{\boldsymbol{A}}(D)\boldsymbol{1}_n, \bar{\lambda}\boldsymbol{1}_n, D\big) - \boldsymbol{b}\big(\dot{\boldsymbol{A}}(D \boxminus x)\boldsymbol{1}_n, \bar{\lambda}\boldsymbol{1}_n, D \boxminus x\big) = \big(\dot{\boldsymbol{A}}(D) - \dot{\boldsymbol{A}}(D \boxminus x)\big)\boldsymbol{1}_n.$$

Second, consider the case $\bar{\lambda} \neq 0$. First, note that, for all $i \in \mathcal{I}(n)$,

$$\begin{aligned}
&[\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)(\boldsymbol{A} - \boldsymbol{A}_x)\boldsymbol{1}_n]_i \\
&\quad = \sum_{j=1}^{n} [\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)]_{i,j} [(\boldsymbol{A} - \boldsymbol{A}_x)\boldsymbol{1}_n]_j \\
&\quad = \sum_{j=1, j\neq x}^{n} [\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)]_{i,j} [\boldsymbol{A}]_{j,x} + [\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)]_{i,x} [\boldsymbol{A}\boldsymbol{1}_n]_x \\
&\quad = \sum_{j=1}^{n} [\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)]_{i,j} [\boldsymbol{A}]_{j,x} + [\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)]_{i,x} [\boldsymbol{A}\boldsymbol{1}_n]_x \\
&\quad = [\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)\boldsymbol{A}]_{i,x} + [\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)]_{i,x} [\boldsymbol{A}\boldsymbol{1}_n]_x \\
&\quad = [\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A})]_{i,x} + [\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)]_{i,x} ([\boldsymbol{A}\boldsymbol{1}_n]_x - 1) \\
&\quad = \det(\boldsymbol{I}_n - \boldsymbol{A})\big[(\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{e}_x\big]_i + \det(\boldsymbol{I}_n - \boldsymbol{A}_x)([\boldsymbol{A}\boldsymbol{1}_n]_x - 1)[\boldsymbol{e}_x]_i,
\end{aligned} \tag{F.13}$$

where the second equality follows from

$$\forall j \in \mathcal{I}(n) \quad [(\boldsymbol{A} - \boldsymbol{A}_x)\boldsymbol{1}_n]_j = \begin{cases} [\boldsymbol{A}\boldsymbol{1}_n]_x & \text{if } j = x, \\ [\boldsymbol{A}]_{j,x} & \text{if } j \neq x, \end{cases}$$

the third equality follows from $[\dot{\boldsymbol{A}}(D)]_{x,x} = 0$, the fifth equality follows from Fact F.6, and the sixth equality follows from Fact F.4 (see the proof of Proposition 3.25 for

Facts F.4 and F.6). Second, note that

$$\begin{aligned}
&\boldsymbol{b}\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n - \dot{\boldsymbol{A}}(D\boxminus x)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D\boxminus x\big)\\
&\quad= \frac{1}{\bar{\lambda}}(\boldsymbol{I}_n - \boldsymbol{A}_x)^{-1}(\boldsymbol{A} - \boldsymbol{A}_x)\mathbf{1}_n\\
&\quad= \frac{1}{\bar{\lambda}}\frac{1}{\det(\boldsymbol{I}_n - \boldsymbol{A}_x)}\operatorname{adj}(\boldsymbol{I}_n - \boldsymbol{A}_x)(\boldsymbol{A} - \boldsymbol{A}_x)\mathbf{1}_n\\
&\quad= \frac{1}{\bar{\lambda}}\frac{\det(\boldsymbol{I}_n - \boldsymbol{A})}{\det(\boldsymbol{I}_n - \boldsymbol{A}_x)}(\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{e}_x + \frac{1}{\bar{\lambda}}([\boldsymbol{A}\mathbf{1}_n]_x - 1)\boldsymbol{e}_x\\
&\quad= \frac{1}{\bar{\lambda}}\frac{1}{b_x(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)}\boldsymbol{b}(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D) + \deg_D^+(x)\boldsymbol{e}_x - \frac{1}{\bar{\lambda}}\boldsymbol{e}_x,
\end{aligned}\tag{F.14}$$

where the third equality follows from (F.13) and the fourth equality from (F.9). We find

$$\begin{aligned}
&\boldsymbol{b}\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D\big) - \boldsymbol{b}\big(\dot{\boldsymbol{A}}(D\boxminus x)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D\boxminus x\big)\\
&\quad= \boldsymbol{b}\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n - \dot{\boldsymbol{A}}(D\boxminus x)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D\boxminus x\big)\\
&\qquad+ \boldsymbol{b}\big(\bar{\lambda}\big(\dot{\boldsymbol{A}}(D) - \dot{\boldsymbol{A}}(D\boxminus x)\big)\boldsymbol{b}\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D\big), \bar{\lambda}\mathbf{1}_n, D\boxminus x\big)\\
&\quad= \frac{1}{\bar{\lambda}}\frac{1}{b_x(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)}\boldsymbol{b}(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D) + \deg_D^+(x)\boldsymbol{e}_x - \frac{1}{\bar{\lambda}}\boldsymbol{e}_x\\
&\qquad+ \frac{b_x\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D\big)}{b_x(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)}\boldsymbol{b}(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D) - \deg_D^+(x)\boldsymbol{e}_x\\
&\quad= \frac{1}{\bar{\lambda}}\left(\frac{1}{b_x(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)}\boldsymbol{b}(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D) - \boldsymbol{e}_x\right)\\
&\qquad+ \frac{b_x\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D\big)}{b_x(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)}\boldsymbol{b}(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D),
\end{aligned}$$

where the first equality is according to Proposition 3.24 and the second equality follows from (F.14) and Proposition 3.25 (see also the proof thereof), in particular,

$$\begin{aligned}
&\boldsymbol{b}\big(\bar{\lambda}\big(\dot{\boldsymbol{A}}(D) - \dot{\boldsymbol{A}}(D\boxminus x)\big)\boldsymbol{b}\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D\big), \bar{\lambda}\mathbf{1}_n, D\boxminus x\big)\\
&\qquad\qquad= \frac{b_x\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D\big)}{b_x(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)}\boldsymbol{b}(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D) - \deg_D^+(x)\boldsymbol{e}_x.
\end{aligned}$$

Let $i \in \mathcal{I}(n) \setminus \{x\}$.

Proof of Result 3.31.1 Suppose $\bar{\lambda} \geq 0$ and $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$. First, consider the case $\bar{\lambda} = 0$. We find $\Delta_x(D\boxminus x, D) = -\deg_D^+(x) \leq 0$ and $\Delta_i(D\boxminus x, D) = -\mathbb{1}_{\mathcal{N}_D^+(i)}(x) \leq 0$. Second, consider the case $\bar{\lambda} > 0$. We find

$$\Delta_x(D\boxminus x, D) = -\,b_x\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D\big) \leq 0$$

because $b_x(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D) \geq 0$ (Result 3.15.1) and

$$\Delta_i(D\boxminus x, D) = -\frac{b_i(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)}{b_x(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)}\left(\frac{1}{\bar{\lambda}} + b_x\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D\big)\right) \leq 0$$

because $b_x(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D) \geq 1$ (Result 3.19.1), $b_i(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D) \geq 0$ (Result 3.19.1), and $b_x(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D) \geq 0$ (Result 3.15.1). ■

Proof of Result 3.31.2 Suppose $\bar{\lambda} \geq 0$, $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$, and $\deg_D^+(x) > 0$. We find

$$\Delta_x(D \boxminus x, D) = -\, b_x(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D) \leq -\deg_D^+(x) < 0$$

because $b_x(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D) \geq \deg_D^+(x)$ (Result 3.15.1). ■

Proof of Result 3.31.3 Suppose $\bar{\lambda} > 0$, $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$, and there exists a walk in D from i to x. We find

$$\Delta_i(D \boxminus x, D) = -\frac{b_i(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)}{b_x(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)} \left(\frac{1}{\bar{\lambda}} + b_x\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D\big) \right) < 0$$

because $b_x(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D) \geq 1$ (Result 3.19.1), $b_i(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D) > 0$ (Result 3.19.2), and $b_x(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D) \geq 0$ (Result 3.15.1). ■

Proof of Lemma 3.32

Assume that $\rho(\dot{\boldsymbol{A}}(D)) < (\sum_{j \in \mathcal{N}_D^+(x)} \deg_D^+(j))^{1/2} =: L(D, x)$. Note that $L(D, x) > 0$. Let the function $f(D, x)\colon \{\tilde{\lambda} \in \mathbb{R}_{++} \mid \rho(\tilde{\lambda}\dot{\boldsymbol{A}}(D)) < 1\} \to \mathbb{R}$ be defined by

$$f(D, x)(t) := \frac{1}{t} + b_x\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, t\mathbf{1}_n, D\big).$$

We find

$$\begin{aligned}
\partial f(D, x)(t) &= -\frac{1}{t^2} + \boldsymbol{e}_x^\top \big(\boldsymbol{I}_n - t\dot{\boldsymbol{A}}(D)\big)^{-1} \dot{\boldsymbol{A}}(D) \boldsymbol{b}\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, t\mathbf{1}_n, D\big) \\
&= -\frac{1}{t^2} + \boldsymbol{e}_x^\top \frac{\big(\boldsymbol{I}_n - t\dot{\boldsymbol{A}}(D)\big)^{-1} - \boldsymbol{I}_n}{t} \boldsymbol{b}\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, t\mathbf{1}_n, D\big) \\
&= \frac{1}{t}\Big(b_x\big(\boldsymbol{b}\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, t\mathbf{1}_n, D\big), t\mathbf{1}_n, D\big) - b_x\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, t\mathbf{1}_n, D\big) - \frac{1}{t} \Big) \\
&= \sum_{j \in \mathcal{N}_D^+(x)} b_j\big(\boldsymbol{b}\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, t\mathbf{1}_n, D\big), t\mathbf{1}_n, D\big) - \frac{1}{t^2},
\end{aligned}$$

where the second equality follows from

$$\big(\boldsymbol{I}_n - t\dot{\boldsymbol{A}}(D)\big)^{-1} = \boldsymbol{I}_n + t\big(\boldsymbol{I}_n - t\dot{\boldsymbol{A}}(D)\big)^{-1}\dot{\boldsymbol{A}}(D) = \boldsymbol{I}_n + t\dot{\boldsymbol{A}}(D)\big(\boldsymbol{I}_n - t\dot{\boldsymbol{A}}(D)\big)^{-1} \tag{F.15}$$

and the last from

$$\begin{aligned}
b_x\big(\boldsymbol{b}\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, t\mathbf{1}_n, D\big), t\mathbf{1}_n, D\big) = b_x\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, t\mathbf{1}_n, D\big) \\
+ t \sum_{j \in \mathcal{N}_D^+(x)} b_j\big(\boldsymbol{b}\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, t\mathbf{1}_n, D\big), t\mathbf{1}_n, D\big).
\end{aligned}$$

Note that

$$\partial f(D,x)(t) \geq \sum_{j\in\mathcal{N}_D^+(x)} \deg_D^+(j) - \frac{1}{t^2}$$

because, for all $j \in \mathcal{N}_D^+(x)$, $b_j(\boldsymbol{b}(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, t\mathbf{1}_n, D), t\mathbf{1}_n, D) \geq b_j(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, t\mathbf{1}_n, D) \geq \deg_D^+(j)$ (Result 3.15.1). It follows that $\partial f(D,x)(t) > 0$ if $t > 1/L(D,x)$. There exists a $t_1 > 0$ such that $1/L(D,x) < t_1$ in case $\rho(\dot{\boldsymbol{A}}(D)) = 0$ and $1/L(D,x) < t_1 < 1/\rho(\dot{\boldsymbol{A}}(D))$ in case $\rho(\dot{\boldsymbol{A}}(D)) > 0$ because $\rho(\dot{\boldsymbol{A}}(D)) < L(D,x)$. In both cases, t_1 lies in the domain of $f(D,x)$. There exists a $t_0 < t_1$ in the domain of $f(D,x)$ such that $\partial f(D,x)(t_0) < 0$ because $\lim_{t\downarrow 0} \partial f(D,x)(t) = -\infty$. The function $\partial f(D,x)$ has a root in the interval (t_0, t_1), denoted by $c(D,x)$, because it is continuous with $\partial f(D,x)(t_0) < 0 < \partial f(D,x)(t_1)$ (Bolzano's theorem). This root is unique because $\partial f(D,x)(t)$ is strictly increasing. Indeed, we find

$$\begin{aligned}
&\partial^2 f(D,x)(t)\\
&\quad= \sum_{j\in\mathcal{N}_D^+(x)} \boldsymbol{e}_j^\top \frac{\partial\big(\boldsymbol{I}_n - t\dot{\boldsymbol{A}}(D)\big)^{-1}\boldsymbol{b}\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, t\mathbf{1}_n, D\big)}{\partial t} + \frac{2}{t^3}\\
&\quad= 2\sum_{j\in\mathcal{N}_D^+(x)} \boldsymbol{e}_j^\top\big(\boldsymbol{I}_n - t\dot{\boldsymbol{A}}(D)\big)^{-1}\big(\boldsymbol{I}_n - t\dot{\boldsymbol{A}}(D)\big)^{-1}\dot{\boldsymbol{A}}(D)\boldsymbol{b}\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, t\mathbf{1}_n, D\big) + \frac{2}{t^3}\\
&\quad\geq 2\sum_{j\in\mathcal{N}_D^+(x)} \boldsymbol{e}_j^\top\dot{\boldsymbol{A}}(D)\dot{\boldsymbol{A}}(D)\mathbf{1}_n + \frac{2}{t^3}\\
&\quad= 2\sum_{j\in\mathcal{N}_D^+(x)}\sum_{k\in\mathcal{N}_D^+(j)} \deg_D^+(k) + \frac{2}{t^3}\\
&\quad> 0,
\end{aligned}$$

where the second equality follows from the fact that $\dot{\boldsymbol{A}}(D)$ and $(\boldsymbol{I}_n - t\dot{\boldsymbol{A}}(D))^{-1}$ commute (see (F.15)), and the first inequality from $(\boldsymbol{I}_n - t\dot{\boldsymbol{A}}(D))^{-1} \geq_c \boldsymbol{I}_n$ (Lemma B.6) and $\boldsymbol{b}(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, t\mathbf{1}_n, D) \geq_c \dot{\boldsymbol{A}}(D)\mathbf{1}_n$ (Result 3.15.1). We conclude that the function $f(D,x)$ is strictly increasing on the set $\{\tilde{\lambda} \in \mathbb{R}_{++} \mid \rho(\tilde{\lambda}\dot{\boldsymbol{A}}(D)) < 1\} \cap (c(D,x), +\infty)$, which is a nonempty interval. ∎

Proof of Proposition 3.35

Suppose $\boldsymbol{\alpha}(D \ominus x) = [\boldsymbol{\alpha}(D \boxminus x)]_{-x}$ and $\boldsymbol{\lambda}(D \ominus x) = [\boldsymbol{\lambda}(D \boxminus x)]_{-x}$.

Proof of Result 3.35.1 There exists a permutation matrix $\boldsymbol{P}_x$ of order n, where $\boldsymbol{P}_x^{-1} = \boldsymbol{P}_x^\top$, such that

$$\boldsymbol{Q}_x := \boldsymbol{P}_x \operatorname{diag}\big(\boldsymbol{\lambda}(D \boxminus x)\big)\dot{\boldsymbol{A}}(D \boxminus x)\boldsymbol{P}_x^{-1} = \begin{pmatrix} 0 & \mathbf{0}_{n-1}^\top \\ \mathbf{0}_{n-1} & \operatorname{diag}\big(\boldsymbol{\lambda}(D \ominus x)\big)\dot{\boldsymbol{A}}(D \ominus x) \end{pmatrix},$$

specifically, $\boldsymbol{P}_x = (\boldsymbol{e}_x, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_{x-1}, \boldsymbol{e}_{x+1}, \ldots, \boldsymbol{e}_n)^\top$. Note that, for all $\mu \in \mathbb{C}$,

$$\det(\mu\boldsymbol{I}_n - \boldsymbol{Q}_x) = \det\begin{pmatrix} \mu & \mathbf{0}_{n-1}^\top \\ \mathbf{0}_{n-1} & \mu\boldsymbol{I}_{n-1} - \operatorname{diag}\big(\boldsymbol{\lambda}(D \ominus x)\big)\dot{\boldsymbol{A}}(D \ominus x) \end{pmatrix}$$

$$= \mu \det\big(\mu \boldsymbol{I}_{n-1} - \operatorname{diag}\big(\boldsymbol{\lambda}(D \ominus x)\big)\dot{\boldsymbol{A}}(D \ominus x)\big),$$

which implies that $\sigma(\boldsymbol{Q}_x) \setminus \{0\} = \sigma(\operatorname{diag}(\boldsymbol{\lambda}(D \ominus x))\dot{\boldsymbol{A}}(D \ominus x)) \setminus \{0\}$. Note also that $\sigma(\boldsymbol{Q}_x) = \sigma(\operatorname{diag}(\boldsymbol{\lambda}(D \boxminus x))\dot{\boldsymbol{A}}(D \boxminus x))$ because similar matrices have the same spectrum. Combining the foregoing two results gives Result 3.35.1. ■

Proof of Result 3.35.2 Suppose $1 \notin \sigma(\operatorname{diag}(\boldsymbol{\lambda}(D \ominus x))\dot{\boldsymbol{A}}(D \ominus x)) \cup \sigma(\operatorname{diag}(\boldsymbol{\lambda}(D))\dot{\boldsymbol{A}}(D))$. Result 3.35.1 implies that the matrices $\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}(D))\dot{\boldsymbol{A}}(D)$, $\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}(D \boxminus x))\dot{\boldsymbol{A}}(D \boxminus x)$, and $\boldsymbol{I}_{n-1} - \operatorname{diag}(\boldsymbol{\lambda}(D \ominus x))\dot{\boldsymbol{A}}(D \ominus x)$ are nonsingular. We find

$$\begin{aligned}
&\boldsymbol{P}_x \boldsymbol{b}\big(\boldsymbol{\alpha}(D \boxminus x), \boldsymbol{\lambda}(D \boxminus x), D \boxminus x\big) \\
&\quad = \boldsymbol{P}_x \big(\boldsymbol{I}_n - \operatorname{diag}\big(\boldsymbol{\lambda}(D \boxminus x)\big)\dot{\boldsymbol{A}}(D \boxminus x)\big)^{-1} \boldsymbol{P}_x^{-1} \boldsymbol{P}_x \boldsymbol{\alpha}(D \boxminus x) \\
&\quad = \big(\boldsymbol{I}_n - \boldsymbol{P}_x \operatorname{diag}\big(\boldsymbol{\lambda}(D \boxminus x)\big)\dot{\boldsymbol{A}}(D \boxminus x)\boldsymbol{P}_x^{-1}\big)^{-1} \boldsymbol{P}_x \boldsymbol{\alpha}(D \boxminus x) \\
&\quad = \begin{pmatrix} 1 & \boldsymbol{0}_{n-1}^\mathsf{T} \\ \boldsymbol{0}_{n-1} & \boldsymbol{I}_{n-1} - \operatorname{diag}\big(\boldsymbol{\lambda}(D \ominus x)\big)\dot{\boldsymbol{A}}(D \ominus x) \end{pmatrix}^{-1} \begin{pmatrix} [\boldsymbol{P}_x \boldsymbol{\alpha}(D \boxminus x)]_1 \\ [\boldsymbol{P}_x \boldsymbol{\alpha}(D \boxminus x)]_{-1} \end{pmatrix} \\
&\quad = \begin{pmatrix} 1 & \boldsymbol{0}_{n-1}^\mathsf{T} \\ \boldsymbol{0}_{n-1} & \big(\boldsymbol{I}_{n-1} - \operatorname{diag}\big(\boldsymbol{\lambda}(D \ominus x)\big)\dot{\boldsymbol{A}}(D \ominus x)\big)^{-1} \end{pmatrix} \begin{pmatrix} [\boldsymbol{\alpha}(D \boxminus x)]_x \\ \boldsymbol{\alpha}(D \ominus x) \end{pmatrix}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
b_x\big(\boldsymbol{\alpha}(D \boxminus x), \boldsymbol{\lambda}(D \boxminus x), D \boxminus x\big) &= \big[\boldsymbol{P}_x \boldsymbol{b}\big(\boldsymbol{\alpha}(D \boxminus x), \boldsymbol{\lambda}(D \boxminus x), D \boxminus x\big)\big]_1 \\
&= [\boldsymbol{\alpha}(D \boxminus x)]_x
\end{aligned}$$

and

$$\begin{aligned}
&\big[\boldsymbol{b}\big(\boldsymbol{\alpha}(D \boxminus x), \boldsymbol{\lambda}(D \boxminus x), D \boxminus x\big)\big]_{-x} \\
&\quad = \big[\boldsymbol{P}_x \boldsymbol{b}\big(\boldsymbol{\alpha}(D \boxminus x), \boldsymbol{\lambda}(D \boxminus x), D \boxminus x\big)\big]_{-1} \\
&\quad = \big(\boldsymbol{I}_{n-1} - \operatorname{diag}\big(\boldsymbol{\lambda}(D \ominus x)\big)\dot{\boldsymbol{A}}(D \ominus x)\big)^{-1} \boldsymbol{\alpha}(D \ominus x) \\
&\quad = \boldsymbol{b}\big(\boldsymbol{\alpha}(D \ominus x), \boldsymbol{\lambda}(D \ominus x), D \ominus x\big).
\end{aligned}$$

Finally, using the preceding result, we find

$$\begin{aligned}
&\boldsymbol{b}\big(\boldsymbol{\alpha}(D \ominus x), \boldsymbol{\lambda}(D \ominus x), D \ominus x\big) - \big[\boldsymbol{b}\big(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D\big)\big]_{-x} \\
&\qquad = \big[\boldsymbol{b}\big(\boldsymbol{\alpha}(D \boxminus x), \boldsymbol{\lambda}(D \boxminus x), D \boxminus x\big) - \boldsymbol{b}\big(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D\big)\big]_{-x}. \quad ■
\end{aligned}$$

Proof of Proposition 3.36

Suppose Conditions C-α and C-λ are satisfied, $\boldsymbol{\alpha}(D \ominus x) = [\boldsymbol{\alpha}(D \boxminus x)]_{-x}$, $\boldsymbol{\lambda}(D \ominus x) = [\boldsymbol{\lambda}(D \boxminus x)]_{-x}$, and $1 \notin \sigma(\operatorname{diag}(\boldsymbol{\lambda}(D \ominus x))\dot{\boldsymbol{A}}(D \ominus x)) \cup \sigma(\operatorname{diag}(\boldsymbol{\lambda}(D))\dot{\boldsymbol{A}}(D))$. It follows that $\boldsymbol{I}_{n-1} - \operatorname{diag}(\boldsymbol{\lambda}(D \ominus x))\dot{\boldsymbol{A}}(D \ominus x)$ and $\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}(D))\dot{\boldsymbol{A}}(D)$ are nonsingular. Result 3.35.1 implies that $\boldsymbol{I}_n - \operatorname{diag}(\boldsymbol{\lambda}(D \boxminus x))\dot{\boldsymbol{A}}(D \boxminus x)$ is nonsingular. We find

$$\begin{aligned}
&\boldsymbol{b}\big(\boldsymbol{\alpha}(D \ominus x), \boldsymbol{\lambda}(D \ominus x), D \ominus x\big) - \big[\boldsymbol{b}\big(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D\big)\big]_{-x} \\
&\quad = \big[\boldsymbol{b}\big(\boldsymbol{\alpha}(D \boxminus x), \boldsymbol{\lambda}(D \boxminus x), D \boxminus x\big) - \boldsymbol{b}\big(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D\big)\big]_{-x} \\
&\quad = \alpha_x(D)[\boldsymbol{e}_x]_{-x} - \frac{b_x\big(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D\big)}{b_x\big(\boldsymbol{e}_x, \boldsymbol{\lambda}(D), D\big)} \big[\boldsymbol{b}\big(\boldsymbol{e}_x, \boldsymbol{\lambda}(D), D\big)\big]_{-x} \\
&\quad = -\frac{b_x\big(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D\big)}{b_x\big(\boldsymbol{e}_x, \boldsymbol{\lambda}(D), D\big)} \big[\boldsymbol{b}\big(\boldsymbol{e}_x, \boldsymbol{\lambda}(D), D\big)\big]_{-x},
\end{aligned}$$

where the first equality follows from Result 3.35.2, the second from Proposition 3.25, and the last from $[\boldsymbol{e}_x]_{-x} = \boldsymbol{0}_{n-1}$. ■

Proof of Proposition 3.37

Suppose α is as in Example 3.2, λ is as in Example 3.5, and $1 \notin \sigma(\bar{\lambda}\dot{\boldsymbol{A}}(D)) \cup \sigma(\bar{\lambda}\dot{\boldsymbol{A}}(D \ominus x))$. It follows that $\boldsymbol{I}_{n-1} - \bar{\lambda}\dot{\boldsymbol{A}}(D \ominus x)$ and $\boldsymbol{I}_n - \bar{\lambda}\dot{\boldsymbol{A}}(D)$ are nonsingular. Result 3.35.1 implies that $\boldsymbol{I}_n - \bar{\lambda}\dot{\boldsymbol{A}}(D \boxminus x)$ is nonsingular. Note that $\boldsymbol{\alpha}(D \ominus x) = [\boldsymbol{\alpha}(D \boxminus x)]_{-x}$ and $\boldsymbol{\lambda}(D \ominus x) = [\boldsymbol{\lambda}(D \boxminus x)]_{-x}$. First, consider the case $\bar{\lambda} = 0$. We find

$$\begin{aligned}
&\boldsymbol{b}\big(\boldsymbol{\alpha}(D \ominus x), \boldsymbol{\lambda}(D \ominus x), D \ominus x\big) - \big[\boldsymbol{b}\big(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D\big)\big]_{-x} \\
&\qquad\qquad = \dot{\boldsymbol{A}}(D \ominus x)\boldsymbol{1}_{n-1} - [\dot{\boldsymbol{A}}(D)\boldsymbol{1}_n]_{-x}.
\end{aligned}$$

Second, consider the case $\bar{\lambda} \neq 0$. We find

$$\begin{aligned}
&\boldsymbol{b}\big(\boldsymbol{\alpha}(D \ominus x), \boldsymbol{\lambda}(D \ominus x), D \ominus x\big) - \big[\boldsymbol{b}\big(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D\big)\big]_{-x} \\
&\quad = \big[\boldsymbol{b}\big(\boldsymbol{\alpha}(D \boxminus x), \boldsymbol{\lambda}(D \boxminus x), D \boxminus x\big) - \boldsymbol{b}\big(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D\big)\big]_{-x} \\
&\quad = \frac{1}{\bar{\lambda}}\left([\boldsymbol{e}_x]_{-x} - \frac{1}{b_x(\boldsymbol{e}_x, \bar{\lambda}\boldsymbol{1}_n, D)}[\boldsymbol{b}(\boldsymbol{e}_x, \bar{\lambda}\boldsymbol{1}_n, D)]_{-x}\right) \\
&\qquad - \frac{b_x(\dot{\boldsymbol{A}}(D)\boldsymbol{1}_n, \bar{\lambda}\boldsymbol{1}_n, D)}{b_x(\boldsymbol{e}_x, \bar{\lambda}\boldsymbol{1}_n, D)}[\boldsymbol{b}(\boldsymbol{e}_x, \bar{\lambda}\boldsymbol{1}_n, D)]_{-x} \\
&\quad = -\frac{1}{b_x(\boldsymbol{e}_x, \bar{\lambda}\boldsymbol{1}_n, D)}\left(\frac{1}{\bar{\lambda}} + b_x(\dot{\boldsymbol{A}}(D)\boldsymbol{1}_n, \bar{\lambda}\boldsymbol{1}_n, D)\right)[\boldsymbol{b}(\boldsymbol{e}_x, \bar{\lambda}\boldsymbol{1}_n, D)]_{-x},
\end{aligned}$$

where the first equality follows from Result 3.35.2, the second from Proposition 3.31, and the last from $[\boldsymbol{e}_x]_{-x} = \boldsymbol{0}_{n-1}$. ■

Proof of Proposition 3.38

Suppose the GKB centrality in D with profile of vertex idiosyncrasies $\boldsymbol{\alpha} >_c \boldsymbol{0}_n$ and profile of localness parameters $\boldsymbol{\lambda}$ exists and is unique and positive, that is, $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) >_c \boldsymbol{0}_n$. Suppose $\|\cdot\|$ is monotonic in the positive orthant.

Proof of Result 3.38.1 The inequalities $\boldsymbol{\alpha} >_c \mathbf{0}_n$ and $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) >_c \mathbf{0}_n$ imply that, for all $i \in \mathcal{I}(n)$, $\mathrm{VDI}(i, \boldsymbol{\alpha}, \boldsymbol{\lambda}, D) > 0$, $\mathrm{ADI}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) > 0$, and $\mathrm{TDI}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) > 0$.

First, consider the case $\boldsymbol{\lambda} \geq_c \mathbf{0}_n$. We have $\boldsymbol{\alpha} \leq_c \boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ because $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) = \boldsymbol{\alpha} + \mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$, $\boldsymbol{\lambda} \geq_c \mathbf{0}_n$, and $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) >_c \mathbf{0}_n$. It follows that, for all $i \in \mathcal{I}(n)$, $\mathrm{VDI}(i, \boldsymbol{\alpha}, \boldsymbol{\lambda}, D) \leq 1$, which in turn implies that $\mathrm{ADI}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) \leq 1$. We have $\|\boldsymbol{\alpha}\| \leq \|\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)\|$ because $\boldsymbol{\alpha} \leq_c \boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ and $\|\cdot\|$ is monotonic in the positive orthant. If follows that $\mathrm{TDI}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) \leq 1$.

Second, consider the case $\boldsymbol{\lambda} \leq_c \mathbf{0}_n$. We have $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) \leq_c \boldsymbol{\alpha}$ because $\boldsymbol{\lambda} \leq_c \mathbf{0}_n$ and $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) >_c \mathbf{0}_n$. It follows that, for all $i \in \mathcal{I}(n)$, $1/\,\mathrm{VDI}(i, \boldsymbol{\alpha}, \boldsymbol{\lambda}, D) \leq 1$. This result in turn implies that $1/\,\mathrm{ADI}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) \leq 1$. Indeed, according to Jensen's inequality,

$$\frac{1}{\mathrm{ADI}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)} = \frac{1}{\dfrac{1}{n}\displaystyle\sum_{i=1}^{n} \frac{\alpha_i}{b_i(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)}} \leq \frac{1}{n}\sum_{i=1}^{n} \frac{b_i(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)}{\alpha_i} \leq 1.$$

We have $\|\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)\| \leq \|\boldsymbol{\alpha}\|$ because $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) \leq_c \boldsymbol{\alpha}$ and $\|\cdot\|$ is monotonic in the positive orthant. If follows that $1/\,\mathrm{TDI}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) \leq 1$. ∎

Proof of Result 3.38.2 The statements follow from Proposition 3.23. ∎

Proof of Proposition 3.39

Suppose Conditions 3.39.1, 3.39.2, and 3.39.3 are satisfied.

Let $\mathcal{Y}^\star \subset \mathbb{R}^n_{++}$ denote the set of all interior NEs in pure strategies of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$. I show that $|\mathcal{Y}^\star| = 1$. Note that, for all $\boldsymbol{y}^\star \in \mathbb{R}^n_{+}$, $\boldsymbol{y}^\star \in \mathcal{Y}^\star$ if and only if

$$\boldsymbol{y}^\star \in \mathbb{R}^n_{++}, \tag{F.16}$$

$$\forall i \in \mathcal{I} \quad \frac{\partial u_i(\boldsymbol{y}^\star)}{\partial y_i} = 0, \tag{F.17}$$

and

$$\forall i \in \mathcal{I} \quad \frac{\partial^2 u_i(\boldsymbol{y}^\star)}{\partial {y_i}^2} < 0. \tag{F.18}$$

We find, for all $\boldsymbol{y} := (y_1, \ldots, y_n) \in \mathbb{R}^n_+$ and for all $i \in \mathcal{I}$,

$$\frac{\partial u_i(\boldsymbol{y})}{\partial y_i} = \alpha_i - y_i + \lambda_i \sum_{j \in \mathcal{I}} \dot{a}_{i,j}(D) y_j \quad \text{and} \quad \frac{\partial^2 u_i(\boldsymbol{y})}{\partial {y_i}^2} = -1 < 0.$$

It follows that (F.17) is equivalent to $(\boldsymbol{I}_n - \mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D))\boldsymbol{y}^\star = \boldsymbol{\alpha}$, which in turn is equivalent to $\boldsymbol{y}^\star = (\boldsymbol{I}_n - \mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D))^{-1}\boldsymbol{\alpha}$, that is, $\boldsymbol{y}^\star = \boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$, because Condition 3.39.2 implies that $\boldsymbol{I}_n - \mathrm{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)$ is nonsingular (Proposition 3.12). Conditions 3.39.1, 3.39.2, and 3.39.3 imply that $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) >_c \mathbf{0}_n$ (Result 3.15.2). The preceding arguments demonstrate that there exists a unique $\boldsymbol{y}^\star \in \mathbb{R}^n_+$, namely, $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$, that satisfies (F.16), (F.17), and (F.18).

Table F.1. Possible types of boundary Nash equilibria of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ (Propositions 3.39 and 3.40)

	Action and set of actions	
Type	0	$\mathbb{R}_{++}$
B-1	$\times$	
B-2	$\times$	$\times$

In the remainder of the proof, I show that $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ has no boundary NEs. Table F.1 gives an overview of all possible types of boundary NEs of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$.

First, I show that $\mathbf{0}_n$ (this is referred to as a boundary NE of type B-1 in Table F.1) cannot be a boundary NE of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$. Suppose, for the sake of contradiction, $\mathbf{0}_n$ is a boundary NE of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$. We must have

$$\forall i \in \mathcal{I} \quad \frac{\partial u_i(\mathbf{0}_n)}{\partial y_i} \leq 0. \tag{F.19}$$

We find, for all $i \in \mathcal{I}$, $\partial u_i(\mathbf{0}_n)/\partial y_i = \alpha_i > 0$ (Condition 3.39.3). The foregoing result contradicts (F.19). Consequently, $\mathbf{0}_n$ cannot be a boundary NE of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$.

Second, I show that a situation where some players play zero and the remaining players play a positive action (this is referred to as a boundary NE of type B-2 in Table F.1) cannot be a boundary NE of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$. Suppose, for the sake of contradiction, $\tilde{\boldsymbol{y}}^\star := (\tilde{y}_1^\star, \ldots, \tilde{y}_n^\star) \in \mathbb{R}_+^n$ is a boundary NE of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ of type B-2. Specifically, suppose there exists a $\mathcal{J} \subset \mathcal{I}$ with $0 < |\mathcal{J}| < n$ such that (i) for all $i \in \mathcal{I} \setminus \mathcal{J}$, $\tilde{y}_i^\star = 0$, and (ii) for all $i \in \mathcal{J}$, $\tilde{y}_i^\star \in \mathbb{R}_{++}$. We must have

$$\forall i \in \mathcal{I} \setminus \mathcal{J} \quad \frac{\partial u_i(\tilde{\boldsymbol{y}}^\star)}{\partial y_i} \leq 0. \tag{F.20}$$

We find

$$\forall i \in \mathcal{I} \setminus \mathcal{J} \quad \frac{\partial u_i(\tilde{\boldsymbol{y}}^\star)}{\partial y_i} = \alpha_i + \lambda_i \sum_{j \in \mathcal{I}} \dot{a}_{i,j}(D)\tilde{y}_j^\star \geq \alpha_i > 0,$$

where the equality follows from $\tilde{y}_i^\star = 0$, the first inequality from $\lambda_i \geq 0$ (Condition 3.39.1) and $\sum_{j \in \mathcal{I}} \dot{a}_{i,j}(D)\tilde{y}_j^\star \geq 0$, and the second inequality from $\alpha_i > 0$ (Condition 3.39.3). The foregoing result contradicts (F.20). Consequently, $\tilde{\boldsymbol{y}}^\star$ cannot be a boundary NE of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ of type B-2. ■

Proof of Proposition 3.40

Suppose Conditions 3.40.1 to 3.40.4 are satisfied.

Let $\mathcal{Y}^\star \subset \mathbb{R}_{++}^n$ denote the set of all interior NEs in pure strategies of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$. I show that $|\mathcal{Y}^\star| = 1$. Note that, for all $\boldsymbol{y}^\star \in \mathbb{R}_+^n$, $\boldsymbol{y}^\star \in \mathcal{Y}^\star$ if and only if (F.16), (F.17), and (F.18) are true. Analogous to the proof of Proposition 3.39, we find that there exists a unique $\boldsymbol{y}^\star \in \mathbb{R}^n_+$, namely, $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$, that satisfies (F.16), (F.17), and (F.18). As regards (F.16), note that Conditions 3.40.2 and 3.40.3 imply that $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) >_c \mathbf{0}_n$ (Result 3.14.2).

In the remainder of the proof, I show that $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ has no boundary NEs. Table F.1 gives an overview of all possible types of boundary NEs of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$.

First, $\mathbf{0}_n$ (this is referred to as a boundary NE of type B-1 in Table F.1) cannot be a boundary NE of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$. See the proof of Proposition 3.39 for the precise argument (Conditions 3.40.2 and 3.40.3 imply that $\boldsymbol{\alpha} >_c \mathbf{0}_n$).

Second, I show that a situation where some players play zero and the remaining players play a positive action (this is referred to as a boundary NE of type B-2 in Table F.1) cannot be a boundary NE of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$. Suppose, for the sake of contradiction, $\tilde{\boldsymbol{y}}^\star := (\tilde{y}_1^\star, \ldots, \tilde{y}_n^\star) \in \mathbb{R}^n_+$ is a boundary NE of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ of type B-2. Specifically, suppose there exists a $\mathcal{J} \subset \mathcal{I}$ with $0 < |\mathcal{J}| < n$ such that (i) for all $i \in \mathcal{I} \setminus \mathcal{J}$, $\tilde{y}_i^\star = 0$, and (ii) for all $i \in \mathcal{J}$, $\tilde{y}_i^\star \in \mathbb{R}_{++}$. We must have

$$\forall i \in \mathcal{I} \setminus \mathcal{J} \quad \frac{\partial u_i(\tilde{\boldsymbol{y}}^\star)}{\partial y_i} \leq 0. \tag{F.21}$$

and

$$\forall i \in \mathcal{J} \quad \frac{\partial u_i(\tilde{\boldsymbol{y}}^\star)}{\partial y_i} = 0. \tag{F.22}$$

First, note that (F.21) is equivalent to

$$\forall i \in \mathcal{I} \setminus \mathcal{J} \quad \alpha_i + \lambda_i \sum_{j \in \mathcal{J}} \dot{a}_{i,j}(D) \tilde{y}_j^\star \leq 0,$$

which in turn is equivalent to

$$[\boldsymbol{\alpha}]_{\mathcal{I} \setminus \mathcal{J}} + [\operatorname{diag}(\boldsymbol{\lambda}) \dot{\boldsymbol{A}}(D)]_{\mathcal{I} \setminus \mathcal{J}, \mathcal{J}} [\tilde{\boldsymbol{y}}^\star]_{\mathcal{J}} \leq_c \mathbf{0}_{|\mathcal{I} \setminus \mathcal{J}|}. \tag{F.23}$$

Second, note that (F.22) is equivalent to

$$\forall i \in \mathcal{J} \quad \tilde{y}_i^\star - \lambda_i \sum_{j \in \mathcal{J}} \dot{a}_{i,j}(D) \tilde{y}_j^\star = \alpha_i,$$

which in turn is equivalent to

$$[\tilde{\boldsymbol{y}}^\star]_{\mathcal{J}} - [\operatorname{diag}(\boldsymbol{\lambda}) \dot{\boldsymbol{A}}(D)]_{\mathcal{J}, \mathcal{J}} [\tilde{\boldsymbol{y}}^\star]_{\mathcal{J}} = [\boldsymbol{\alpha}]_{\mathcal{J}}.$$

Condition 3.40.4 implies that $\boldsymbol{I}_{|\mathcal{J}|} - [\operatorname{diag}(\boldsymbol{\lambda}) \dot{\boldsymbol{A}}(D)]_{\mathcal{J}, \mathcal{J}}$ is nonsingular (Lemma B.3). It follows that

$$[\tilde{\boldsymbol{y}}^\star]_{\mathcal{J}} = \left(\boldsymbol{I}_{|\mathcal{J}|} - [\operatorname{diag}(\boldsymbol{\lambda}) \dot{\boldsymbol{A}}(D)]_{\mathcal{J}, \mathcal{J}} \right)^{-1} [\boldsymbol{\alpha}]_{\mathcal{J}}. \tag{F.24}$$

Combining results (F.23) and (F.24) gives

$$[\boldsymbol{\alpha}]_{\mathcal{I} \setminus \mathcal{J}} + [\operatorname{diag}(\boldsymbol{\lambda}) \dot{\boldsymbol{A}}(D)]_{\mathcal{I} \setminus \mathcal{J}, \mathcal{J}} \left(\boldsymbol{I}_{|\mathcal{J}|} - [\operatorname{diag}(\boldsymbol{\lambda}) \dot{\boldsymbol{A}}(D)]_{\mathcal{J}, \mathcal{J}} \right)^{-1} [\boldsymbol{\alpha}]_{\mathcal{J}} \leq_c \mathbf{0}_{|\mathcal{I} \setminus \mathcal{J}|}.$$

The foregoing result contradicts Condition 3.40.4. Consequently, $\tilde{\boldsymbol{y}}^\star$ cannot be a boundary NE of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ of type B-2. ∎

Proof of Proposition 3.44

Suppose Conditions C-α and C-λ are satisfied and $\rho(\text{diag}(|\boldsymbol{\lambda}|)\dot{\boldsymbol{A}}(D)) < 1$. Let $x \in \mathcal{I}$. Note that $1 \notin \sigma(\text{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) \cup \sigma(\text{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D \boxminus x))$ (Lemma 3.26). Note also that $b_x(\boldsymbol{e}_x, \boldsymbol{\lambda}, D) \neq 0$ (Proposition 3.25). We find

$$\begin{aligned}
\Upsilon(\boldsymbol{\omega}, D \boxminus x) - \Upsilon(\boldsymbol{\omega}, D) &= \sum_{i=1}^{n} \omega_i [\boldsymbol{\Delta}(D \boxminus x, D)]_i \\
&= \sum_{i=1}^{n} \omega_i \left(\alpha_x [\boldsymbol{e}_x]_i - \frac{b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)}{b_x(\boldsymbol{e}_x, \boldsymbol{\lambda}, D)} [\boldsymbol{b}(\boldsymbol{e}_x, \boldsymbol{\lambda}, D)]_i \right) \\
&= \sum_{i=1}^{n} \omega_i \alpha_x \delta_{i,x} - \frac{b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)}{b_x(\boldsymbol{e}_x, \boldsymbol{\lambda}, D)} \sum_{i=1}^{n} \omega_i [\boldsymbol{b}(\boldsymbol{e}_x, \boldsymbol{\lambda}, D)]_i \\
&= \omega_x \alpha_x - \frac{b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)}{b_x(\boldsymbol{e}_x, \boldsymbol{\lambda}, D)} \langle \boldsymbol{\omega}, \boldsymbol{b}(\boldsymbol{e}_x, \boldsymbol{\lambda}, D) \rangle,
\end{aligned}$$

where the second equality follows from (3.8). ∎

Proof of Proposition 3.45

Suppose Conditions C-α and C-λ are satisfied, $\boldsymbol{\alpha} >_c \boldsymbol{0}_n$ and $\boldsymbol{\lambda} \geq_c \boldsymbol{0}_n$, and $\rho(\text{diag}(\boldsymbol{\lambda})\dot{\boldsymbol{A}}(D)) < 1$. Note that $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ has a unique and interior NE, namely, $\boldsymbol{b}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)$ (Proposition 3.39). Let $x \in \mathcal{I}$. In what follows, $\Delta\Upsilon^I(x, \boldsymbol{\omega}, D)$ is abbreviated as $\Delta\Upsilon^I$. We find, according to Definition CS and Proposition 3.44,

$$\Delta\Upsilon^I = -\omega_x \big(b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) - \alpha_x\big) - b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) \sum_{j \in \mathcal{I} \setminus \{x\}} \omega_j S_{D,j,x}(\boldsymbol{\lambda}) \leq 0,$$

because $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n) \in [0,1]^n$, $b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) \geq \alpha_x$, $b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) > 0$, and for all $j \in \mathcal{I} \setminus \{x\}$, $S_{D,j,x}(\boldsymbol{\lambda}) \geq 0$ (Result 3.19.1).

Proof of Result 3.45.1 Suppose $\omega_x > 0$, $\lambda_x > 0$, and $\mathcal{N}_D^+(x) \neq \emptyset$. We find $b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) - \alpha_x = \lambda_x \sum_{j \in \mathcal{N}_D^+(x)} b_j(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) > 0$. It follows that $\Delta\Upsilon^I < 0$. ∎

Proof of Result 3.45.2 Suppose Condition $(*)$ is satisfied. It follows that $\Delta\Upsilon^I < 0$ because $\omega_i > 0$, $b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) > 0$ and $S_{D,i,x}(\boldsymbol{\lambda}) > 0$ (Results 3.19.1 and 3.19.2). ∎

Proof of Result 3.45.3 Suppose Conditions I-α-λ and $(*)$ are satisfied. We find

$$\frac{\partial \Delta\Upsilon^I}{\partial \alpha_x} = -\omega_x \big(b_x(\boldsymbol{e}_x, \boldsymbol{\lambda}, D) - 1\big) - b_x(\boldsymbol{e}_x, \boldsymbol{\lambda}, D) \sum_{j \in \mathcal{I} \setminus \{x\}} \omega_j S_{D,j,x}(\boldsymbol{\lambda}) < 0,$$

where the equality follows from Proposition 3.22 and the inequality from $\omega_i > 0$, $\omega_x \geq 0$, $b_x(\boldsymbol{e}_x, \boldsymbol{\lambda}, D) \geq 1$ (Result 3.19.1), and $S_{D,i,x}(\boldsymbol{\lambda}) > 0$ (Results 3.19.1 and 3.19.2). ∎

Proof of Result 3.45.4 Suppose Conditions I-α-λ and $(*)$ are satisfied and there exists a walk $(k_0, \ldots, k_q)$ in D of length q from x to i such that $(\lambda_{k_0}, \ldots, \lambda_{k_{q-1}}) >_c \mathbf{0}_q$. We find

$$\frac{\partial \Delta Y^I}{\partial \alpha_i} = -\omega_x b_x(\boldsymbol{e}_i, \boldsymbol{\lambda}, D) - b_x(\boldsymbol{e}_i, \boldsymbol{\lambda}, D) \sum_{j \in \mathcal{I} \setminus \{x\}} \omega_j S_{D,j,x}(\boldsymbol{\lambda}) < 0,$$

where the equality follows from Proposition 3.22 and the inequality from $\omega_i > 0$, $\omega_x \geq 0$, $b_x(\boldsymbol{e}_i, \boldsymbol{\lambda}, D) > 0$ (Result 3.19.2), and $S_{D,i,x}(\boldsymbol{\lambda}) > 0$ (Results 3.19.1 and 3.19.2). ■

Proof of Result 3.45.5 Suppose Conditions I-α-λ and $(*)$ are satisfied and $\mathcal{N}_D^+(x) \neq \emptyset$. We find

$$\frac{\partial \Delta Y^I}{\partial \lambda_x} = -\omega_x \frac{\partial b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)}{\partial \lambda_x} - \frac{\partial b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)}{\partial \lambda_x} \sum_{j \in \mathcal{I} \setminus \{x\}} \omega_j S_{D,j,x}(\boldsymbol{\lambda}) < 0,$$

where the equality follows from, for all $j \in \mathcal{I} \setminus \{x\}$, $\partial S_{D,j,x}(\boldsymbol{\lambda})/\partial \lambda_x = 0$ (see the proof of Proposition 3.30), and the inequality from

$$\frac{\partial b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)}{\partial \lambda_x} = b_x(\boldsymbol{e}_x, \boldsymbol{\lambda}, D) \sum_{k \in \mathcal{N}_D^+(x)} b_k(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) > 0$$

(Proposition 3.23 and Result 3.19.1), $\omega_i > 0$, and $S_{D,i,x}(\boldsymbol{\lambda}) > 0$ (Results 3.19.1 and 3.19.2) ■

Proof of Result 3.45.6 Suppose Conditions I-α-λ and $(*)$ are satisfied. We find

$$\begin{aligned}\frac{\partial \Delta Y^I}{\partial \lambda_i} &= -\omega_x \frac{\partial b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)}{\partial \lambda_i} - \frac{\partial b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)}{\partial \lambda_i} \sum_{j \in \mathcal{I} \setminus \{x\}} \omega_j S_{D,j,x}(\boldsymbol{\lambda}) \\ &\qquad - b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D) \sum_{j \in \mathcal{I} \setminus \{x\}} \omega_j \frac{\partial S_{D,j,x}(\boldsymbol{\lambda})}{\partial \lambda_i} \\ &< 0,\end{aligned}$$

where the inequality follows from $\partial b_x(\boldsymbol{\alpha}, \boldsymbol{\lambda}, D)/\partial \lambda_i \geq 0$ (Proposition 3.23 and Result 3.19.1) and $\partial S_{D,i,x}(\boldsymbol{\lambda})/\partial \lambda_i > 0$ (see the proof of Proposition 3.30). ■

Proof of Corollary 3.46

Note that $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) = \rho(|\bar{\lambda}|\dot{\boldsymbol{A}}(D))$ (Lemma B.9). The statement follows from Proposition 3.44 and (3.26). ■

Proof of Proposition 3.47

Suppose Conditions 3.47.1 and 3.47.2 are satisfied. Let $\mathcal{I}_0^+(D)$ denote the set of all players with zero out-degree in D, and let $\mathcal{J} := \mathcal{I} \setminus \mathcal{I}_0^+(D)$.

Player i's utility at the action profile $\boldsymbol{y} := (y_1, \ldots, y_n) \in \mathbb{R}_+^n$ is given by

$$u_i(\boldsymbol{y}) = \begin{cases} -\dfrac{1}{2}y_i^2 & \text{if } \deg_D^+(i) = 0, \\ \Big(\deg_D^+(i) + \bar{\lambda} \displaystyle\sum_{j \in \mathcal{I}} \dot{a}_{i,j}(D) y_j\Big) y_i - \dfrac{1}{2}y_i^2 & \text{if } \deg_D^+(i) > 0, \end{cases}$$

and satisfies $\partial^2 u_i(\boldsymbol{y}) / \partial {y_i}^2 = -1 < 0$. It follows that player i's best reply, which is denoted by $\mathrm{BR}_i(\boldsymbol{y})$, satisfies

$$\mathrm{BR}_i(\boldsymbol{y}) = \begin{cases} 0 & \text{if } \deg_D^+(i) = 0, \\ \deg_D^+(i) + \bar{\lambda} \sum_{j \in \mathcal{I}} \dot{a}_{i,j}(D) y_j & \text{if } \deg_D^+(i) > 0. \end{cases}$$

Condition 3.47.2 implies that the system of best replies has a unique solution, which is given by $\boldsymbol{y}^\star := \boldsymbol{b}(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D)$. Note that $[\boldsymbol{y}^\star]_{\mathcal{I} \setminus \mathcal{J}} = \mathbf{0}_{|\mathcal{I} \setminus \mathcal{J}|}$ and $[\boldsymbol{y}^\star]_{\mathcal{J}}$ satisfies

$$[\boldsymbol{y}^\star]_{\mathcal{J}} = [\dot{\boldsymbol{A}}(D)\mathbf{1}_n]_{\mathcal{J}} + \bar{\lambda}[\dot{\boldsymbol{A}}(D)]_{\mathcal{J},\mathcal{J}}[\boldsymbol{y}^\star]_{\mathcal{J}}.$$

Note also that $\rho(\bar{\lambda}[\dot{\boldsymbol{A}}(D)]_{\mathcal{J},\mathcal{J}}) \leq \rho(\bar{\lambda})\dot{\boldsymbol{A}}(D)$ because $\bar{\lambda}\dot{\boldsymbol{A}}(D)$ is nonnegative and $\bar{\lambda}[\dot{\boldsymbol{A}}(D)]_{\mathcal{J},\mathcal{J}}$ is a principal submatrix of $\bar{\lambda}\dot{\boldsymbol{A}}(D)$ (see, for example, Berman and Plemmons 1994, Corollary 1.6 on p. 28). We find

$$[\boldsymbol{y}^\star]_{\mathcal{J}} = \big(\boldsymbol{I}_{|\mathcal{J}|} - \bar{\lambda}[\dot{\boldsymbol{A}}(D)]_{\mathcal{J},\mathcal{J}}\big)^{-1}[\dot{\boldsymbol{A}}(D)\mathbf{1}_n]_{\mathcal{J}} \geq_c [\dot{\boldsymbol{A}}(D)\mathbf{1}_n]_{\mathcal{J}} >_c \mathbf{0}_{|\mathcal{J}|},$$

where the equality follows from $\rho(\bar{\lambda}[\dot{\boldsymbol{A}}(D)]_{\mathcal{J},\mathcal{J}}) \leq \rho(\bar{\lambda})\dot{\boldsymbol{A}}(D)$ and Condition 3.47.2, the first inequality from Lemma B.6, and the second inequality from, for all $i \in \mathcal{J}$, $[\dot{\boldsymbol{A}}(D)\mathbf{1}_n]_i = \deg_D^+(i) > 0$. ∎

Proof of Proposition 3.48

Suppose α is as in Example 3.2, λ is as in Example 3.5 with $\bar{\lambda} \geq 0$, and $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$. Note that $1 \notin \sigma(\bar{\lambda}\dot{\boldsymbol{A}}(D)) \cup \sigma(\bar{\lambda}\dot{\boldsymbol{A}}(D \boxminus x))$ (Lemma 3.26).

First, consider the case $\bar{\lambda} = 0$. We find

$$\begin{aligned} Y(\boldsymbol{\omega}, D \boxminus x) - Y(\boldsymbol{\omega}, D) &= \sum_{i=1, i \neq x}^{n} \omega_i \Delta_i(D \boxminus x, D) + \omega_x \Delta_x(D \boxminus x, D) \\ &= -\sum_{i=1, i \neq x}^{n} \omega_i \mathbb{1}_{\mathcal{N}_D^+(i)}(x) - \omega_x \deg_D^+(x) \\ &= -\sum_{i=1, i \neq x}^{n} \omega_i \mathbb{1}_{\mathcal{N}_D^-(x)}(i) - \omega_x \deg_D^+(x) \end{aligned}$$

$$= -\sum_{i \in \mathcal{N}_D^-(x)} \omega_i - \omega_x \deg_D^+(x),$$

where the second equality follows from (3.14), the third from $x \in \mathcal{N}_D^+(i)$ if and only if $i \in \mathcal{N}_D^-(x)$, and the last from $x \notin \mathcal{N}_D^-(x)$ (because D has no self-loops).

Second, consider the case $\bar{\lambda} > 0$. We find

$$\begin{aligned}
&\Upsilon(\boldsymbol{\omega}, D \boxminus x) - \Upsilon(\boldsymbol{\omega}, D) \\
&\quad = \sum_{i=1}^{n} \omega_i [\boldsymbol{\Delta}(D \boxminus x, D)]_i \\
&\quad = \frac{1}{\bar{\lambda}} \left(\sum_{i=1}^{n} \omega_i [\boldsymbol{e}_x]_i - \frac{1}{b_x(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)} \sum_{i=1}^{n} \omega_i \big[\boldsymbol{b}(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)\big]_i \right) \\
&\qquad - \frac{b_x\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D\big)}{b_x(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)} \sum_{i=1}^{n} \omega_i \big[\boldsymbol{b}(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)\big]_i \\
&\quad = \frac{1}{\bar{\lambda}} \left(\omega_x - \frac{\big\langle \boldsymbol{\omega}, \boldsymbol{b}(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)\big\rangle}{b_x(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)} \right) - \frac{b_x\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D\big)\big\langle \boldsymbol{\omega}, \boldsymbol{b}(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)\big\rangle}{b_x(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)} \\
&\quad = \frac{1}{\bar{\lambda}} \left(\omega_x - \frac{b_x(\boldsymbol{\omega}, \bar{\lambda}\mathbf{1}_n, D^\top)}{b_x(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)} \right) - \frac{b_x\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D\big)\, b_x(\boldsymbol{\omega}, \bar{\lambda}\mathbf{1}_n, D^\top)}{b_x(\boldsymbol{e}_x, \bar{\lambda}\mathbf{1}_n, D)},
\end{aligned}$$

where the second equality follows from (3.13) and the last from (3.26). ■

Proof of Proposition 3.49

Suppose Conditions C-α and C-λ are satisfied, $\rho(\operatorname{diag}(|\boldsymbol{\lambda}(D)|)\dot{\boldsymbol{A}}(D)) < 1$, and for all $x \in \mathcal{I}$, $\boldsymbol{\alpha}(D \ominus x) = [\boldsymbol{\alpha}(D \boxminus x)]_{-x}$ and $\boldsymbol{\lambda}(D \ominus x) = [\boldsymbol{\lambda}(D \boxminus x)]_{-x}$. Note that $\boldsymbol{\lambda}(D) = \boldsymbol{\lambda}(D \boxminus x)$ and $1 \notin \sigma(\operatorname{diag}(\boldsymbol{\lambda}(D \ominus x))\dot{\boldsymbol{A}}(D \ominus x)) \cup \sigma(\operatorname{diag}(\boldsymbol{\lambda}(D))\dot{\boldsymbol{A}}(D))$ (Lemma 3.26 and Result 3.35.1). We find

$$\begin{aligned}
&\Upsilon([\boldsymbol{\omega}]_{-x}, D \ominus x) - \Upsilon(\boldsymbol{\omega}, D) \\
&\quad = \Big\langle [\boldsymbol{\omega}]_{-x}, \boldsymbol{b}\big(\boldsymbol{\alpha}(D \ominus x), \boldsymbol{\lambda}(D \ominus x), D \ominus x\big)\Big\rangle - \Big\langle \boldsymbol{\omega}, \boldsymbol{b}\big(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D\big)\Big\rangle \\
&\quad = \Big\langle [\boldsymbol{\omega}]_{-x}, \Big[\boldsymbol{b}\big(\boldsymbol{\alpha}(D \boxminus x), \boldsymbol{\lambda}(D \boxminus x), D \boxminus x\big)\Big]_{-x}\Big\rangle - \Big\langle \boldsymbol{\omega}, \boldsymbol{b}\big(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D\big)\Big\rangle \\
&\quad = \sum_{i=1}^{n} \omega_i\, b_i\big(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D \boxminus x\big) - \sum_{i=1}^{n} \omega_i\, b_i\big(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D\big) \\
&\qquad - \omega_x\, b_x\big(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D \boxminus x\big) \\
&\quad = \Upsilon(\boldsymbol{\omega}, D \boxminus x) - \Upsilon(\boldsymbol{\omega}, D) - \omega_x \alpha_x(D),
\end{aligned}$$

where the second equality follows from Proposition 3.35, the third from $\boldsymbol{\alpha}(D \boxminus x) = \boldsymbol{\alpha}(D)$ (Condition C-α) and $\boldsymbol{\lambda}(D \boxminus x) = \boldsymbol{\lambda}(D)$ (Condition C-λ), and the last from $b_x(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D \boxminus x) = \alpha_x(D)$ (because $\mathcal{N}_{D \boxminus x}^+(x) = \emptyset$). ■

Proof of Corollary 3.50

The statement follows from Proposition 3.49. ■

Proof of Proposition 3.51

Suppose α is as in Example 3.2, λ is as in Example 3.5 with $\bar{\lambda} \geq 0$, and $\rho(\bar{\lambda}\dot{\boldsymbol{A}}(D)) < 1$. Note that $1 \notin \sigma(\bar{\lambda}\dot{\boldsymbol{A}}(D \ominus x)) \cup \sigma(\bar{\lambda}\dot{\boldsymbol{A}}(D))$ (Lemma 3.26 and Result 3.35.1). We find

$$\begin{aligned}
&\Upsilon([\boldsymbol{\omega}]_{-x}, D \ominus x) - \Upsilon(\boldsymbol{\omega}, D) \\
&\quad = \big\langle [\boldsymbol{\omega}]_{-x}, \boldsymbol{b}\big(\boldsymbol{\alpha}(D \ominus x), \boldsymbol{\lambda}(D \ominus x), D \ominus x\big)\big\rangle - \big\langle \boldsymbol{\omega}, \boldsymbol{b}\big(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D\big)\big\rangle \\
&\quad = \big\langle [\boldsymbol{\omega}]_{-x}, \big[\boldsymbol{b}\big(\boldsymbol{\alpha}(D \boxminus x), \boldsymbol{\lambda}(D \boxminus x), D \boxminus x\big)\big]_{-x}\big\rangle - \big\langle \boldsymbol{\omega}, \boldsymbol{b}\big(\boldsymbol{\alpha}(D), \boldsymbol{\lambda}(D), D\big)\big\rangle \\
&\quad = \sum_{i=1}^{n} \omega_i\, b_i\big(\dot{\boldsymbol{A}}(D \boxminus x)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D \boxminus x\big) - \sum_{i=1}^{n} \omega_i\, b_i\big(\dot{\boldsymbol{A}}(D)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D\big) \\
&\qquad - \omega_x\, b_x\big(\dot{\boldsymbol{A}}(D \boxminus x)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D \boxminus x\big) \\
&\quad = \Upsilon(\boldsymbol{\omega}, D \boxminus x) - \Upsilon(\boldsymbol{\omega}, D),
\end{aligned}$$

where the second equality follows from Proposition 3.35 and the last equality from $b_x(\dot{\boldsymbol{A}}(D \boxminus x)\mathbf{1}_n, \bar{\lambda}\mathbf{1}_n, D \boxminus x) = [\dot{\boldsymbol{A}}(D \boxminus x)\mathbf{1}_n]_x = 0$ (because $\mathcal{N}^{+}_{D \boxminus x}(x) = \emptyset$). ■

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Curriculum Vitae

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